Proposition. Let p be an odd prime, then the Legendre symbol $\left(\frac{2}{p}\right) = \begin{cases} 1, & p \equiv \pm 1 \pmod 8 \\ -1, & p \equiv \pm 3 \pmod 8). \end{cases}$

Proof. We can rewrite the product of all even residues modulo p as $2\cdot 4\cdot 6\cdot\ldots\cdot(p-1) \equiv 2\cdot 4\cdot 6\cdot\ldots\cdot 2\lfloor\frac{p-1}{4}\rfloor\cdot$ $\cdot\left(-2\cdot\lfloor\frac{p-1}{4}\rfloor + (-1)^{\frac{p-1}{2}}\right)\cdot\ldots\cdot(-3)\cdot(-1)$ (substitute every factor a greater that $\frac{p-1}{2}$ by $p-a$) ($\star$)

Examples.
- $p=11$, then $2\cdot 4\cdot 6\cdot 8\cdot 10 \equiv 2\cdot 4\cdot(-5)\cdot(-3)\cdot(-1) \pmod{11}$
- $p=13$, then $2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12 \equiv 2\cdot 4\cdot 6\cdot(-5)(-3)(-1) \pmod{13}$

Notice that the l.h.s. of ($\star$) is

$$2\cdot 4\cdot 6\cdot\ldots\cdot(p-1) \equiv 2^{\frac{p-1}{2}}\cdot 1\cdot 2\cdot 3\cdot\ldots\cdot\left(\frac{p-1}{2}\right) = 2^{\frac{p-1}{2}}\cdot\left(\frac{p-1}{2}\right)!,$$

while the r.h.s. can be simplified as

$$2\cdot 4\cdot\ldots\cdot 2\left\lfloor\frac{p-1}{4}\right\rfloor\cdot\underbrace{(-1)\cdot(-3)\cdot\ldots\cdot\left(-2\left\lfloor\frac{p-1}{4}\right\rfloor+(-1)^{\frac{p-1}{2}}\right)}_{\lceil\frac{p-1}{4}\rceil \text{ factors}} =$$

$$= (-1)^{\lceil\frac{p-1}{4}\rceil}\cdot 1\cdot 2\cdot 3\cdot\ldots\cdot\left(\frac{p-1}{2}\right) = (-1)^{\lceil\frac{p-1}{4}\rceil}\cdot\left(\frac{p-1}{2}\right)!$$

Therefore, ($\star$) is equivalent to the equality

$$2^{\frac{p-1}{2}}\left(\frac{p-1}{2}\right)! = (-1)^{\lceil\frac{p-1}{4}\rceil}\cdot\left(\frac{p-1}{2}\right)! \iff 2^{\frac{p-1}{2}} = (-1)^{\lceil\frac{p-1}{4}\rceil}$$

As $2^{\frac{p-1}{2}} = \left(\frac{2}{p}\right)$ (Euler's property), it remains to check that $(-1)^{\lceil\frac{p-1}{4}\rceil} = \begin{cases} 1, & p \equiv \pm 1 \pmod 8 \\ -1, & p \equiv \pm 3 \pmod 8 \end{cases}$, which is straightforward

$\square$