Summer 2020

Name:____ **Total**______/100

MATH 4581: Statistics and Stochastic Processes

Test 2

Problem 1. Let $X_1(t)$, $X_2(t)$ and $X_3(t)$ be independent Poisson processes with means $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$, respectively. Let $X(t)$ be the merged process $X(t) = X_1(t) + X_2(t) + X_3(t)$.

(a) [5 **pts**] Find the probability that $X(1) = 2$ and $X(2) = 4$.

Solution: $X(t)$ is a Poisson process with mean $\lambda = 1 + 2 + 3 = 6$, so

$P(X(1) = 2) = \frac{6^2}{2!}e^{-6} = 0.045$,

$P(X(2) = 4) = \frac{(6\cdot 2)^4}{4!}e^{-6\cdot 2} = 0.005$.

(b) [10 **pts**] Given that $X(1) = 2$, find the probability that $X_1(1) = 1$.

Solution:

$P(X_1(1) = 1 \mid X(1) = 2) = \frac{P((X_1(1)=1)\cap(X(1)=2))}{P(X(1)=2)} = \frac{P(X_1(1)=1)P(X_2(1)=1)P(X_3(1)=0)+P(X_1(1)=1)P(X_2(1)=0)P(X_3(1)=1)}{P(X(1)=2)} = \frac{\frac{1^1}{1!}\cdot\frac{2^1}{1!}+\frac{1^1}{1!}\cdot\frac{3^1}{1!}}{\frac{6^2}{2!}} = \frac{5}{18} = .$

Problem 2. Consider an $M/M/1$ system with $\lambda = 6$ and $\mu = 18$ (per minute).

(a) [5 **pts**] Find L, the average number of customers in the system and W, the average amount of time a customer spends in the system.

Solution:

$L = \frac{\lambda}{\mu-\lambda} = 0.5$,

$W = \frac{1}{\mu-\lambda} = \frac{1}{12}$.

(b) [5 **pts**] Give the density function $f_T(t)$.

Solution:

$f_T(t) = (\mu - \lambda)e^{-(\mu-\lambda)t} = 12e^{-12t}$.

(c) [5 **pts**] Find the probability that the time a customer spends in the system exceeds 10 seconds.

Solution:

$P(T > 1/6) = \int_{1/6}^{\infty} 12e^{-12t}dt = -e^{-12t}\Big|_{1/6}^{\infty} = e^{-2}$.

Problem 3. Use Ito's formula to compute the differentials of the following functions ($B(t)$ is a standard Brownian motion):

(a) [5 **pts**] $X(t, B(t)) = t + B^5(t)$

Solution:

$\frac{\partial X}{\partial t} = 1$, $\frac{\partial X}{\partial B(t)} = 5B^4(t)$ and $\frac{\partial^2 X}{\partial B(t)^2} = 20B^3(t)$, hence

$dX(t) = (1 + 10B^3(t))dt + 5B^4(t)dB(t)$.

(b) [5 **pts**] $Y(t, B(t)) = \sin(t^2 + B^2(t))$

Solution:

$\frac{\partial Y}{\partial t} = 2t\cos(t^2 + B^2(t))$, $\frac{\partial Y}{\partial B(t)} = 2B(t)\cos(t^2 + B^2(t))$ and $\frac{\partial^2 Y}{\partial B(t)^2} = 2\cos(t^2 + B^2(t)) - 4B^2(t)\sin(t^2 + B^2(t))$, hence

$dY(t) = (2t\cos(t^2 + B^2(t)) + \cos(t^2 + B^2(t)) - 2B^2(t)\sin(t^2 + B^2(t)))dt + 2B(t)\cos(t^2 + B^2(t))dB(t)$.

Problem 4. Let $B(t)$ be a standard Brownian motion.

(a) [5 **pts**] Find $P(B(9) < 2)$

Solution:

$P(B(9) < 2) = P(Z < \frac{2}{3}) = 0.748$.

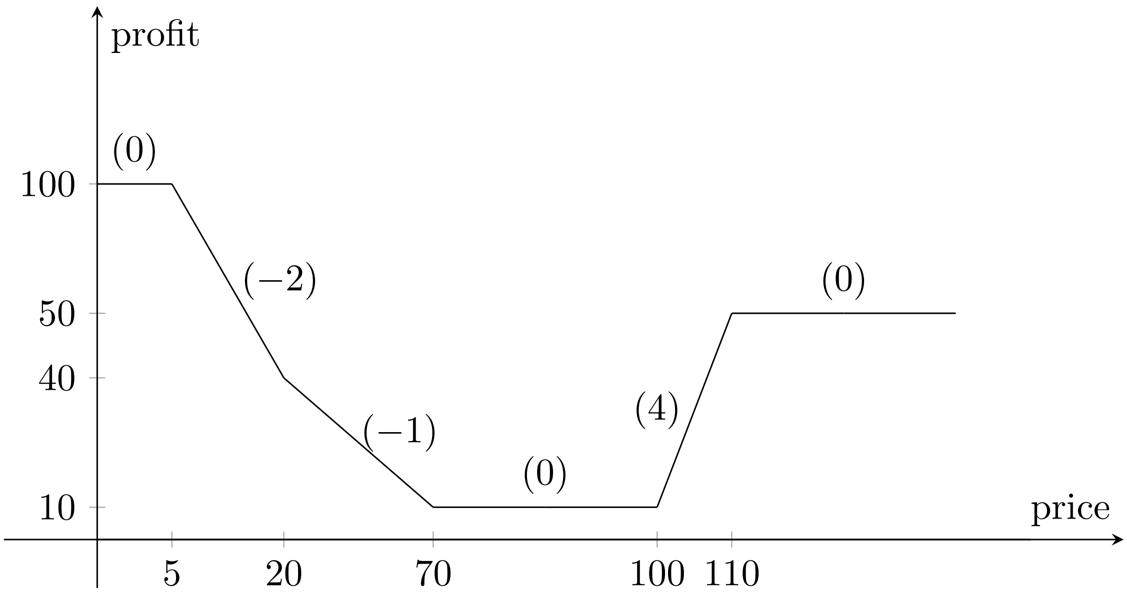
(b) [10 **pts**] Find $P(B(10) - B(6) > 4 \mid B(5) - B(2) = 2020)$.

Solution:

$P(B(10) - B(6) > 4 \mid B(5) - B(2) = 2020) = P(B(10) - B(6) > 4) = P(Z > 2) = 1 - P(Z \leq 2) = 1 - 0.977 = 0.023.$

Problem 5. For a given shape of the profit curve, design the portfolio and draw the graph of the profit as a function of price. The profit line is horizontal $\mathcal{P} = \$100$ until the price is \$5. Then the profit line has slope -2 until price \$20. At that point, it has slope -1 until price \$70. Then, the line is horizontal until price \$100. Next, it has slope 4 until price \$110. After that, it is horizontal.

(a) [5 **pts**] Draw the graph of the profit as a function of price.



(b) [5 **pts**] Design the portfolio with the above behavior using only call options.

Solution:

$\mathcal{P} = \$100 - 2C_5 + C_{20} + C_{70} + 4C_{100} - 4C_{110}$.

(c) [5 **pts**] Design the portfolio with the above behavior using only put options.

Solution:

$\mathcal{P} = \$100 - 4P_{110} + 4P_{100} + P_{70} + P_{20} - 2P_5$.

Problem 6 [15 **pts**] Peter read in the press this morning that, for an expiration date of a year from now (with 5% interest) that $C_{60}(70, t) = 9$ and $P_{60}(70, t) = 4$. How can he use this information to make some money?

Solution:

Notice that $P_{60}(70, t) + S = 4 + 70 = 74 > 66.07 = 9 + 60 \cdot e^{-0.05} = C_{60}(70, t) + Ee^{-r(T-t)}$, so Peter should go short on a Put option and a share of the stock, buy a Call option and put the remaining money acquired from selling in the bank. After a year passes he will gain approximately $74 - 66.07 = \$7.93$.

Problem 7 Let $X(t)$ be the price of a stock at time t. Assume that the current price of the stock is \$50 and it is modeled by a geometric Brownian motion with drift parameter $\mu = -0.1$ and volatility $\sigma = 0.49$.

(a) [10 **pts**] Find the probability that the price of the stock in two years is between \$30 and \$60.

Solution:

From the data given, $X(t) = 50e^{-0.1 \cdot t + 0.49B(t)}$, hence, $P(30 < X(2) < 60) = P(30 < 50e^{-0.1 \cdot 2 + 0.49B(2)} < 60) = P(0.6 < e^{-0.2+0.49B(2)} < 1.2) = P(\ell n 0.6 < -0.2 + 0.49B(2) < \ell n 1.2) = P(\frac{0.2+\ell n 0.6}{0.49} < B(2) < \frac{0.2+\ell n 1.2}{0.49}) = P(\frac{0.2+\ell n 0.6}{0.49\sqrt{2}} < Z < \frac{0.2+\ell n 1.2}{0.49\sqrt{2}}) = \phi(0.552) - \phi(-0.449) = 0.71 - 0.327 = 0.383.$

(b) [5 **pts**] If the yearly interest rate is $r = 0.05$, what should the selling price of a European 2 year Call option with strike price \$35 be, so there is no arbitrage opportunity?

Solution:

As $\mu = -0.1, \sigma = 0.49, r = 0.05, T = 2$ and $E = 35$, we compute

$b = \frac{\ell n(50/35)+2(0.05-\frac{0.49^2}{2})}{0.49\sqrt{2}} = 0.313.$

The price for the Call option is established at

$50\phi(0.49\sqrt{2}+0.313) - 35e^{-0.1}\phi(0.313) = 50 \cdot 0.843 - 35 \cdot e^{-0.1} \cdot 0.623 = \$22.42.$