Summer 2020

Name:____ **Total**_____/100

MATH 4581: Statistics and Stochastic Processes

Final Exam

Problem 1 [10 **pts**] Records kept at a racetrack showed the following distribution of winners as a function of their starting-post position. All 144 races were run with a field of six horses. Test the goodness-of-fit hypothesis that horses are equally likely to win from any starting position at significance level $\alpha = 0.05$.

Starting Post	1	2	3	4	5	6
Number of Winners	36	25	23	24	20	16

Problem 2 Given the data below:

x_i	10	42	38	15	22
y_i	450	1050	900	525	710

(a) [5 **pts**] Find the equation of regression line. Test at the 5% level of significance if the slope of regression line is 0.

(b) [5 **pts**] Find the 95% confidence interval for the predicted value of y when $x = 40$. Show work.

Problem 3 Let X and Y be independent random variables with moment generating functions $M_X(t) = \frac{1}{1-t}$ and $M_Y(t) = \frac{1}{1-2t}$.

(a) [5 **pts**] Find the moment-generating function of the random variable $W = 2X - 3Y + 2020$.

(b) [5 **pts**] Using the result in (a), find the expected value $\mathbb{E}(W)$.

Problem 4 Consider the $M/M/?/k$ system $(k > 1)$.

(a) [5 **pts**] What is the minimal value of $'?'$, so there will never be any customers waiting to be served.

(b) [5 **pts**] Let $'?' = k - 2$, i.e. consider the system $M/M/k-2/k$ with $k > 2$. Find an expression for L_Q in terms of the steady state probabilities $p_0, p_1, p_2, \ldots, p_k$ (DO NOT COMPUTE THE p_i's).

Problem 5 Consider the $M/M/2$ system with average arrival rate 2 people per minute and mean service time 30 seconds.

(a) [5 **pts**] Compute the steady state probabilities.

(b) [5 **pts**] Find L, the average number of customers in the system and W, the average amount of time a customer spends in the system.[1]

Problem 6.

(a) [5 **pts**] Use Ito's formula to compute the differential of $X(t, B(t)) = 2.2t + 3B^2(t)$ ($B(t)$ is a standard Brownian motion):

(b) [5 **pts**] Assume we use the function $X(t, B(t)) = 2.2t + 3B^2(t)$ to model the price of a stock and at this moment $t = 1$. Find the probability that the price of the stock in three years is less than \$29.8.

Problem 7 Let $X(t)$ be the price of a stock at time t. Assume that the current price of the stock is \$50 and it is modeled by a geometric Brownian motion with drift parameter $\mu = -0.1$ and volatility $\sigma = .7$.

(a) [5 **pts**] Find the probability that the price of the stock in three years is more than \$40.

[1] **Hint:** You may use the formula $1 + 2q + 3q^2 + \ldots = \frac{1}{(1-q)^2}$ for $|q| < 1$.

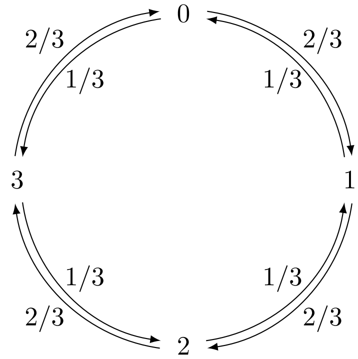
(*b*) [5 **pts**] If the yearly interest rate is $r = .05$, what should the selling price of a European 3 year Call option with strike price \$35 be, so there is no arbitrage opportunity?

Problem 8.

(a) [5 **pts**] Find $P(X(1) = 4)$ if the random variable X has a Poisson distribution such that $P(X(1) = 1) = P(X(1) = 2)$.

(b) [5 **pts**] Assume that the number of hits, X, that a baseball team makes in a nine-inning game has a Poisson distribution. If the probability that a team makes zero hits is $1/3$, what are their chances of getting two or more hits?

Problem 9. Consider the Markov chain with states $S = \{0, 1, 2, 3\}$ and transition probabilities $p_{ii+1} = 2/3$, $p_{ii-1} = 1/3$ and $p_{ii} = p_{ij} = 0$ for $|i - j| > 1$ (here we work in the arithmetics modulo 4, i.e. $0 \equiv 4$, so the state corresponding to $3 + 1$ is 0 and the state corresponding to $0 - 1$ is 3).



(a) [2 **pts**] Write down the transition matrix P.

(b) [3 **pts**] Find the stationary distribution $w = (w_0, w_1, w_2, w_3)$.

(c) [5 **pts**] Find the mean first passage time from state 0 to state 2.

(d) [5 **pts**] Find the mean recurrence time for state 2.

Problem 10**.** Let $B(t)$ be a standard Brownian motion.

(a) [2 **pts**] Find $P(B(9) - B(7) > 2)$

(b) [3 **pts**] Find $P(B(8) - B(6) > \sqrt{2} \mid B(5) - B(2) = 20^{e^{\pi}} \text{ and } B(15) - B(9) < 50)$.