Intro to affine Grassmannians.

\[ \Theta = \mathbb{C}[t_1, \ldots, t_d] - \text{power series} \]

\[ K = \text{Frac}(\Theta) = \mathbb{C}((t_1)) - \text{formal Laurent polynomials}. \]

**Def.** An \( \Theta \)-lattice in \( K^n \) is a projective finitely generated \( \Theta \)-submodule \( \Lambda \), s.t. \( \Lambda \otimes K \cong K^n \).

Such lattices are points in the affine Grassmannian.

**Goal:** Endow with topology!
Let \( G_{N} := \{ \Lambda | t^{-N} \Lambda \supseteq \Lambda \supseteq t^{N} \Lambda_{0} \} \), where \( \Lambda_{0} = \emptyset \).

Remark: \( G_{N} < G_{N+1} < \cdots \)

\( \Theta_{N} = \lim_{\to} G_{N} \)

Notice that \( \frac{t^{-N} \Lambda_{0}}{t^{N} \Lambda_{0}} = C^{2nN} \)

There is a map \( \varphi : G_{N} \hookrightarrow G_{(2n)N} \)

\[ \cup_{k \in \{1, 2, \ldots, 2nN-1\}} G_{(k, 2nN)} \]

\( \varphi(\Lambda) = \frac{\Lambda}{t^{N} \Lambda_{0}} \)

Remark: \( \varphi \) is not surjective, since to be an \( \Theta \)-submodule, a subspace must be \( t \)-stable.

Recall: \( G_{(2n)N} \) is a projective variety (via Plücker embedding), \( t \)-stability is a closed condition, so we get an induced structure of proj. variety on \( G_{N} \).

Example: \( N = 0 \), \( G_{0} = \{ \Lambda | \Lambda_{0} \supseteq \Lambda \supseteq \Lambda_{0} \} = \)
Conclusion: \( \text{Lie: } \text{Gr}_n \hookrightarrow \text{Gr}(2nN) \) is a closed embedding, giving \( \text{Gr}_n \) a structure of proj. scheme and \( \text{Gr} = \text{lim} \text{Gr}_n \) the structure of ind-proj. scheme.

Cartan decomposition / Affine Schubert cells.

Recall: \( G \)-classical Lie group

Bruhat decomposition:

\[
G = \text{L} \uparrow \text{B} \downarrow \text{W} \downarrow \text{B}
\]

Example. \( G = \mathfrak{sl}_n \), \( \text{B} = (\begin{smallmatrix} \ast & \ast \cr 0 & \ast \end{smallmatrix}) \), \( \text{W} = \mathfrak{S}_n \).

For instance, \( (12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \).

Acting by row and column transformations, we can bring any matrix to a unique permutation matrix.
Cartan decomposition:

\[ G(K) = \bigsqcup G(\theta) t^\lambda G(\theta). \]

\( \lambda \)-dominant coweights

In case \( G = G_{\text{un}} \), \( t^\lambda = (t^{\lambda_1}, 0, t^{\lambda_n}) \) with \( \lambda_i \in \mathbb{Z} \) and \( \lambda_1 > \lambda_2 > \cdots > \lambda_n \).

**Proof:** Gauss-Jordan elimination (Smith normal form).

**Rmk:** \( G(\theta) \) is the analog of \( pCG \) maximal parabolic

and is called **parahoric** = 'Iwahori + parabolic'

Analog of \( \beta \) is \( \iota: \chi: G(\theta) \to G \), \( \iota = \Pi^{-1}(\beta) \).

\( t_i \to 0 \) stabilizes a full flag of lattices
'Old' Grassmannian: \( P = \left( \otimes_1^n X \right) \subset G \subset G / p \),
\( B \subset G / p \rightarrow \text{Schubert cells} \)
(closures of) \( B \)-orbits.

'New' Grassmannian:
\( G(\Theta) \subset G(\mathbb{K}) / G(\Theta) \)

Affine Schubert cells are
\( X_\lambda = G(\Theta) \cdot \epsilon \lambda \).

Fact. \( \overline{X_\lambda} = \bigsqcup \overline{X_\mu} \), \( \mu \) is dominant.

\( \mu \prec \lambda \) means that \( \mu - \lambda \in X_+ \) (positive wt).

For \( B \subset \mathfrak{g} \), \( \mu \prec \lambda \) means \( \mu_i \leq \lambda_i \),
\( \mu_1 + \mu_2 \leq \lambda_1 + \lambda_2 \)
\( \vdots \)
\( \mu_1 + \cdots + \mu_n \leq \lambda_1 + \cdots + \lambda_n \).

Rmk. \( \overline{X_\lambda} \) is closed if \( \lambda \) is a minuscule wt (not greater than any \( \mu \in X_+ \)).
Example. The minuscule wts for $GL_n$ are

$$x_k = \left(1, 1, \ldots, 1, 0, 0, \ldots, 0\right)_{\scriptscriptstyle K \atop \scriptstyle n-k}$$

$\text{Prop.}\ n, \ \chi_k \chi_k = \chi_n = 6\gamma(n-k, n).$

Indeed, this follows from a computation

$$\begin{pmatrix}
\begin{array}{cccc}
\Phi_1(t) & \cdots & \Phi_n(t) \\
\vdots & \ddots & \vdots \\
\Phi_1(t) & \cdots & \Phi_n(t)
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
t \\
0 \\
\vdots \\
0
\end{array}
\end{pmatrix}$$

$$= \begin{pmatrix}
\begin{array}{c}
t \Phi_1(t) & \cdots & t \Phi_{n-k}(t) & \Phi_{n-k+1}(t) & \cdots & \Phi_n(t) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
t \Phi_1(t) & \cdots & t \Phi_{n-k}(t) & \Phi_{n-k+1}(t) & \cdots & \Phi_n(t)
\end{array}
\end{pmatrix}$$

$$t x_k \cdot Q(t) = \begin{pmatrix}
\begin{array}{cccc}
t Q_{11}(t) & \cdots & t Q_{1n}(t) \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
t Q_{k_1}(t) & \cdots & t Q_{k_{n-k}}(t) \\
0 & \cdots & 0 \\
\vdots & \cdots & \vdots \\
0 & \cdots & 0 \\
Q_{11}(t) & \cdots & Q_{1n}(t) \\
\vdots & \cdots & \vdots \\
\vdots & \cdots & \vdots \\
\vdots & \cdots & \vdots \\
Q_{k_1}(t) & \cdots & Q_{k_{n-k}}(t) \\
\vdots & \cdots & \vdots \\
0 & \cdots & 0 \\
0 & \cdots & 0
\end{array}
\end{pmatrix}$$
Conclusion: the action of $G(\Theta)$ factors through the action of $t \cdot G(\Theta)$

\[ g \in G(\Theta) \]
\[ G = g_0 + t \cdot g, (t) \]
\[ G \rightarrow G(\Theta) = G + t \cdot G(\Theta) \]

$U$ acts trivially

\[ p = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \]

We get that $X_{\chi_k} \cong G/p \cong G_{T}(n-k,n)$.

The stabilizer of $t^2$ for left $G(\Theta)$-action is $G(\Theta) \cap t^2 \cdot G(\Theta) \cdot t^{-1}$

In the example above, $t \cdot G(\Theta) \cdot t^{-1}$ is

\[
\begin{pmatrix}
\begin{pmatrix}
t & & \\
& \ddots & \\
& & 1
\end{pmatrix}
& \begin{pmatrix}
0 & & \\
& \ddots & \\
& & 0
\end{pmatrix}

\begin{pmatrix}
P_1(t) & \cdots & P_m(t)
\end{pmatrix}

\begin{pmatrix}
t^{-1} & 0 \\
0 & t^{-1}
\end{pmatrix}
\end{pmatrix}
\]
After intersecting with $G(9)$, we get that $\text{Stab}_{\mathfrak{t} \lambda \nu}$ consists of matrices of the form

$$g = \begin{pmatrix}
  P_\nu(t) & \cdots & P_{\lambda \nu}(t) \\
  \vdots & & \vdots \\
  t^{-1}P_{\lambda \nu}(t) & \cdots & t^{-1}P_{\lambda \nu}(t)
\end{pmatrix} \in G(9)$$

$$G(9)/\text{Stab}_{\mathfrak{t} \lambda \nu} \cong G/p, \quad p = \begin{pmatrix}
  x & 0 \\
  0 & x
\end{pmatrix}$$

Coffee break
Recall: \( \rho = \frac{1}{2} \sum_{\lambda \in \Phi^+} \lambda \)

**Prop-n.** \( \dim X_\lambda = (2 \rho, \lambda) \),

**Pf:** \( X_\lambda \cong G(\Theta)/\bigl( G(\Theta) \cap t^\lambda G(\Theta) t^{-\lambda} \bigr) \), hence, \( X_\lambda \) is smooth and the dimension of \( X_\lambda \) equals the dimension of \( X_\lambda \) at any point \( x_0 \):

\[
\dim T_{x_0}(X_\lambda) = \dim \left( \bigoplus_{\lambda \in \Phi^+} \frac{G(\Theta)}{t(\lambda, \Theta) \cdot G(\Theta)} \right) = \sum_{\lambda \in \Phi^+} (2 \rho, \lambda) = (2 \rho, \lambda).
\]

**Rmk.** \( \lambda \) is dominant, so \((L_j, \lambda) < 0\) for any \( L_j \) and \( G(\Theta)/t(\lambda, \Theta) \cdot G(\Theta) = 0 \).

**Example.** \( \lambda_k = (1,1, \ldots, 1, 0, \ldots, 0) \), a minuscule coweight for \( \overline{G} = G_{kn} \).

As \( 2 \rho = \sum \xi_i - \xi_j \),

\[
\sum_{\mathbf{k} \preceq \mathbf{i} \preceq j \preceq \mathbf{n}} \xi_i - \xi_j + \sum_{\mathbf{i} \preceq \mathbf{k} \preceq \mathbf{j} \preceq \mathbf{n}} \xi_i - \xi_j + \sum_{\mathbf{k} \preceq \mathbf{i} \preceq \mathbf{j} \preceq \mathbf{n}} \xi_i - \xi_j,
\]

where \( \xi_i(\xi_j) = \delta_{i,j} \).
we have \((2g, \lambda_k) = \left( \sum_{1 \leq i < k < j \leq n} (e_i - e_j, \lambda_k) = k(n-k) \right) \dim \mathfrak{g} = (n-k,n)\).

**Nilcone inside affine Grassmannian.**

**Def:** The subvariety \(N = \{ A \in \mathfrak{g} | A^n = 0 \} \) is called the nilpotent cone.

**Remark:** The definition above works for \( \mathfrak{g} = \mathfrak{gl}_n \) or \( \mathfrak{g} = \mathfrak{sl}_n \).

**Example.** \( \mathfrak{g} = \mathfrak{sl}_2 = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right\} \)

\( A \in \mathfrak{g} \) is nilpotent \( \iff A^2 = 0 \left( \overset{\text{Cayley-Hamilton}}{\iff} \right) \)

\( \chi_A(t) = t^2 \). As \( \mathfrak{sl}_2 \) consists of traceless matrices, \( \chi_A(t) = t^2 \iff \det A = -x^2 - yz = 0 \), i.e.

\[ N = \mathbb{C}[x,y,z] / (x^2 + yz) \text{ is a cone} \]

This is where the name 'nilpotent cone' or 'nilcone' comes from.
If \( \sigma_1 = \sigma_1^N \), then an operator \( A \in \mathcal{N} \) iff \\
\( X_\mu(t) = t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 = t^n \), i.e., the \( n \) coefficients \( a_0, a_1, \ldots, a_{n-1} \) (which are polynomials in the matrix entries of \( A \) all vanish). \\
This allows to conclude \\
\( \dim \mathcal{N} = \dim \mathcal{O}_1 - n = n^2 - n \). \\
The following construction is attributed to G. Lusztig. \\
Let \( \mu = (n, 0, 0, \ldots, 0) \). It is not hard to check that \\
\( X_\mu = G(\mathfrak{g}) \cdot t^n \) \( \forall \lambda \in \mathfrak{n} \supset \mathcal{N} \), \( \dim \mathfrak{n}/\lambda = n \). \\
Consider the map \\
\( \Psi: \mathcal{N} \to \mathcal{O}_1 \) \\
\( A \mapsto A_0 / (t-A) A_0 \). \\
Remark. This is the same construction as the one used in the proof of existence of Jordan canonical form: given a matrix \( A \in \mathfrak{gl}_n \) and \( V = \mathbb{C}^n \), we make \( V \) into a \( (\mathbb{C}[t]) - \)module.
with the action of $f(x) \in C[x]$ being via $f(A)$.

Remark. Notice that $\dim X_\mu = (2p, \mu) = n(n-1) = \dim N$, hence, $\Phi$ is an open embedding.

Coffee break

Valuation.

Let $\Lambda = \text{span} g \{v_1, \ldots, v_n\}$ be a lattice, then $\det \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \in K^\times$.

Define the map $\text{val} : \mathcal{G}_r \rightarrow \mathbb{Z}$ via $\text{val}(\Lambda) = \min \{n | t^n \text{ occurs in } \det(\text{basis})\}$.

Properties: 1. Independent of the choice of basis
2. Constant on left $G(\Theta)$-orbits. (Schubert cells)
Reason: any matrix \( g \in G(\mathbb{F}) \) has \( \det g \in C[t, t^*] \) (is invertible), i.e., \( \det g = a_0 + a_1 t + \ldots \) with \( a_0 \neq 0 \). It follows that multiplication by \( g \) (left or right) does not change the minimal power of \( t \) in the determinant.

Complete picture for \( GL_n \):

As shown above, we have a map

\[ Val : \{ \text{connected com-?} \} \rightarrow \mathbb{Z} \]

Example. \( n = 2 \):

\[ Val^{-1}(0) = \begin{array}{c}
\text{Gr}_{(0,0)} \quad \text{Gr}_{(1,1)} \\
\end{array} \]

\[ \dim \text{Gr}_{(m, -m)} = (1, 1-1, (m, -m)) = 2m \]

\[ Val^{-1}(1) = \begin{array}{c}
\text{Gr}_{(1,0)} \quad \text{1p} \\
\end{array} \]

\[ \dim \text{Gr}_{(m, -m+1)} = 2m+1. \]

((1,0) is a minuscule weight)
Rmk. \( \text{Val}^{-1}(2k) \) is \( \text{GL}_2(F) \)-equivariantly isomorphic to \( \text{Val}^{-1}(0) \) and \( \text{Val}^{-1}(2k+1) \) is \( \text{GL}_2(F) \)-equivariantly isomorphic to \( \text{Val}^{-1}(0) \) for any \( k \in \mathbb{Z} \). The isomorphisms are given by multiplication by the matrix \( (t^k 0) \) and its inverse \( (0 t^{-k}) \).

In other words, we get the bijection

\[
\begin{cases}
\text{iso-classes of connected components of Gr} \\
\text{for } \text{GL}_2
\end{cases}
\xleftrightarrow{\quad}\mathbb{Z}/2
\]

parity of valuation

Similarly one gets

\[
\begin{cases}
\text{iso-classes of connected components of Gr} \\
\text{for } \text{GL}_n
\end{cases}
\xleftrightarrow{1:1}\mathbb{Z}/n
\]

valuation (mod \( n \))

Slodowy slices.

Let \( g \) be a reductive Lie algebra and \( x \in N \subseteq \mathfrak{g} \) a nilpotent element.
A transversal slice $S_x$ in $X$ to the (adjoint) orbit of $x$ is a locally closed subvariety $S_x \subset \mathfrak{g}$ such that

- $x \in S_x$;
- the morphism $G \times S_x \to \mathfrak{g}$, $(g,s) \mapsto \text{ad}(g)(s)$ is smooth;
- $\dim S_x = \text{codim}(G \cdot x)$.

In case $x \in N$ such a slice is obtained as the affine space complimentary to the tangent space of the orbit $G \cdot x$ in $\mathfrak{g}$.

The recipe is as follows.

**Step 1.** We will need the Jacobson-Morozov theorem.

**Thm.** There exists a Lie algebra homomorphism $\mathfrak{h} \ni \mathfrak{l}_2 \to \mathfrak{g}$ with $\mathfrak{l}(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = X$. All such homomorphisms are conjugate under the centralizer $Z_\mathfrak{g}(X)$. 
The result above allows to complete $x$ to an $\mathfrak{sl}_2$-triple $<x,y,h>$, which will be denoted by $\hat{x}$.

**Step 2. Decompose $\hat{x}$ into the sum of irreducible representations w.r.t. adjoint $\mathfrak{g}$-action:**

$$\hat{x} = \bigoplus_{i=1}^{k} V_i$$

As $T_x(\mathfrak{g},x) = x + [x,y]$, the complement to $T_x(\mathfrak{g},x)$ in $\mathfrak{g}$ is $x + \cdots = x + \ker(\text{ad}y)$ (consists of lowest weight vectors in $V_i$'s). 

**Step 3. A slice to $x$ inside $\mathfrak{N}$ is $S_x \cap \mathfrak{N}$.**

Example. $\mathfrak{g}_1 = \mathfrak{sl}_2$, $x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. 
Step 1. As $x$ is a positive root, $y = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the corresponding negative root and

$$h = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Step 2. Let $A \in \mathbb{F}^{l_3}$, then

$$[y, A] = \begin{pmatrix} -a_{13} & 0 & 0 \\ -a_{23} & 0 & 0 \\ a_{11} - a_{33} & a_{12} & a_{13} \end{pmatrix}.$$

Thus, $A \in \ker (ady)$ is of the form

$$A = \begin{pmatrix} a & 0 & 1 \\ b & -2a & 0 \\ d & c & a \end{pmatrix}.$$

Step 3. Now we find the intersection

$$S_x \cap S^* = \{ A^x (a, 0, 1) | \chi_A(t) = t^3 \},$$

where $\chi_A(t)$ is the characteristic polynomial of $A$, i.e.

$$\chi_A(t) = \det (A - t \cdot I).$$

The coefficient of $t^3$ is $\text{tr} A = 0$ ($A \in \mathbb{F}^{l_3}$).
The coefficient of $t$ is $2a^2+2a^2-a^2+d$. The constant term is $\det(A) = -2a^3 + 2ad + bc$.

Hence, $S_{x \cap N} = \mathbb{C}[a,b,c]/(bc-b^3)$ is a Kleinian singularity of type $A_2$ (the Dynkin diagram of $sl_3$).

We will need a little bit of preparation in order to formulate a more general result.

**Def.** An element $x \in \mathfrak{g}$ is called **regular** if its adjoint $G$-orbit is of maximal possible dimension. This is equivalent to $\dim Z_G(x) = \text{rk} \mathfrak{g} \mathfrak{g}$ (here $Z_G(x)$ is the centralizer of $x$).

An element $x \in \mathfrak{g}$ is called **subregular** if $\dim Z_G(x) = \text{rk} \mathfrak{g} \mathfrak{g} + 2$.

**Example.** Let $\mathfrak{g} = sl_n$, $x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}$ and $y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}$. A direct calculation shows that

\[ Z_G(x) = \begin{pmatrix} 0 & a_{11} & a_{21} & \cdots & a_{n-1,1} \\ 0 & 0 & a_{22} & \cdots & a_{n-1,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \]

\[ Z_G(y) = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & 0 \\ 0 & a_1 & a_2 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix} \]
As $rk(g_{ln}) = n-1$, $\dim Z_G(x) = n-1$ and $\dim Z_G(y) = n+1$, $x$ is regular and $y$ is subregular.

**Thm (Dynkin).** If $y$ is simple, all subregular elements belong to the same conjugacy class.

Now we can state an interesting result.

**Thm (Brieskorn).** Let $y$ be a simple Lie algebra of type $A$, $D$ or $E$ and $x \in N < y$ a subregular element. Then the variety $S_{xN}N$ is a Kleinian singularity of type 'prescribed' by the Dynkin diagram of $y$.

Next we will show how the shadowy slices (inside the nilcone) are realized in the affine Grassmannian.

**Birkhoff decomposition.**

Apart from the Cartan decomposition

$$G(k) = \bigcup_{\lambda \in \text{dom. coweights}} G(\Theta) t^\lambda G(\Theta)$$
there is the Birkhoff's decomposition:

\[ G(K) = \bigcup_{\text{dom. coweights}} G[t^*] t^* G(\Theta) \]

The existence of this decomposition is equivalent to Grothendieck's thm classifying locally free sheaves (vector bundles) on the projective line \( \mathbb{P}^1 \).

**Thm (Grothendieck).** Let \( E \) be a rank \( n \) locally free sheaf on \( \mathbb{P}^1 \), then 

\[ E = \bigoplus_{i=1}^n \mathcal{O}(s_i), \quad s_i \in \mathbb{P}^1. \]

Recall that the line bundle \( \mathcal{O}(k) \) on \( \mathbb{P}^1 \) is given by two modules \( M_0 = \mathbb{C}[t] \) and \( M_1 = \mathbb{C}[t^*] \) (on the two affine charts \( \mathbb{A}^1 \)) and transition function being multiplication by \( t^k \).

Similarly, a rank \( n \) locally free sheaf on \( \mathbb{P}^1 \) is given by two modules \( M_0 = \mathbb{C}[t]^n \) and \( M_1 = \mathbb{C}[t^*]^n \) (over \( \mathbb{C}[t] \) and \( \mathbb{C}[t^*] \), respectively) together with a transition matrix \( g \in G_l n(K) \). Notice that \( g \) is
defined up to the change of basis in $\mathfrak{m}_0$ and $\mathfrak{m}_1$, i.e. action of $G[t^{-1}]$ on the left and $G[t]^{-1}$ on the right. It follows that the Birkhoff's decomposition and Grothendieck's thm are equivalent.

Rmk. The attentive reader may have noticed that in Birkhoff's decomposition we act by $G[t]$ (the matrix entries are power series), while $G[t]$ above stands for matrices of polynomials, so instead of the decomposition above we rather need

$$G[t, t^{-1}] = \bigcup_{x \in \text{dom. CSweights}} G[t^{-1}] t^x G[t],$$

which also holds true and bears Birkhoff's name.

Slices in affine Grassmannian.

Let $Gr^\lambda = G[t^{-1}] \cdot t^\lambda \subset Gr$.

Thm. (1) $Gr^\mu \cap Gr^\lambda = \emptyset$ if $\mu > \lambda$.

(2) $Gr^\mu \cap Gr^\lambda \cong G \cdot t^\lambda$. 

Let $Gr^\mu = G[t^{-1}] \cdot t^\mu \subset Gr$. 

Thm. (1) $Gr^\mu \cap Gr^\lambda = \emptyset$ if $\mu > \lambda$.

(2) $Gr^\mu \cap Gr^\lambda \cong G \cdot t^\lambda$. 
\textbf{Remark.} The proof is a straightforward calculation. The variety $G \cdot t^\mu$ is the fixed point set for the action of one-dimensional torus $\mathbb{C}^*$ on $Gr_x$ via rescaling $t$. This torus is called the \textit{rotation torus}.

Let $G_i \subset G[t^{-1}]$ be the kernel of the evaluation map $\varphi: G[t^{-1}] \rightarrow G$

\[ t^{-1} \mapsto 0 \]

and $\overline{Gr^\mu} := G_i \cdot t^\mu$.

Then $\overline{Gr^\mu} \cap \overline{Gr^\mu} = t^\mu$ is a single point.

\textbf{Prop.} Let $\mu \leq \lambda$, then $\overline{Gr^\mu} \cap \overline{Gr^\mu} \cap Gr_x$ intersects $Gr_x$ transversally for any $\mu \leq \nu \leq \lambda$.

In particular, for $\lambda = (n, 0, 0, \ldots, 0)$ and $\mu \leq \lambda$, one gets $\overline{Gr^\mu} \cap \overline{Gr^\mu} \approx S_x \cap N_x$, where the Jordan form of the nilpotent matrix $X$ has partition type $\mu$. 