Spring 2023

MATH 0200: Preparation for Scientific Calculus

Review Midterm Exam 1

Solutions

1. Explain why each of the following statements is false.

   (*a*) The graph of $g(x) = f(x+2) + 2$ can be obtained from the graph of f by shifting it 2 units up and 2 units to the right.

   Solution: the graph is shifted 2 units up and 2 units to the **left**.

   (*b*) The polynomial $p(x) = x^3 - x^2 - 10x - 8$ has zeros at -1 and 3.

   Solution: $p(3) = 3^3 - 3^2 - 10 \cdot 3 - 8 = -20 \neq 0$.

2. Compute $f^{-1}(y)$ for the function $f(x) = \dfrac{2x+1}{3x-4}$.

   Solution:

Step 1. Write $y = \dfrac{2x+1}{3x-4}$.

Step 2. Express x in terms of y: $y = \dfrac{2x+1}{3x-4} \Leftrightarrow (3x-4)y = 2x+1 \Leftrightarrow 3xy - 4y = 2x+1 \Leftrightarrow 3xy - 2x = 4y + 1 \Leftrightarrow x(3y-2) = 4y+1 \Leftrightarrow x = \dfrac{4y+1}{3y-2}$.

Step 3. The inverse function is $f^{-1}(x) = \dfrac{4x+1}{3x-2}$.

Step 4. Check: $f^{-1} \circ f(x) = \dfrac{4 \cdot \dfrac{2x+1}{3x-4} + 1}{3 \cdot \dfrac{2x+1}{3x-4} - 2} = \dfrac{8x+4+3x-4}{6x+3-6x+8} = \dfrac{11x}{11} = x \quad \checkmark$

3. Compute the compositions $f \circ g$ and $g \circ f$ for the functions $f(x) = \sqrt{x}, g(x) = \dfrac{x+1}{x+2}$. Find $(f \circ g)(2)$.

   Solution: $(f \circ g)(x) = \sqrt{\dfrac{x+1}{x+2}}$ and $g \circ f(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}+2}$, so $f \circ g(2) = \sqrt{\dfrac{2+1}{2+2}} = \dfrac{\sqrt{3}}{2}$.

4. List all vertical and horizontal asymptotes for the rational function

$$f(x) = \frac{9x+5}{x^2 - x - 6}.$$

   Solution: $\deg(9x+5) = 1 < 2 = \deg(x^2 - x - 6)$, hence the horizontal asymptote is $y = 0$. We solve the quadratic equation $x^2 - x - 6$:

   $x = \dfrac{1 \pm \sqrt{1+24}}{2} = 3$ or -2.

   Vertical asymptotes: $x = -2$ and $x = 3$.

5. Find the equation of the circle centered at $(5,1)$ and containing the point $(-2,3)$.

   Solution: first we find the radius:

   $r = dist((5,1),(-2,3)) = \sqrt{(5-(-2))^2+(1-3)^2} = \sqrt{53}$.

   The equation of the circle with given center and radius is

$$(x-5)^2+(y-1)^2 = 53.$$

6. Find the equation of the line containing the point $(-4,-5)$ and parallel to the line through the points $(7,1)$ and $(5,6)$.

   Solution: the slope is $m = \dfrac{6-1}{5-7} = -2.5$, giving rise to the equation $y-(-5) = -2.5(x-(-4)) \Leftrightarrow y = -2.5x-5$.

7. Let $f(x) = 4+5\ell og_3(7x+2)$.

   (*a*) Find the domain of f.

   Solution: $7x+2>0 \Leftrightarrow x > -\dfrac{2}{7}$.

   (*b*) Compute $f^{-1}(y)$.

   Solution:

   Step 1. Write $y = 4+5\ell og_3(7x+2)$.

   Step 2. Express x in terms of y: $y = 4+5\ell og_3(7x+2) \Leftrightarrow \dfrac{y-4}{5} = \ell og_3(7x+2) \Leftrightarrow 3^{\dfrac{y-4}{5}} \Leftrightarrow 3^{\ell og_3(7x+2)} \Leftrightarrow 3^{\frac{y-4}{5}} = 7x+2 \Leftrightarrow x = \dfrac{3^{\frac{y-4}{5}}-2}{7}$.

   Step 3. The inverse function is $f^{-1}(x) = \dfrac{3^{\frac{x-4}{5}}-2}{7}$.

   Step 4. Check: $f^{-1}\circ f(x) = \dfrac{3^{\frac{4+5\ell og_3(7x+2)-4}{5}}-2}{7} = \dfrac{3^{\frac{5\ell og_3(7x+2)}{5}}-2}{7} = \dfrac{7x+2-2}{7} = x$ ✓

8. Evaluate the expression

$$\ell og_2\left(\frac{x^2y^3}{4}\right)$$

   if $\ell og_2 x = 7$ and $\ell og_2 y = 3$.

   Solution: $\ell og_2\left(\frac{x^2y^3}{4}\right) = \ell og_2(x^2y^3) - \ell og_2 4 = \ell og_2 x^2 + \ell og_2 y^3 - 2 = 2\ell og_2 x + 3\ell og_2 y - 2 = 2\cdot 7+3\cdot 3-2 = 21$.

9. Find all solutions to the equation

$$\ell og_2(x+5) - \ell og_2(x-1) = 2.$$

   Solution: $\ell og_2(x+5)-\ell og_2(x-1) = \ell og_2\left(\dfrac{x+5}{x-1}\right) = 2 \Leftrightarrow 2^{\ell og_2\left(\dfrac{x+5}{x-1}\right)} = 2^2 \Leftrightarrow \dfrac{x+5}{x-1} = 4 \Leftrightarrow x+5 = 4x-4 \Leftrightarrow 3x = 9 \Leftrightarrow x = 3$.

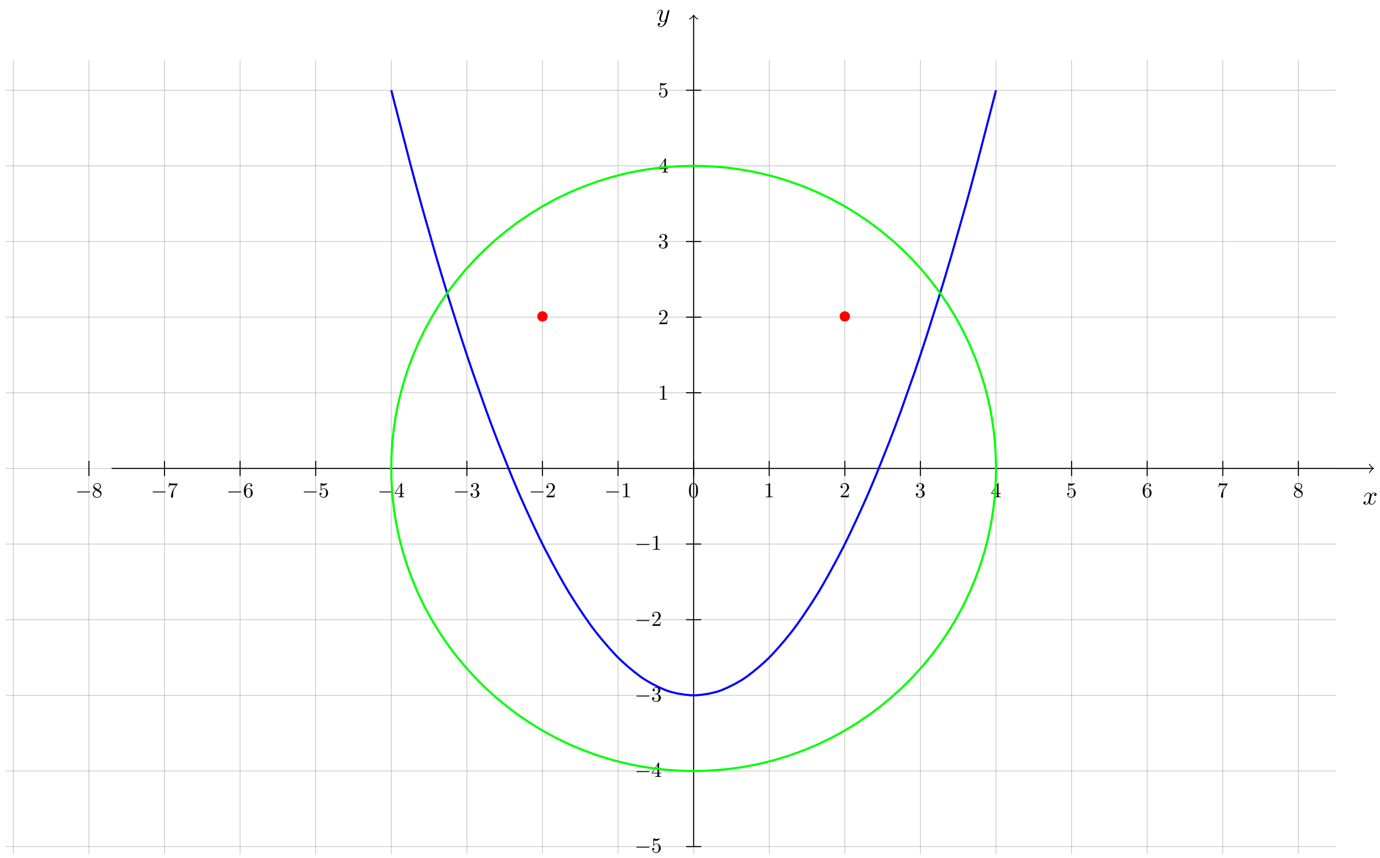
10. Plot the following items on the same coordinate plane below:

   (a) The circle $x^2 + y^2 = 16$.

   (b) The points $(2, 2)$ and $(-2, 2)$.

   (c) The parabola $y = \frac{1}{2}x^2 - 3$ with domain $[-4, 4]$.

   Solution:



11. Suppose a bank wants to advertise that \$1000 deposited in its savings account will grow to \$1050 in one year. This bank compounds interest daily (365 times per year). What minimal annual interest rate must the bank pay?

   Solution: the rate r must satisfy the inequality $1000 \cdot \left(1 + \frac{r}{365}\right)^{365} \geq 1050$, equivalently,

   $r \geq 365 \cdot (\sqrt[365]{1.05} - 1) \approx 0.049$ or 4.9%.