Lecture 9

Exponents

MATH 0200

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Outline

Positive integer exponents

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- $(a-b)^2 = (a-b) \cdot (a-b) = a \cdot a - a \cdot b - b \cdot a + b \cdot b = a^2 - 2ab + b^2.$

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- $(ab)^m = \underbrace{ab \cdot \ldots \cdot ab}_{m} = \underbrace{a \cdot \ldots \cdot a}_{m} \cdot \underbrace{b \cdot \ldots \cdot b}_{m} = a^m b^m$ (in the last equality we have used that $ab = ba$ and that the order in which the multiplications are performed does not change the product).

Negative integer exponents

Let a be a number and m a positive integer. Notice that $a^m = a^{m+0} = a^m \cdot a^0$, equivalently, $a^m(a^0 - 1) = 0$. If $a \neq 0$, then we must have (define) $a^0 = 1$.

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- $23^{-1} = \frac{1}{23}$;
- $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$.

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Notice that the equation $x^m = -1$ has a single real solution $x = -1$ if m is an odd positive integer and no real solutions if m is an even positive integer.
It follows that $\sqrt[m]{a}$ has domain $[0, \infty)$ for even positive integers m and $(-\infty, \infty)$ for odd positive integers m.

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- $(-512)^{\frac{1}{12}} = \sqrt[12]{-512}$ DNE, since 12 is even and $-512 < 0$;
- $4^{-\frac{3}{2}} = \frac{1}{(\sqrt{4})^3} = \frac{1}{2^3} = \frac{1}{8}$.

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1. $a^m a^n = a^{m+n}$;
2. $(a^m)^n = a^{mn}$;
3. $a^{-m} = \frac{1}{a^m}$;
4. $a^{m-n} = \frac{a^m}{a^n}$;
5. $a^{\frac{1}{m}} = \sqrt[m]{a}$;
6. $a^m b^m = (ab)^m$;
7. $a^0 = 1$ for any $a \neq 0$ and the expression 0^0 is undefined.

Question

Evaluate $90 \cdot 27^{-\frac{2}{3}}$.