Lecture 6

Inverse functions

MATH 0200

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Outline

1. Inverse functions

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Definition

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Remark

The inverse of a function is **NOT** the multiplicative inverse, in other words,

$$f^{-1} \neq \frac{1}{f}.$$

Example

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Said differently, we would like to find the function $C(x)$ with $F(C(x)) = x$ (since conversion from Fahrenheit to Celsius and then back to Fahrenheit should give the initial value).

Let's denote $C(x)$ by a dummy variable y. Then we need to find y from the equation $F(y) = \frac{9}{5}y + 32 = x$.

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Therefore, $C(x) = \frac{5}{9}(x - 32)$.
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$$C(F(x)) = \frac{5}{9}\left(\frac{9}{5}x + 32 - 32\right) = \frac{5}{9} \cdot \frac{9}{5}x = x.$$

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Example

Consider the function $f(x) = x^2$, then $f(1) = 1^2 = 1$, but $f(-1) = (-1)^2 = 1$ as well. If f^{-1} existed then what would $f^{-1}(1)$ be equal to? Recall that f^{-1} is a function, so we can not have $f^{-1}(1) = \{1, -1\}$. Therefore, the function $f(x) = x^2$ does not have an inverse.

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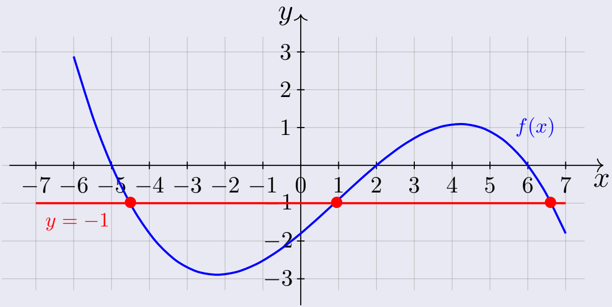
Remark

A function f is invertible if and only if it is one-to-one.

Example

The function $f(x)$ below is not one-to-one.



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Step 3. Substitute x by $f^{-1}(x)$ and y by x to obtain a formula for the inverse function.

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3. We get $f^{-1}(x) = \dfrac{1-3x}{x+1}$.

We check: $f^{-1} \circ f(x) = \dfrac{1 - 3 \cdot \dfrac{1-x}{3+x}}{\dfrac{1-x}{3+x} + 1} = \dfrac{\dfrac{3 + x - (3 - 3x)}{3+x}}{\dfrac{1 - x + 3 + x}{3+x}} =$

$$\dfrac{\dfrac{x+3x}{3+x}}{\dfrac{4}{3+x}} = \frac{4x}{4} = x \quad \checkmark$$

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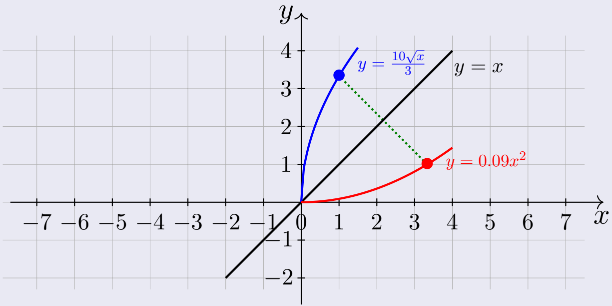

Question

Let $f(x)$ be given by the table below

x	$f(x)$
-2	3
0	1
3	-2

What are the range of f^{-1} and the value of $f^{-1}(-2)$?