Summer 2020

Name:____ **Total**_____/100

MATH 4581: Statistics and Stochastic Processes

Final Exam

Problem 1[10 **pts**] Records kept at a racetrack showed the following distribution of winners as a function of their starting-post position. All 144 races were run with a field of six horses. Test the goodness-of-fit hypothesis that horses are equally likely to win from any starting position at significance level $\alpha = 0.05$.

Starting Post	1	2	3	4	5	6
Number of Winners	36	25	23	24	20	16

Solution:

Number of Winners	36	25	23	24	20	16
Expected Number of Winners	24	24	24	24	24	24

As $df = n - 1 = 5$ the critical value is $\chi^2_{0.05,5} = 11.07$. Since the test statistic $X^2 = \frac{(36-24)^2}{24} + \ldots + \frac{(16-24)^2}{24} = 9.42 < 11.07$ or the p-value $0.094 > 0.05$, we accept H_0.

Problem 2 Given the data below:

x_i	10	42	38	15	22
y_i	450	1050	900	525	710

(a) [5 **pts**] Find the equation of regression line. Test at the 5% level of significance if the slope of regression line is 0.

Solution:

The regression line is $y = 17.652x + 278.648$, as $p = 0.0014 < 0.05$, we reject H_0.

(b) [5 **pts**] Find the 95% confidence interval for the predicted value of y when $x = 40$. Show work.

Solution:

As $df = n - 2 = 3$, the critical value is $t_{0.025,3} = 3.182$. Also $\hat{y}(40) = 984.714, \bar{x} = 25.4, \sum_{i=1}^{5}(x_i - \bar{x})^2 = 791.2$.

$$CI = \hat{y}(40) \pm t_{0.025,3} \cdot s\sqrt{1 + \frac{1}{5} + \frac{(40-25.4)^2}{\sum_{i=1}^{5}(x_i-\bar{x})^2}} = 984.714 \pm 3.182 \cdot 43.04\sqrt{\frac{6}{5} + \frac{213.16}{791.2}} = 984.714 \pm 166.01.$$

Problem 3 Let X and Y be independent random variables with moment generating functions $M_X(t) = \frac{1}{1-t}$ and $M_Y(t) = \frac{1}{1-2t}$.

(a) [5 **pts**] Find the moment-generating function of the random variable $W = 2X - 3Y + 2020$.

Solution:

$M_W(t) = M_{2X}(t)M_{-3Y}(t)e^{2020t} = e^{2020t}M_X(2t)M_Y(-3t) = \frac{e^{2020t}}{(1-2t)(1+6t)}$.

(b) [5 **pts**] Using the result in (a), find the expected value $\mathbb{E}(W)$.

Solution:

$\mathbb{E}(W) = 2\mathbb{E}(X) - 3\mathbb{E}(Y) + 2020.$

$\mathbb{E}(X) = M_X'(0) = \left.\frac{1}{(1-t)^2}\right|_{t=0} = 1.$

$\mathbb{E}(Y) = M_Y'(0) = \left.\frac{2}{(1-2t)^2}\right|_{t=0} = 2.$

$\mathbb{E}(W) = 2 - 6 + 2020 = 2016.$

Problem 4 Consider the $M/M/?/k$ system ($k > 1$).

(a) [5 **pts**] What is the minimal value of $'?'$, so there will never be any customers waiting to be served.

Solution:

$'?' = k$

(b) [5 **pts**] Let $'?' = k - 2$, i.e. consider the system $M/M/k-2/k$ with $k > 2$. Find an expression for L_Q in terms of the steady state probabilities $p_0, p_1, p_2, \ldots, p_k$ (DO NOT COMPUTE THE p_i's).

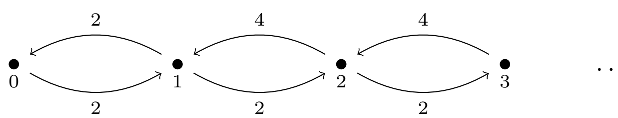
Solution:

$L_Q = p_{k-1} + 2p_k$.

Problem 5 Consider the $M/M/2$ system with average arrival rate 2 people per minute and mean service time 30 seconds.

(a) [5 **pts**] Compute the steady state probabilities.

Solution:



The balance equations

$$\begin{cases} 2p_0 = p_1 \\ 4p_1 = 2p_0 + 4p_2 \\ 6p_2 = 2p_1 + 4p_3 \\ \ldots \end{cases} \tag{1}$$

give $p_1 = p_0$, $p_2 = \frac{1}{2}p_0$, $p_n = \frac{1}{2^{n-1}}p_0, n \geq 2$. Using that $p_0 + p_1 + \ldots = 1$, we get $p_0 + p_1 + \ldots = p_0(1 + \sum_{n=1}^{\infty} \frac{1}{2^n}) = p_0(1 + \frac{1}{1-\frac{1}{2}}) = 3p_0 = 1$. Therefore, $p_1 = p_0 = \frac{1}{3}$ and $p_n = \frac{1}{3 \cdot 2^{n-1}}, n \geq 2$.

(b) [5 **pts**] Find L, the average number of customers in the system and W, the average amount of time a customer spends in the system.[1]

Solution:

$L = \sum_{n=0}^{\infty} np_n = \frac{1}{3}(1 + 2 \cdot \frac{1}{2} + 3 \cdot \left(\frac{1}{2}\right)^2 + \ldots) = \frac{1}{3} \cdot \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4/3.$

Little's formula gives $W = L/\lambda = 2/3$.

Problem 6.

(a) [5 **pts**] Use Ito's formula to compute the differential of $X(t, B(t)) = 2.2t + 3B^2(t)$ ($B(t)$ is a standard Brownian motion):

Solution:

$\frac{\partial X}{\partial t} = 2.2$, $\frac{\partial X}{\partial B(t)} = 6B(t)$ and $\frac{\partial^2 X}{\partial B(t)^2} = 6$, hence

$dX(t) = 5.2dt + 6B(t)dB(t)$.

(b) [5 **pts**] Assume we use the function $X(t, B(t)) = 2.2t + 3B^2(t)$ to model the price of a stock and at this moment $t = 1$. Find the probability that the price of the stock in three years is less than \$29.8.

Solution:

$P(X(4) < 29.8) = P(2.2 \cdot 4 + 3B^2(4) < 29.8) = P(8.8 + 3B^2(4) < 29.8) = P(B^2(4) < 7) = P(-\frac{\sqrt{7}}{2} < Z < \frac{\sqrt{7}}{2}) = \phi(\frac{\sqrt{7}}{2}) - \phi(-\frac{\sqrt{7}}{2}) = 1 - 2\phi(-\frac{\sqrt{7}}{2}) = 1 - 0.186 = 0.814$.

Problem 7 Let $X(t)$ be the price of a stock at time t. Assume that the current price of the stock is \$50 and it is modeled by a geometric Brownian motion with drift parameter $\mu = -0.1$ and volatility $\sigma = .7$.

(a) [5 **pts**] Find the probability that the price of the stock in three years is more than \$40.

Solution:

From the data given, $X(t) = 50e^{-0.1 \cdot t + 0.49B(t)}$, hence, $P(X(3) > 40) = P(50e^{-0.1 \cdot 2 + 0.7B(3)} > 40) = P(e^{-0.3+0.7B(3)} > 0.8) = P(-0.3 + 0.7B(3) > \ell n0.8) = P(B(3) > \frac{\ell n0.8+0.3}{0.7}) = 1 - P(Z < \frac{\ell n0.8+0.3}{0.7\sqrt{3}}) = 1 - P(Z < 0.063)1 - 0.525 = 0.475$.

[1] **Hint:** You may use the formula $1 + 2q + 3q^2 + \ldots = \frac{1}{(1-q)^2}$ for $|q| < 1$.

(b) [5 **pts**] If the yearly interest rate is $r = .05$, what should the selling price of a European 3 year Call option with strike price \$35 be, so there is no arbitrage opportunity?

Solution:

As $\mu = -0.1, \sigma = 0.7, r = 0.05, T = 3$ and $E = 35$, we compute

$b = \frac{\ell n(50/35)+3(0.05-\frac{0.7^2}{2})}{0.7\sqrt{3}} = -0.188.$

The price for the Call option is established at

$50\phi(0.7\sqrt{3} - 0.188) - 35e^{-0.15}\phi(-0.188) = 50 \cdot 0.847 - 35 \cdot e^{-0.15} \cdot 0.425 = \$29.55.$

Problem 8.

(a) [5 **pts**] Find $P(X(1) = 4)$ if the random variable X has a Poisson distribution such that $P(X(1) = 1) = P(X(1) = 2)$.

Solution:

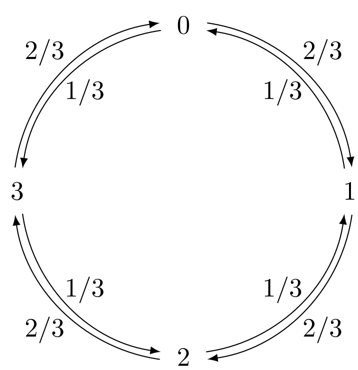
The equality $P(X(1) = 1) = P(X(1) = 2)$ gives $\lambda e^{-\lambda} = \frac{\lambda^2}{2}e^{-\lambda}$, which simplifies to $\lambda(\lambda - 2) = 0$. As $\lambda > 0$, the only solution is $\lambda = 2$, hence, $P(X(1) = 4) = \frac{2^4}{4!}e^{-2} = 0.09$.

(b) [5 **pts**] Assume that the number of hits, X, that a baseball team makes in a nine-inning game has a Poisson distribution. If the probability that a team makes zero hits is 1/3, what are their chances of getting two or more hits?

Solution:

$P(X = 0) = e^{-\lambda} = 1/3$, which gives $\lambda = -\ell n(1/3) = \ell n(3)$, hence, $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{1}{3} - \frac{\ell n(3)}{3} = 0.3$.

Problem 9. Consider the Markov chain with states $S = \{0, 1, 2, 3\}$ and transition probabilities $p_{ii+1} = 2/3$, $p_{ii-1} = 1/3$ and $p_{ii} = p_{ij} = 0$ for $|i - j| > 1$ (here we work in the arithmetics modulo 4, i.e. $0 \equiv 4$, so the state corresponding to $3 + 1$ is 0 and the state corresponding to $0 - 1$ is 3).



(a) [2 **pts**] Write down the transition matrix P.

Solution:

$$P = \begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 2/3 & 0 & 1/3 & 0 \end{pmatrix}.$$

(b) [3 **pts**] Find the stationary distribution $w = (w_0, w_1, w_2, w_3)$.

Solution:

Let $w = (x,\ y,\ z,\ t)$ be the fixed probability vector ($x + y + z + t = 1$). The matrix equation $(x,\ y,\ z,\ t)P = (x,\ y,\ z,\ t)$ reads

$$\begin{cases} y + 2t = 3x \\ 2x + z = 3y \\ 2y + t = 3z, \end{cases} \tag{2}$$

which gives $w = (1/4, 1/4, 1/4, 1/4)$.

(c) [5 **pts**] Find the mean first passage time from state 0 to state 2.

Solution:

Make state 2 absorbing to get the MC with $P' = \begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1 & 0 \\ 2/3 & 0 & 1/3 & 0 \end{pmatrix}$ or $P' = \begin{pmatrix} 0 & 2/3 & 1/3 & 0 \\ 1/3 & 0 & 0 & 2/3 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ in a canonical form. Hence, $Q' = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 0 \\ 2/3 & 0 & 0 \end{pmatrix}$ and $N' = (I - Q')^{-1} = \begin{pmatrix} 1.8 & 1.2 & 0.6 \\ 0.6 & 1.4 & 0.2 \\ 1.2 & 0.8 & 1.4 \end{pmatrix}$.

Thus $m_{02} = 1.8 + 1.2 + 0.6 = 3.6$.

(d) [5 **pts**] Find the mean recurrence time for state 2.

Solution:

$r_2 = 1 + p_{20}m_{02} + p_{21}m_{12} + p_{23}m_{32} = 1 + \frac{1}{3} \cdot 2.2 + \frac{2}{3} \cdot 3.4 = 4$.

Problem 10**.** Let $B(t)$ be a standard Brownian motion.

(a) [2 **pts**] Find $P(B(9) - B(7) > 2)$

Solution:

$P(B(9) - B(7) > 2) = 1 - P(Z < \sqrt{2}) = 1 - 0.921 = 0.079$.

(b) [3 **pts**] Find $P(B(8) - B(6) > \sqrt{2} \mid B(5) - B(2) = 20^{e^\pi}$ and $B(15) - B(9) < 50)$.

Solution:

As the intervals do not overlap, $P(B(8) - B(6) > \sqrt{2} \mid B(5) - B(2) = 20^{e^\pi}$ and $B(15) - B(9) < 50) = P(B(8) - B(6) > \sqrt{2}) = 1 - P(Z < 1) = 1 - 0.841 = 0.159$.