**Total**______/50

MATH 4581: Statistics and Stochastic Processes

# Homework

## Markov Chains

**Problem** 1 [5 **pts**] A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability $p$ that the digit that enters this stage will be changed when it leaves and a probability $q = 1-p$ that it won't. Form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1. What is the matrix of transition probabilities?

**Solution:**

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}.$$

**Problem** 2 [5 **pts**] In the Dark Ages, Harvard, Dartmouth, and Yale admitted only male students. Assume that, at that time, the son of a Harvard man always went to Harvard, 40% of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70% percent went to Dartmouth, 20% percent to Harvard, and 10% percent to Yale. We form a Markov chain with the steps corresponding to generations. Write the transition matrix $P$ and use it to find the probability that the grandson of a man from Yale went to Harvard.

**Solution:**

| | Harvard | Yale | Dartmouth |
|---|---|---|---|
| Harvard | 1 | 0 | 0 |
| Yale | 0.3 | 0.4 | 0.3 |
| Dartmouth | 0.2 | 0.1 | 0.7 |

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{pmatrix}.$$

$P^2_{21} = 0.3 \cdot 1 + 0.4 \cdot 0.3 + 0.3 \cdot 0.2 = 0.48 = 48\%.$

**Problem** 3 [10 **pts**]. A process moves on the integers $1, 2, 3, 4,$ and 5. It starts at 1 and, on each successive step, moves to an integer greater than its present position, moving with equal probability to each of the remaining larger integers. State five is an absorbing state. Find the expected number of steps to reach state five from state 1.

**Solution:**

$$P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$Q = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$N = (I-Q)^{-1} = \begin{pmatrix} 1 & 1/4 & 1/3 & 1/2 \\ 0 & 1 & 1/3 & 1/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$m_{15} = 1 + 1/4 + 1/3 + 1/2 = 25/12.$

**Problem** 4 Smith is in jail and has 2 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability .4 and loses $A$ dollars with probability .6. Find the probability that he wins 8 dollars before losing all of his money if

(a) [10 **pts**] he bets 2 dollars each time (timid strategy).

**Solution:** The transition matrix is

| | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 2 | 0.6 | 0 | 0.4 | 0 | 0 |
| 4 | 0 | 0.6 | 0 | 0.4 | 0 |
| 6 | 0 | 0 | 0.6 | 0 | 0.4 |
| 8 | 0 | 0 | 0 | 0 | 1 |

$$P_{timid} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The canonical form is

$$P_{timid} = \begin{pmatrix} 0 & 0.4 & 0 & 0.6 & 0 \\ 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$Q_{timid} = \begin{pmatrix} 0 & 0.4 & 0 \\ 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 \end{pmatrix}.$$

$$R_{timid} = \begin{pmatrix} 0.6 & 0 \\ 0 & 0 \\ 0 & 0.4 \end{pmatrix}.$$

$$N_{timid} = (I - Q_{timid})^{-1} = \begin{pmatrix} \frac{19}{13} & \frac{10}{13} & \frac{4}{13} \\ \frac{15}{13} & \frac{25}{13} & \frac{10}{13} \\ \frac{9}{13} & \frac{15}{13} & \frac{19}{13} \end{pmatrix}.$$

$(N_{timid}R_{timid})_{12} = \frac{4}{13} \cdot 0.4 = 0.123.$

(b) [8 **pts**] he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars (bold strategy).

**Solution:** The transition matrix is

| | 0 | 2 | 4 | 8 |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 2 | 0.6 | 0 | 0.4 | 0 |
| 4 | 0.6 | 0 | 0 | 0.4 |
| 8 | 0 | 0 | 0 | 1 |

$$P_{bold} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.6 & 0 & 0.4 & 0 \\ 0.6 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The canonical form is

$$P_{bold} = \begin{pmatrix} 0 & 0.4 & 0.6 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$Q_{bold} = \begin{pmatrix} 0 & 0.4 \\ 0 & 0 \end{pmatrix}.$$

$$R_{bold} = \begin{pmatrix} 0.6 & 0 \\ 0.6 & 0.4 \end{pmatrix}.$$

$$N_{bold} = (I - Q_{bold})^{-1} = \begin{pmatrix} 1 & 0.4 \\ 0 & 1 \end{pmatrix}.$$

$(N_{bold}R_{bold})_{12} = 0.16$

(c) [2 **pts**] Which strategy gives Smith the better chance of getting out of jail?

**Solution:** As $0.16 > 0.123$ the second strategy is preferrable.

**Problem** 5. A rat runs through the maze shown in Figure 1. At each step it leaves the room it is in by choosing at random one of the doors out of the room.

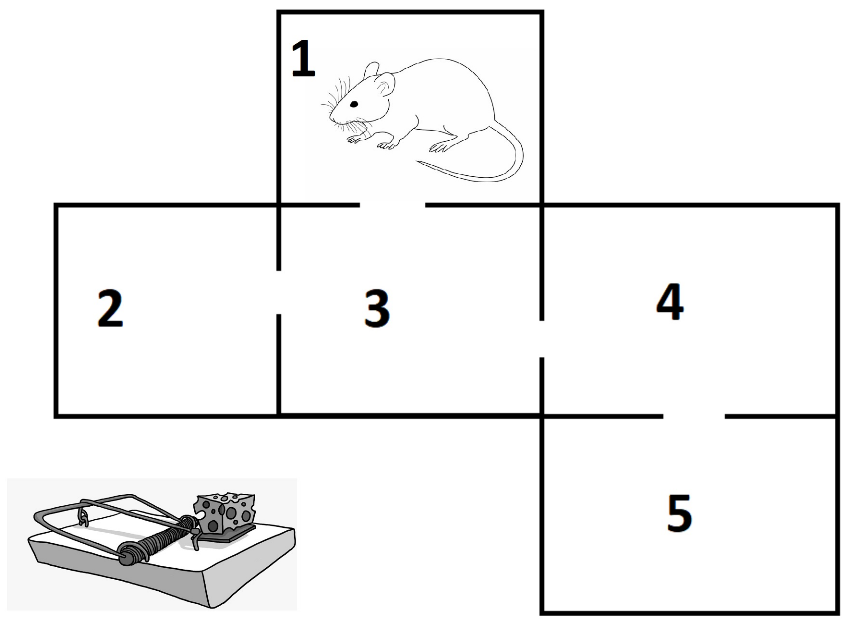


Figure 1: Rat run

(a) [5 **pts**] Give the transition matrix P for this Markov chain.

**Solution:**

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(b) [5 **pts**] Show that it is an ergodic chain.

**Solution:** The directed graph below shows the interconnections between the rooms. It is clear that any two vertices are connected by a path, which is equivalent to ergodicity of the chain.

$$
\begin{tikzcd}
2 \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[d, shift left] \\
\end{tikzcd}
$$

$$
\begin{tikzcd}
& 1 \arrow[d, shift left] & \\
2 \arrow[r, shift left] & 3 \arrow[l, shift left] \arrow[u, shift left] \arrow[r, shift left] & 4 \arrow[l, shift left] \arrow[d, shift left] \\
& & 5 \arrow[u, shift left]
\end{tikzcd}
$$

(c) [5 **pts**] Find the fixed probability vector $w$.

**Solution:** Let $w = (x,\ y,\ z,\ t,\ 1 - x - y - z - t)$ be the fixed probability vector. The matrix equation $(x,\ y,\ z,\ t,\ 1 - x - y - z - t)P = (x,\ y,\ z,\ t,\ 1 - x - y - z - t)$ reads

$$
\begin{cases}
\frac{1}{3}z = x \\
\frac{1}{3}z = y \\
x + y + \frac{1}{2}t = z \\
\frac{1}{2}t = 1 - x - y - z - t.
\end{cases}
\tag{1}
$$

Therefore the fixed vector is $w = (\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8})$.

(d) [5 **pts**] You are allowed to put a rat trap in one of the rooms and the rat is assumed to be caught with probability 1 upon entrance. Explain how to use the trap to modify the Markov chain so that it allows to find the expected number of steps before reaching Room 5 for the first time, starting in Room 1. Find this number.

**Solution:** Put the trap in room 5. The new Markov chain has transition matrix

$$
P' = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1/3 & 1/3 & 0 & 1/3 & 0 \\
0 & 0 & 1/2 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

$$
Q' = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1/3 & 1/3 & 0 & 1/3 \\
0 & 0 & 1/2 & 0
\end{pmatrix}.
$$

$$
N' = (I - Q')^{-1} = \begin{pmatrix}
3 & 2 & 6 & 2 \\
2 & 3 & 6 & 2 \\
2 & 2 & 6 & 2 \\
1 & 1 & 3 & 2
\end{pmatrix}.
$$

$m_{15} = 3 + 2 + 6 + 2 = 13.$