Mirkovic-Vilonen cycles and polytopes
Satake isomorphism

Let $\mathcal{O} = \mathbb{C}[[t]]$ be a ring of formal power series, $G$ be a connected, reductive algebraic group over $\mathcal{K} = \text{Frac}(\mathcal{O})$ and $K$ a maximal compact subgroup (for instance, $K = G(\mathcal{O})$).

The Hecke ring $\mathcal{H} = \mathcal{H}(G, K)$ is by definition the ring of all locally constant, compactly supported functions $f : G \to \mathbb{Z}$ which are $K$-biinvariant:

$$f(kx) = f(xk') = f(x)$$

for all $k, k' \in K$. The multiplication in $\mathcal{H}$ is via convolution

$$f \ast g(z) = \int_{G} f(x)g(x^{-1}z)dx$$

where $dx$ is the unique Haar measure on $G$, s.t. $K$ has volume 1.
Remark. Each function $f \in \mathcal{H}$ is constant on double cosets $K \times K$, since it is also compactly supported, it is a finite linear combination of the characteristic functions $\text{char}(K \times K)$ of double cosets. Hence these characteristic functions give a $\mathbb{Z}$-basis for $\mathcal{H}$.

Theorem. There is an isomorphism of rings

$$(\mathcal{H}, \ast) \simeq (\text{Rep}(G^\vee) \otimes \mathbb{C}, \otimes).$$

Example. Let $G = \mathbb{C}^*$. The Cartan decomposition gives

$$G(\mathcal{K}) = \mathcal{K}^* = \bigsqcup_{m \in \mathbb{Z}} O^*t^mO^*$$

with $Gr_G \simeq \mathbb{Z} = \bigcup_{n \geq 0} (-n, n] \cap \mathbb{Z}$ and $K = O^*$. 

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We have $\mathcal{H}(G(K), K)) = \mathcal{H}(K^*, O) = \text{Fun}^c_{\mathcal{O}^* \times \mathcal{O}^*}(K^*, C) = \text{Fun}^c(Z, C)$. Next notice that $G^\vee = G = C^*$ and there is a ring isomorphism

$$\varphi: \text{Fun}^c(Z, C) \to (\text{Rep}(C^*) \otimes C, \otimes).$$

If the range of $\psi \in \text{Fun}^c(Z, C)$ is a subset of $Z_{\geq 0}$, then $\varphi(\psi) = \bigotimes_{i \in Z} V_i^{\oplus \psi(i)}$, where $V_i$ is the one-dimensional representation of $C^*$ (the action of $1 \in C^*$ on $V_i$ is via $i$th primitive root of unity). For instance, let $\chi_i \in \mathcal{H}$ be the characteristic function of $i \in Z$, i.e. $\chi_i(k) = \delta_{i,k}$. Hence

$$\chi_i \ast \chi_j(a) = \sum_{s \in Z} \chi_i(s)\chi_j(a-s) = \delta_{i+j,a} = \chi_{i+j}(a).$$

On the other hand, $V_i \otimes V_j = V_{i+j}$, so $\varphi(\chi_i \ast \chi_j) = \varphi(\chi_{i+j}) = \varphi(\chi_i) \otimes \varphi(\chi_j)$.
Geometric Satake isomorphism

There is an isomorphism of tensor categories

\[ \text{Perv}_{G(\mathcal{O})}(Gr_G, \mathbb{k}) \simeq \text{Rep}(G^\vee), \]

where \( \mathbb{k} \) is a Noetherian commutative ring with unit and of finite global dimension (\( \mathbb{k} = \mathbb{C}, \mathbb{Z}, \overline{\mathbb{F}}_q, \ldots \)).

Recall that for a smooth manifold \( M \) of dimension \( n \), the cohomology of \( M \) satisfies the Poincare duality, i.e.

\[ H^i(M, \mathbb{C}) \simeq H^{n-i}(M, \mathbb{C}). \]

Moreover, there is a 'sheaf way' to get the cohomology of \( M \). Let \( \mathbb{C}_M \) be the sheaf of locally constant functions on \( M \): for any open connected \( U \subset M \) one has \( \mathbb{C}_M(U) = \mathbb{C} \). Then there is an isomorphism of graded algebras

\[ H^*(M, \mathbb{C}_M) \simeq H^*(M, \mathbb{C}). \]
Question. What if the manifold $X$ is not smooth?

In case $X$ admits a 'good enough' stratification, Goresky and Macpherson found the 'right' version of homology that satisfies Poincare duality. They called it intersection homology and following a request of Deligne 'sheafified' it to get IC sheaves. Analogously to the case of smooth manifolds, there is an isomorphism

$$IH^*(X) \cong H^*(X, IC(X)).$$

The sheaves $IC(\overline{X}_\lambda)$ for affine Schubert cells $X_\lambda = G(\mathcal{O})t^\lambda$ play a fundamental role in the geometric Satake correspondence, namely, $IC(\overline{X}_\lambda)$ corresponds to the irreducible highest weight representation of $G^\vee$ given by the coweight $\lambda$. 
Mirkovic-Vilonen cycles

Fix $T \subset B \subset G$ and let $N \subset B$ be the unipotent radical with $N(K) \subset G(K)$. If $G = GL_n$ and $B$ consists of upper-triangular matrices, then

$$N(K) = \begin{pmatrix}
1 & * & \ldots & * \\
0 & 1 & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}$$

with each $*$ being an element of $K$. Let $S_\lambda := N(K)t^\lambda$.

**Remark.** $S_\lambda$ is neither of finite dimension nor of finite codimension in $Gr_G$.

**Theorem.** The intersection $S_\nu \cap X_\lambda \neq \emptyset$ if and only if $t^\nu \in \overline{X}_\lambda$, in which case $\dim(S_\nu \cap X_\lambda) = \rho(\nu + \lambda)$. 

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For any \( n \in N(K) \) we have \( \lim_{s \to 0} 2^\vee \rho(s)n = I \) (the action is via conjugation and all elements above diagonal have positive \( s \)-weights). It follows that \( S_\nu \) can be alternatively defined as

\[
S_\nu = \{ x \in Gr_G \mid \lim_{s \to 0} 2^\vee \rho(s)x = t^\nu \}.
\]

**Theorem.** For each \( \lambda \) we have a decomposition

\[
IH_*(\overline{X}_\lambda) = \bigoplus_{\nu \preceq \lambda} H_{top}(\overline{X}_\lambda \cap S_\nu).
\]

**Definition.** The Mirkovic-Vilonen cycles are the irreducible components of \( \overline{X}_\lambda \cap S_\nu \).

**Theorem.** Mirkovic-Vilonen cycles give a basis of \( H_{top}(\overline{X}_\lambda \cap S_\nu) \).
Example. Consider $G = \text{GL}_n$ and the minuscule weight $\lambda_k = (1, \ldots, 1, 0, \ldots, 0)_k$. Recall that $\overline{X}_{\lambda_k} = X_{\lambda_k} \simeq \text{Gr}(n - k, n)$ and there is the Plucker embedding

$$\mathcal{P} : \text{Gr}(n - k, n) \hookrightarrow \mathbb{P}^{n\choose n-k}_k - 1,$$

given by $\mathcal{P}(W) = w_1 \wedge w_2 \wedge \ldots \wedge w_{n-k}$ for any $W = \text{span}(w_1, w_2, \ldots, w_{n-k}) \in \text{Gr}(n - k, n)$. Next we find the fixed points for the action of the one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ with $\lambda(s) = \text{diag}(s^{n-1}, s^{n-2}, \ldots, s, 1)$ (this subgroup contracts $N(K)$ to a point and can be used instead of $2\mathbb{P}$). This group naturally acts on $V$ giving rise to an action on $\Lambda^{n-k}(V)$ and, hence, on $\mathbb{P}^{n\choose n-k}_k - 1 = \mathbb{P}(\Lambda^{n-k}(V))$. There are $\binom{n}{n-k}$ fixed points corresponding to the 'coordinate wedges' $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_{n-k}}$ with $1 \leq i_1 < i_2 < \ldots < i_{n-k} \leq n$. Such a fixed point, in turn, corresponds to the point

$$\text{diag}(1, \ldots, 1, t, 1, \ldots, 1, t, \ldots, 1, t) \in X_{\lambda_k} \cap S_{\nu_{i_1, \ldots, i_{n-k}}},$$

with $\nu_{i_1, \ldots, i_{n-k}} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \prec \lambda_k$. 

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As one of the descriptions of the Schubert cells (in the usual Grassmannian) is via attracting loci w.r.t. the one-parameter subgroup $\lambda$-action, we conclude that the MV cycles $X_\lambda \cap S_{\nu_{i_1 i_2 \ldots i_{n-k}}}$ are exactly the Schubert cells. Moreover, there is a natural one-to-one correspondence between these cells and the basis of $\Lambda^{n-k}(V)$, an irreducible representation of $GL_n$, via

$$(X_\lambda \cap S_{\nu_{i_1 i_2 \ldots i_{n-k}}}) \leftrightarrow e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_{n-k}}.$$

One more example.

**Example.** Let $G = SL_2$ and consider the dominant weights $\lambda = m$ and $\nu_k = k \leq m$. For $n = \left( \begin{array}{cc} 1 & f(t, t^{-1}) \\ 0 & 1 \end{array} \right) \in N(K)$ we compute $n \cdot \left( \begin{array}{cc} t^k & 0 \\ 0 & t^{-k} \end{array} \right) = \left( \begin{array}{cc} t^k & t^{-k}f(t, t^{-1}) \\ 0 & t^{-k} \end{array} \right)$, while $\left( \begin{array}{cc} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{array} \right) \cdot \left( \begin{array}{cc} t^m & 0 \\ 0 & t^{-m} \end{array} \right) = \left( \begin{array}{cc} t^m g_{11}(t) & t^{-m} g_{12}(t) \\ t^m g_{21}(t) & t^{-m} g_{22}(t) \end{array} \right)$ for $g = \left( \begin{array}{cc} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{array} \right) \in SL_2(\mathcal{O})$. It follows that (up to right $SL_2(\mathcal{O})$-action)

$$X_\lambda \cap S_{\nu_k} = \left\{ \left( \begin{array}{cc} t^k & t^{-k}f(t, t^{-1}) \\ 0 & t^{-k} \end{array} \right) \mid f(t, t^{-1}) = \sum_{i \geq k-m} a_i t^i \right\}.$$
As
\[
\begin{pmatrix}
t^k & t^{-k} f(t, t^{-1}) \\
0 & t^{-k}
\end{pmatrix}
\begin{pmatrix}
g_{11}(t) & g_{12}(t) \\
g_{21}(t) & g_{22}(t)
\end{pmatrix}
= 
\begin{pmatrix}
t^k g_{11}(t) + t^{-k} g_{21}(t) f(t, t^{-1}) & t^k g_{12}(t) + t^{-k} g_{22}(t) f(t, t^{-1}) \\
t^{-k} g_{21}(t) & t^{-k} g_{22}(t)
\end{pmatrix},
\]

we see that the elements of MV cycle \( \overline{X}_\lambda \cap S_{\nu_k} \) are given by Laurent polynomials \( f(t, t^{-1}) = \sum_{i = k-m}^{2k} a_i t^i \). In particular, \( \dim(\overline{X}_\lambda \cap S_{\nu_k}) = m - k + 2k = m + k \), which checks out to be equal to \( \rho(\lambda + \nu_k) = 0.5(1 \cdot (k + m) + (-1) \cdot (-k - m)) = k + m \). Moreover, it is clear that these varieties are irreducible. So, each weight \( \nu_k \) with \( 0 \leq k \leq m \) corresponds to a unique MV cycle.

**Awakeness test.** Which irreducible representation of \( PGL_2 = SL_2^\gamma \) did we just get?
Answer. \( S^m(\mathbb{C}^2) = \mathbb{C}\langle x^m, x^{m-1}y, \ldots, y^m \rangle = \Gamma(\mathbb{P}^1, \mathcal{O}(m)) \).
Mirkovic-Vilonen polytopes

**Fact.** There exists a very ample line bundle \( \mathcal{L} \) on \( Gr \) giving an embedding

\[
\varphi : Gr \hookrightarrow \mathbb{P}(\Gamma(Gr, \mathcal{L})^*)
\]

via \( \varphi(x) = \{ s \in \Gamma(Gr, \mathcal{L}) \mid s(x) = 0 \}^* \).

Henceforth we will identify \( Gr \) with its image \( \varphi(Gr) \) and denote \( W := \Gamma(Gr, \mathcal{L})^* \).

Let \( T \subset G \) be a maximal torus and \( T_K \subset T \) a maximal compact subtorus (for \( G = GL_n \), we have \( T = (\mathbb{C}^*)^n \) and \( T_K = (S^1)^\times_n \)). The torus \( T_K \) acts on \( Gr \) by conjugation and, hence, on \( W \) as well. One can choose an inner product on \( W \) invariant under \( T_K \). Better said, an invariant symplectic form on \( W_\mathbb{R} \) given by

\[
\omega(v_1, v_2) := (v_1, iv_2),
\]

where \((\cdot, \cdot)\) stands for the chosen inner product.
The action of $T_K$ on $W$ gives rise to a weight decomposition $W = \bigoplus W_{\nu}$. The moment map for $T_K \curvearrowright Gr$, i.e. $\mu : Gr \to \mathfrak{t}^*_\mathbb{R} \simeq \mathfrak{t}_\mathbb{R}$ (the last identification is via the Killing form) induced by the action of $T_K$ on $Gr$ is given by

$$\mu(x) = \sum_{\nu} \frac{|v_\nu|^2}{|v|^2},$$

where (the image under $\varphi$ of) $x$ is $x = \sum_{\nu} v_{\nu}$.

**Definition.** The image of a MV cycle under the moment map above is called a **Mirkovic-Vilonen polytope**.

**Remark.**

1. The vertices of the MV polytope for MV cycle $S_\nu \cap \overline{X}_\lambda$ are the points $\mu(t^n)$ for coweights $t^n \in S_\nu \cap \overline{X}_\lambda$ (the fixed points for $T$-action). This easily follows from contractability of $N(K)$ by $2^\vee$-action.

2. $\mu(\overline{X}_\lambda) = \text{conv}(W \cdot \lambda)$ as the points $t^{W \cdot \lambda}$ are fixed and the images of other fixed points are contained in $\text{conv}(W \cdot \lambda)$ (as $X_\eta \subset \overline{X}_\lambda \Leftrightarrow \eta \prec \lambda$).
The following results are due to Anderson.

**Theorem.**  1. If $V_\lambda$ is an irreducible representation of $G$. The multiplicity of a $\nu$-weight space is equal to the number of MV polytopes $P_{\nu-\lambda}$ with $P + \lambda \subseteq \text{conv}(W \cdot \lambda)$.

2. Let $V_\lambda, V_\mu$ be irreducible representations of $G$ and $\nu$ a dominant weight. The multiplicity of $V_\nu$ in $V_\lambda \otimes V_\mu$ is equal to the number of MV polytopes $P_{\nu-\lambda-\mu}$ with $P + \lambda \subseteq \text{conv}(W \cdot \lambda) \cap \text{conv}(W \cdot (-\mu) + \nu)$. 
Example. \(G = SL_2\) Let \(\lambda = n, \mu = m\) with \(m \geq n\) and \(\nu = k = n + m - 2\ell \geq 0\). We compute \(\text{conv}(W \cdot \lambda) = \text{conv}(n, s \cdot n) = \text{conv}(n, s(n + 1/2) - 1/2) = \text{conv}(n, -n - 1) = [-n - 1, n]\).

\[
\text{conv}(W \cdot \lambda) = [-n - 1, n].
\]

Similarly,

\[
\text{conv}(W \cdot (-\mu) + \nu) = [k - m, m - 1 + k].
\]

The MV polytope \(P\) is the interval \([k - n - m, 0]\), which, shifted by \(\lambda = n\), becomes the interval

\[
P_{\nu} = [k - m, n].
\]

The containment in part (2) of the Theorem is equivalent to satisfaction of the inequalities \(-n - 1 \leq k - m \leq n \iff m - n - 1 \leq k \leq n + m \iff 0 \leq \ell \leq n\). This recovers the Clebsch-Gordan rule:

\[
V_m \otimes V_n \cong \bigoplus_{0 \leq \ell \leq n} V_{m+n-2\ell}.
\]