Summer 2020

Name:____ **Total**______/50

MATH 4581: Statistics and Stochastic Processes

Quiz 1

Problem 1

(a) [5 **pts**] Find the moment-generating function of the random variable X having the distribution $P(X=k)=\frac{1}{8}\binom{3}{k}$ with $k\in\{0,1,2,3\}$.

Solution: $M_X(t)=\mathbb{E}(e^{tX})=\frac{1}{8}(e^{0\cdot t}+3e^{1\cdot t}+3e^{2t}+e^{3t})=\frac{1}{8}(1+3e^{t}+3e^{2t}+e^{3t})$.

(b) [10 **pts**] Using the result in (a), find the expected value $\mathbb{E}(X)$, variance $Var(X)$ and standard deviation $\sigma(X)$.

Solution: $\mathbb{E}(X)=M_X(t)'\Big|_{t=0}=\frac{1}{8}(3e^{t}+6e^{2t}+3e^{3t})\Big|_{t=0}=\frac{12}{8}=1.5.$

$\mathbb{E}(X^2)=M_X(t)''\Big|_{t=0}=\frac{1}{8}(3e^{t}+12e^{2t}+9e^{3t})\Big|_{t=0}=\frac{24}{8}=3.$

$Var(X)=\mathbb{E}(X^2)-(\mathbb{E}(X))^2=3-2.25=0.75.$

$\sigma(X)=\sqrt{Var(X)}=0.866.$

Problem 2

(a) [5 **pts**] Check that the function $f_Y(y)=\frac{1}{3}$ for $0\le y\le 2$ and $f_Y(y)=\frac{2y}{15}$ for $2\le y\le 3$ is a probability distribution on the interval $[0,3]$.

Solution: $\int_0^3 f_Y(y)dy=\int_0^2\frac{1}{3}dy+\int_2^3\frac{y}{15}ydy=\frac{2}{3}+\frac{2}{15}\frac{y^2}{2}\Big|_2^3+\frac{9-4}{15}=1.$

(b) [10 **pts**] Find the moment-generating function of the random variable Y having the distribution $f_Y(y)$ as above.

Solution: $M_Y(t)=\int_0^3 f_Y(y)e^{ty}dy=\int_0^2\frac{1}{3}e^{ty}dy+\int_2^3\frac{ye^{ty}}{15}ydy\overset{*}{=}\frac{1}{3t}e^{ty}\Big|_0^2+\frac{1}{15}(\frac{ye^{ty}}{t}\Big|_2^3-\int_2^3\frac{e^{ty}}{t}dy)=\frac{1}{3t}(e^{2t}-1)+\frac{1}{15}(\frac{3e^{3t}-2e^{2t}}{t}-\frac{e^{ty}}{t^2}\Big|_2^3)=\frac{1}{3t}(e^{2t}-1)+\frac{1}{15}(\frac{3e^{3t}-2e^{2t}}{t}-\frac{e^{3t}-e^{2t}}{t^2})$, where we used integration by parts in equality $(*)$.

Problem 3[5 **pts**] Let X and Y be independent random variables. Express the moment-generating function of $W=3X-2Y+2020$ in terms of $M_X(t)$ and $M_Y(t)$, the moment-generating functions of X and Y. Express the moment-generating function of $W=3X-2Y+2020$ in terms of $M_X(t)$ and $M_Y(t)$, the moment-generating functions of X and Y.

Solution: Using property (3) of moment-generating functions we get $M_W(t)=M_X(3t)M_{-2Y+2020}(t)$ and property allows to write the second factor as $M_{-2Y+2020}(t)=e^{2020t}M_Y(-2t)$ giving rise to the final answer $M_W(t)=e^{2020t}M_X(3t)M_Y(-2t)$.

Problem 4 In the dataset "Popular Kids", students in grades $5-7$ were asked whether good grades, athletic ability, or popularity was most important to them. A two-way table separating the students by grade and by choice of most important factor is shown below

Goals	Grade 5	Grade 6	Grade 7	Total
Grades	49	50	69	168
Popularity	24	36	38	98
Sports	19	22	28	69
Total	92	108	135	335

Table 1: Observed values

(a) [7 **pts**] Fill in the table of expected values.

The expected value in cell $ij=\frac{(\text{\# in row i})(\text{\# in column j})}{\text{Total number of elements}}$.

Goals	Grade 5	Grade 6	Grade 7
Grades	46.14	54.16	67.70
Popularity	26.91	31.59	39.49
Sports	18.95	22.24	27.81

Table 2: Expected values

(b) [8 **pts**] Use the χ^2 test and either the critical value or p-value to decide if there is a statistically significant difference at the level $\alpha = 5\%$ between the preferences of three groups.

Solution:

I. Since $p = 0.824 > 0.05$, accept H_0.

II. Find that $df = 2 \cdot 2 = 4$, hence the critical value $\chi^2_{0.05,4} = 9.49$. Since $\chi^2 = 1.51 < 9.49$, accept H_0.