Summer 2020

MATH 4581: Statistics and Stochastic Processes

Tropical Island (bonus problems)

Problem 1. The ticket prices established by 'Magnificient Airlines' for the flights between Paris, London, New York and Tokyo are shown in the picture below. John enjoys travelling and his top priority is to minimize expenditures.

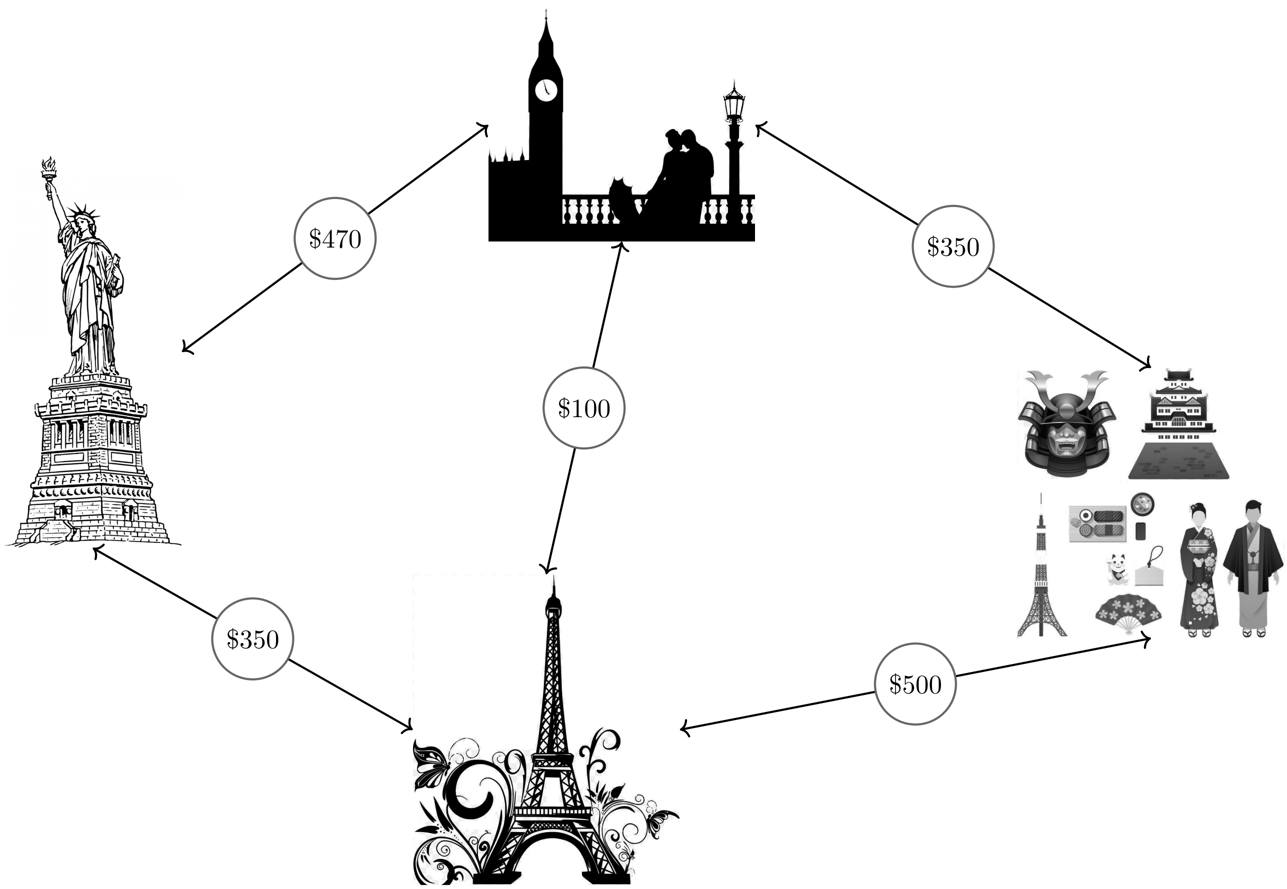

Figure 1: Ticket prices

(a) [1 **pt**] John is in New York and wants to go to London. Which route should he choose?

(b) [1 **pt**] John is in Tokyo and wants to return to New York. He has only \$810 left. Will he be able to complete his journey? If 'yes' give the route, if 'no' explain.

On the tropical island people have different laws of addition and multiplication. Namely, they use $min\{a, b\}$ instead of the sum $a + b$ and $a + b$ (usual addition) instead of the product ab. For example the polynomial $y = x^3 + 7$ for them is the piecewise linear function $min\{3x, 7\}$. From now on we switch to tropical arithmetics.

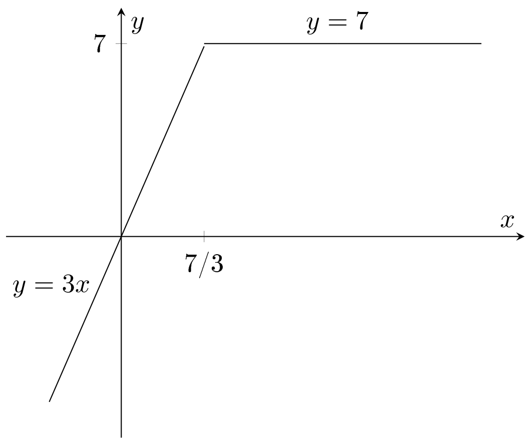

Figure 2: Polynomial $y = x^3 + 7$ tropically.

Problem 2 **[**3 **pts].** Draw the tropical version of the polynomial $y = 5x^2 + x + 1$. Clearly label the line segments.

We return to the first problem. Let us summarize the prices for the tickets in the following table. Here the agreement is that if there is no connection between two cities the price is set to be ∞ with

(1) $min\{\infty, \infty\} = \infty + \infty = \infty$

(2) $min\{a, \infty\} = a$ for any $a \neq \infty$ and

(3) $a + \infty = \infty$ for any a.

From/to	NY	London	Paris	Tokyo
NY	\$0	\$470	\$350	∞
London	\$470	\$0	\$100	\$350
Paris	\$350	\$100	\$0	\$500
Tokyo	∞	\$350	\$500	\$0

Table 1: Table of prices

This can be encoded in the matrix $J = \begin{pmatrix} 0 & 470 & 350 & \infty \\ 470 & 0 & 100 & 350 \\ 350 & 100 & 0 & 500 \\ \infty & 350 & 500 & 0 \end{pmatrix}$.

Problem 3.

(a) [2 **pts**] Using the usual laws of matrix multiplication subject to tropical operations, compute $J^{2,trop}$ and $J^{3,trop}$, the second and third tropical powers of J.

(b) [1 **pt**] Look at the values $J^{2,trop}_{1,2}$ and $J^{3,trop}_{4,1}$. Why did they come up as answers to (a) and (b) in Problem 1?

(c)* [3 **pts**] Will the matrices $J^{k,trop}$ eventually stabilize? In other words, is there a k, s.t. $J^{k,trop} = J^{k+1,trop} = \ldots$ Justify your answer.

Our short exrcusion to the tropical island ends here, but you can read more on Tropical Geometry and some of its applications in the excellent survey [1].

References

[1] D. Speyer and B. Sturmfels, *Tropical mathematics*, Math. Mag. **82** (2009), no. 3, 163–173.