MATH 0200: PREPARATION FOR SCIENTIFIC CALCULUS

Review Midterm Exam 2

Solutions

1. Find the length of the circular arc of the unit circle connecting the point $P_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and the point whose radius corresponds to 1 radian.

   Solution: $P_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ corresponds to the angle $-\frac{\pi}{4}$ radians, the length of the arc is $\frac{\pi}{4} + 1$.

2. For a 14-inch pizza , find the area of a slice with angle 4/7 radians.

   Solution: The radius is $r = 14/2 = 7$ inches, giving the area of the slice $\mathcal{S}$ $= \frac{7^2 \cdot 4/7}{2} = 14\text{in}^2$.

3. Suppose y is a number such that $\tan(y) = -\frac{5}{7}$. Find $\tan(-y)$.

   Solution: $\tan(-y) = -\tan(y) = \frac{5}{7}$.

4. Suppose that $\sin(\alpha) = \frac{3}{5}$ and α is in the second quadrant. Use trigonometric identities to find the exact values of the following quantities.

   (a) $\cos(\alpha)$

   Solution: $\cos(\alpha) = \pm\sqrt{1 - \sin^2(\alpha)} = \pm\sqrt{1 - \left(\frac{3}{5}\right)^2} = \pm\sqrt{1 - \left(\frac{9}{25}\right)} = \pm\sqrt{\left(\frac{16}{25}\right)} = \pm\frac{4}{5}$. As α is in the second quadrant, $\cos(\alpha) = -\frac{4}{5}$.

   (b) $\sin(2\alpha)$

   Solution: $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) = -2 \cdot \frac{3}{5} \cdot \frac{4}{5} = -\frac{24}{25}$.

   (c) $\cos(2\alpha)$

   Solution: $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 2 \cdot \frac{16}{25} - 1 = \frac{7}{25}$.

   (d) $\tan(2\alpha)$

   Solution: $\tan(2\alpha) = \frac{\sin(2\alpha)}{\cos(2\alpha)} = -\frac{24}{7}$.

5. Let α be an angle in the first quadrant, and suppose $\sin(\alpha) = a$. Evaluate the following expressions in terms of a. For example, $\sin(\alpha + \pi) = -a$. Your answers need to be expressions that involve a.

   (a) $\sin\left(\alpha + \frac{3\pi}{2}\right)$

   Solution: $\sin\left(\alpha + \frac{3\pi}{2}\right) = \sin\left(\alpha - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \alpha\right) = -\cos(\alpha) = -\sqrt{1 - a^2}$.

(b) $\cos\left(\frac{\pi}{2}-\alpha\right)$

Solution: $\cos\left(\frac{\pi}{2}-\alpha\right)=\sin(\alpha)=a.$

(c) $\sin(\pi-\alpha)$

Solution: $\sin(\pi-\alpha)=\sin(\alpha)=a.$

(d) $\sin\left(\frac{\pi}{2}-\alpha\right)$

Solution: $\sin\left(\frac{\pi}{2}-\alpha\right)=\cos(\alpha)=\sqrt{1-a^2}.$

(e) $\cos(2\pi-\alpha)$

Solution: $\cos(2\pi-\alpha)=\cos(\alpha)=\sqrt{1-a^2}.$

6. A 12-foot ladder leans against a vertical wall forming an angle of 60° with the ground. How high above the ground does the ladder touch the wall?

Solution: $h=12\sin(60°)=6\sqrt{3}.$

7. Use trigonometric identities to find the exact value of each expression.

(a) $\cos(48°)\cos(12°)-\sin(48°)\sin(12°)$

Solution: $\cos(48°)\cos(12°)-\sin(48°)\sin(12°)=\cos(48°+12°)=\cos(60°)=\frac{1}{2}.$

(b) $\dfrac{\tan(78°)+\tan(112°)}{1-\tan(78°)\tan(112°)}$

Solution: $\dfrac{\tan(78°)+\tan(112°)}{1-\tan(78°)\tan(112°)}=\tan(78°+112°)=\tan(180°)=0.$

8. Show (without using a calculator) that

$$\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{4\pi}{21}\right)+\cos\left(\frac{\pi}{7}\right)\sin\left(\frac{4\pi}{21}\right)=\frac{\sqrt{3}}{2}.$$

Solution: $\sin\left(\frac{\pi}{7}\right)\cos\left(\frac{4\pi}{21}\right)+\cos\left(\frac{\pi}{7}\right)\sin\left(\frac{4\pi}{21}\right)=\sin\left(\frac{\pi}{7}+\frac{4\pi}{21}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}.$

9. (a) Use the half angle formula to find the value of $\sin\left(\frac{\pi}{8}\right)$.

Solution: $\sin\left(\frac{\pi}{8}\right)=\sqrt{\dfrac{1-\cos\left(\frac{\pi}{4}\right)}{2}}=\sqrt{\dfrac{1-\frac{1}{\sqrt{2}}}{2}}=\sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}}.$

(b) Find the area of a regular 16-sided polygon whose vertices are 16 equally spaced points on a circle with radius 3.

Solution: as $\sin\left(\frac{2\pi}{16}\right)=\sin\left(\frac{\pi}{8}\right)=\sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}}$, the area is

$$\mathcal{S}=\frac{16\cdot 3^2\cdot\sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}}}{2}=72\cdot\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}.$$

10. Find the perimeter of a regular 12-sided polygon whose vertices are 12 equally spaced points on a circle with radius 3.[1]

    Solution: the length of a side is found using the law of cosines:

$$x^2 = 3^2 + 3^2 - 2 \cdot 3 \cdot 3 \cos\left(\frac{2\pi}{12}\right) = 18 - 18\frac{\sqrt{3}}{2} \Leftrightarrow x^2 = 18 - 9\sqrt{3} \Leftrightarrow x = 3\sqrt{2 - \sqrt{3}}.$$

    The perimeter is $\mathcal{P} = 12x = 36\sqrt{2 - \sqrt{3}}$.

11. Find the smallest number t such that $\cos(3^t) = 0$.

    Solution: $\cos(3^t) = 0 \Leftrightarrow 3^t = \arccos(0) = \frac{\pi}{2} \Leftrightarrow t = \ell og_3\left(\frac{\pi}{2}\right)$.

[1] **Hint:** use the law of cosines