Summer 2020

Name:____ **Total**______/50

MATH 4581: Statistics and Stochastic Processes

Quiz 2

Problem 1 Consider the $M/M/1$ system with average arrival rate 5 people per hour and mean service time 6 minutes.

(a) [5 **pts**] Compute the steady state probabilities.

Solution: We have a birth–death process with $\lambda = 5$ and $\mu = 60 : 6 = 10$. Next compute $\rho = \frac{\lambda}{\mu} = \frac{1}{2}$, so $p_n = \rho^n p_0 = \left(\frac{1}{2}\right)^n p_0$. As the probabilities must sum up to 1, it follows that

$$1 = \sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n p_0 = p_0 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{p_0}{1 - \frac{1}{2}} = 2p_0,$$

from which we deduce that $p_0 = \frac{1}{2}$ and $p_n = \rho^n p_0 = \left(\frac{1}{2}\right)^n p_0 = \left(\frac{1}{2}\right)^{n+1}$.

(b) [5 **pts**] Find the average number of customers in the system and the average amount of time a customer spends in the system.

Solution: $L = \frac{\rho}{1-\rho} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$.
Using Little's formula we obtain $W = \frac{1}{\lambda} L = \frac{1}{5}$ hour or 12 minutes.

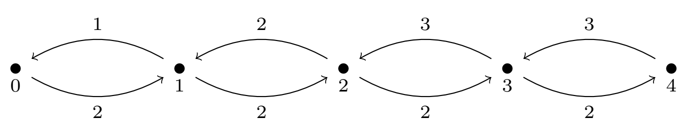
Problem 2 On Monday morning in the Bank of Wonderland customers arrive at the average rate of 8 per 6 minutes. Three tellers are operating with mean service time of 90 seconds. In addition, the bank president Alice does not allow more than 4 customers inside the bank simultaneously.

(a) [5 **pts**] Determine the type of the queue described above, i.e. fill in the missing numbers indicated by the question marks: $M/M/?/?$.

Solution: $M/M/3/4$.

(b) [10 **pts**] Find the steady state probabilities. Show your work.

Solution: $\lambda = 2/90$ seconds, $\mu = 1/90$ seconds.



The balance equations form the system

$$\begin{cases} 2p_0 = p_1 \\ (1+2)p_1 = 2p_0 + 2p_2 \\ (2+2)p_2 = 2p_1 + 3p_3 \\ (2+3)p_3 = 2p_2 + 3p_4 \\ 3p_4 = 2p_3, \end{cases} \tag{1}$$

from which we establish that $p_2 = p_1 = 2p_0$, $p_3 = \frac{4}{3}p_0$ and $p_4 = \frac{8}{9}p_0$. The equality

$$1 = \sum_{n=0}^{4} p_n = p_0(1 + 2 + 2 + \frac{4}{3} + \frac{8}{9}) = \frac{65}{9}p_0$$

allows to conclude $p_0 = \frac{9}{65}$, $p_2 = p_1 = \frac{18}{65}$, $p_3 = \frac{12}{65}$ and $p_4 = \frac{8}{65}$.

(c) [5 **pts**] Find L the average number of customers in the system. Show your work.

Solution: $L = \sum_{n=0}^{4} np_n = \frac{18}{65} + \frac{36}{65} + \frac{36}{65} + \frac{32}{65} = \frac{122}{65} \approx 1.877.$

(d) [5 **pts**] Find W, the average amount of time a customer has to wait in the system and W_Q, the average amount of time a customer has to wait in the queue.[1]

Solution: $W = \frac{L}{\lambda} = 1.877/(\frac{4}{3}) = 1.408$ minutes with $\mu_a = \sum_{i=1}^{4} p_i\mu_i = p_1 + 2p_2 + 3p_3 + 3p_4 = 2.631$ and $W_Q = W - \frac{1}{\mu_a} = 0.554$.

Problem 3 Northeastern students send or receive an average of 48 text messages per day.

(a) [2 **pts**] How many text messages does a Northeastern student receive or send on average per hour?

Solution: $\lambda = 48 : 24 = 2$.

(b) [5 **pts**] What is the probability that a Northeastern student receives or sends four messages per hour?

Solution: Let X be the number of text messages sent or received by a student per hour. Since X is a Poisson random variable, $P(X = 4) = \frac{2^4}{4!}e^{-2} = 0.09$.

(c) [8 **pts**] What is the probability that a Northeastern student receives or sends at least three messages per hour?

Solution: $P(X > 3) = 1 - P(X \leq 3) = 1 - \frac{2^0}{0!}e^{-2} - \frac{2^1}{1!}e^{-2} - \frac{2^2}{2!}e^{-2} = 1 - (1 + 2 + 2)e^{-2} = 1 - 5e^{-2} = 0.323$.

[1] **Hint**: find W using Little's formula, then $W_Q = W - \frac{1}{\mu_a}$, where $\mu_a = \sum_{i=1}^{4} p_i\mu_i$ is the average departure rate and $\frac{1}{\mu_a}$ is the mean service time.