Lecture 31

Complex numbers (applications)

MATH 0200

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Outline

Zeros of polynomials

Example

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More generally, recall that the zeros of a polynomial $f(x) = ax^2 + bx + c$ are $\frac{-b+\sqrt{D}}{2a}$ and $\frac{-b-\sqrt{D}}{2a}$, where $D = b^2 - 4ac$ is the discriminant of $f(x)$. If $D < 0$, then $\sqrt{D}$ is not a real number, so there are no real zeros of $f(x)$, but there are two complex zeros, namely, $\frac{-b+i\sqrt{-D}}{2a}$ and $\frac{-b-i\sqrt{-D}}{2a}$.

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$D = (-4)^2 - 4 \cdot 5 = 16 - 20 = -4 = (2i)^2$ and the zeros
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Remark

Notice that the numbers $2 + i$ and $2 - i$ are conjugate. This is not a coincidence. If a complex number z is a zero of a polynomial $P(x)$ with real coefficients, then its conjugate $\overline{z}$ is a zero of $P(x)$ as well.

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This follows from the fact a real number is equal to its conjugate, hence, $P(\overline{z}) = 0 = \overline{0} = \overline{P(z)}$.

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Well, it is possible to compute $(a+bi)^{10}$ directly, but would be computationally intense. Instead, we should use the polar form of z. Recall that the magnitude of z^{10} is $|z|^{10} = (\sqrt{a^2+b^2})^{10} = (a^2+b^2)^5$ and the argument is $Arg(z^{10}) = 10Arg(z)$ (modulo 2π).

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We first compute $|z| = \sqrt{\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{3+1}{2}} = \sqrt{2}$ and $Arg(z) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. It follows that $|z^{14}| = 2^7$ and $Arg(z^{14}) = \frac{14\pi}{6} = 2\pi + \frac{2\pi}{6} = 2\pi + \frac{\pi}{3} \equiv \frac{\pi}{3}$ (modulo 2π).

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Therefore, $z^{14} = 2^7(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)) = 2^7\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2^6(1+\sqrt{3}i) = 64(1+\sqrt{3}i)$.

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Answer:

(a) $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

(b) $Arg(z) = \arctan\left(\frac{1}{1}\right) = \arctan(1) = \frac{\pi}{4}$.

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(c) $|z^{10}| = \sqrt{2}^{10} = 2^5 = 32$.

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(c) $|z^{10}| = \sqrt{2}^{10} = 2^5 = 32$.

(d) $Arg(z^{10}) = 10Arg(z) = \frac{10\pi}{4} = 2\pi + \frac{2\pi}{4} = 2\pi + \frac{\pi}{2} \equiv \frac{\pi}{2}$.

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Answer:

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(b) $Arg(z) = \arctan\left(\frac{1}{1}\right) = \arctan(1) = \frac{\pi}{4}$.

(c) $|z^{10}| = \sqrt{2}^{10} = 2^5 = 32$.

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We conclude that $(1 + i)^{10} = 32i$.

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$$Arg(i^n) = \frac{\pi n}{2} = \begin{cases} 0, n\%4 = 0 \\ \frac{\pi}{2}, n\%4 = 1 \\ \pi, n\%4 = 2 \\ \frac{3\pi}{2}, n\%4 = 3 \end{cases} \text{ with } i^n = \begin{cases} 1, n\%4 = 0 \\ i, n\%4 = 1 \\ -1, n\%4 = 2 \\ -i, n\%4 = 3. \end{cases}$$

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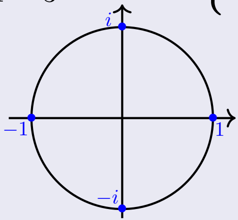

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Answer: as $2023\%4 = 3$, we conclude that $i^{2023} = i^3 = -i$.

Formulas for $\cos(a+b)$ and $\sin(a+b)$ revisited

Let $z = \cos(a) + i\sin(a)$ and $w = \cos(b) + i\sin(b)$ be two complex numbers. Observe that $|z| = |w| = 1$, so both z and w are on the unit circle and form angles a and b with the positive $x-$axis, respectively.

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Notice that $|zw| = |z| \cdot |w| = 1 \cdot 1 = 1$, while $Arg(zw) = Arg(z) + Arg(w) = a + b$. It follows that

$$zw = \cos(a+b) + i\sin(a+b).$$

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Also, $zw = (\cos(a) + i\sin(a))(\cos(b) + i\sin(b)) =$
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$(\cos(a)\cos(b) - \sin(a)\sin(b)) + i(\cos(a)\sin(b) + \sin(a)\cos(b)).$

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We have established that

- $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and
- $\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$.