Spring 2023

MATH 0200: Preparation for Scientific Calculus

Review for the Final Exam

Solutions

1. Solve $e^{2x} + e^x = 6$.

   Solution: $t = e^x > 0$, get $t^2 + t - 6 = 0 \Leftrightarrow t = (-1 \pm 5)/2 = -3$ or 2. As $t > 0$, we get $e^x = 2 \Leftrightarrow x = \ell n(2)$.

2. Solve $\dfrac{\ell n(11x)}{\ell n(4x)} = 2$.

   Solution: $\dfrac{\ell n(11x)}{\ell n(4x)} = 2 \Leftrightarrow \ell n(11x) = 2\ell n(4x) \Leftrightarrow e^{\ell n(11x)} = e^{2\ell n(4x)} \Leftrightarrow 11x = (4x)^2 = 16x^2 \Leftrightarrow x(11 - 16x) = 0 \Leftrightarrow x = 0$ or $x = 16/11$. As $x = 0$ is not in the domain, the answer is $x = 16/11$.

3. Solve $\ell n(\ell n(x)) = 5$.

   Solution: $\ell n(\ell n(x)) = 5 \Leftrightarrow e^{\ell n(\ell n(x))} = e^5 \Leftrightarrow \ell n(x) = e^5 \Leftrightarrow x = e^{e^5}$.

4. How many digits does $5^{999} \cdot 17^{222}$ have?

   Solution: $\ell g(5^{999} \cdot 17^{222}) = \ell g(5^{999}) + \ell g(17^{222}) = 999\ell g(5) + 222\ell g(17) \approx 971.4$, the number of digits is 972.

5. Suppose a savings account pays 5% interest per year, compounded four times per year. If the savings account starts with \$600, how many years would it take for the savings account to exceed \$1400?

   Solution: $600\left(1 + \dfrac{0.05}{4}\right)^{4t} \geq 1400 \Leftrightarrow 1.0125^{4t} = 14/6 = 7/3 \Leftrightarrow 4t \geq \ell og_{1.0125}(7/3) \Leftrightarrow t \geq 0.25\ell og_{1.0125}(7/3) \approx 17$ years.

6. A baseball card bought for \$50 increases by 10% in value each year. How long does it take for the card to quadruple in price?

   Solution: $50(1.1)^t = 4 \cdot 50 = 200 \Leftrightarrow (1.1)^t = 4 \Leftrightarrow t = \ell og_{1.1}4 \approx 14.5$ years.

7. (*a*) A baby alligator  is born 9 inches long and grows by 5% each month. How old will the alligator  be when it doubles its initial length?

   Solution: $9(1 + 0.05)^t = 18 \Leftrightarrow 9 \cdot 1.05^t = 18 \Leftrightarrow 1.05^t = 2 \Leftrightarrow t = \ell og_{1.05}2 \approx 14.21$ months or a year and 2 months.

   (*b*) Our alligator has eaten a slice  of a 20-inch pizza  with angle 2 radians. The weight of the alligator grows by approximately 250 grams per square inch of pizza eaten. How much weight did the alligator gain  ...???

Solution: the area of the slice is $\dfrac{2 \cdot 10^2}{2} = 100$ giving the increase in weight by $100 \cdot 250 = 25000$ grams or 25 kilos...

8. (a) Show that $\cos(15°) = \dfrac{\sqrt{2+\sqrt{3}}}{2}$

   Solution: $\cos(15°) = \dfrac{\sqrt{1+\cos(30°)}}{2} = \dfrac{\sqrt{2+\sqrt{3}}}{2}$.

   (b) Show that $\sin(15°) = \dfrac{\sqrt{2-\sqrt{3}}}{2}$

   Solution: $\sin(15°) = \dfrac{\sqrt{1-\cos(30°)}}{2} = \dfrac{\sqrt{2-\sqrt{3}}}{2}$.

9. Suppose that $\sin(\alpha) = -\dfrac{2}{7}$ and α is in the second quadrant. Use trigonometric identities to find the exact values of the following quantities.

   (a) *Solution:* $\cos(\alpha) = -\sqrt{1-\sin^2(\alpha)} = -\dfrac{\sqrt{45}}{7}$.

   (b) *Solution:* $\sin(2\alpha) = \sin(\alpha)\cos(\alpha) = \dfrac{4\sqrt{45}}{49}$.

   (c) *Solution:* $\cos(2\alpha) = 1 - 2\sin^2(\alpha) = 1 - 2 \cdot \dfrac{4}{49} = \dfrac{41}{49}$.

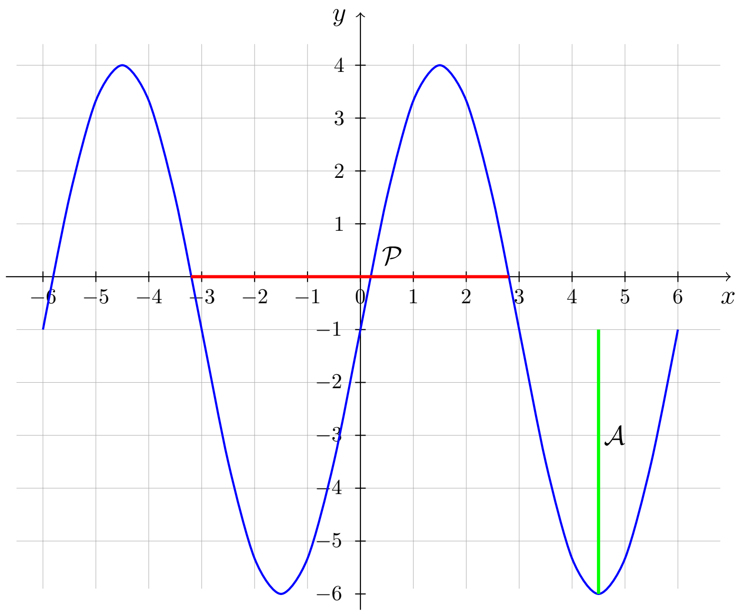
10. Find the smallest number x such that $\cos(e^x + 1) = 0$.

    Solution: notice that $e^x \geq 1$, as $e^x + 1 = \dfrac{\pi}{2} + \pi k$, the minimal such number is found as $e^x + 1 = \dfrac{\pi}{2} \Leftrightarrow e^x = \dfrac{\pi}{2} - 1 \Leftrightarrow x = \ell n\left(\dfrac{\pi}{2} - 1\right)$.

11. Find the amplitude and period of the given function $f(x)$ on the given interval $[a, b]$. Sketch the graph and mark any line segments corresponding to amplitude and period.

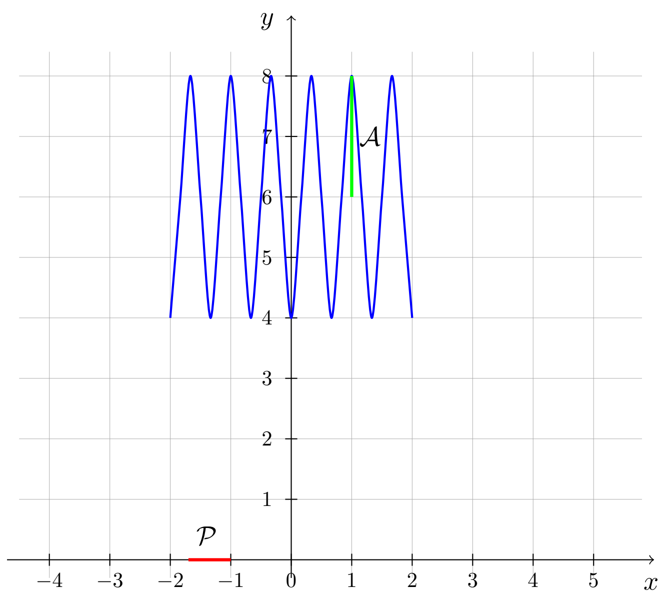
    (a) $f(x) = 5\sin\left(\dfrac{\pi x}{3}\right) - 1$ on the interval $[-6, 6]$.

    Solution: $min = 5 \cdot (-1) - 1 = -6, max = 5 \cdot 1 - 1 = 4, \mathcal{A} = \dfrac{max - min}{2} = 5, \mathcal{P} = \dfrac{2\pi}{\pi/3} = 6$.



(b) $f(x) = -2\cos(3\pi x) + 6$ on the interval $[-2, 2]$.

Solution: $min = -2 \cdot 1 + 6 = 4, max = -2 \cdot (-1) + 6 = 8, \mathcal{A} = \dfrac{max - min}{2} = 2, \mathcal{P} = \dfrac{2\pi}{3\pi} = 2/3.$



12. Find the value of t for which the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

   (a) $\mathbf{u} = (2\ell n(t), -3)$ and $\mathbf{v} = (1, 6)$.

   Solution: $\mathbf{u} \cdot \mathbf{v} = 2\ell n(t) - 18 = 0 \Leftrightarrow \ell n(t) = 9 \Leftrightarrow t = e^9$.

   (b) $\mathbf{u} = (56, 2)$ and $\mathbf{v} = (-1, 7^t)$.

   Solution: $\mathbf{u} \cdot \mathbf{v} = -56 + 2 \cdot 7^t = 0 \Leftrightarrow 7^t = 28 \Leftrightarrow t = \ell og_7(28)$.

   (c) $\mathbf{u} = \left(-\dfrac{\pi}{3}, 2\right)$ and $\mathbf{v} = (1, \arccos(t))$.

Solution: $\mathbf{u}\cdot\mathbf{v} = -\left(\frac{\pi}{3}\right) + 2\arccos(t) = 0 \Leftrightarrow \arccos(t) = \left(\frac{\pi}{6}\right) \Leftrightarrow t = \frac{\sqrt{3}}{2}$.

13. Rewrite the following equations in polar coordinates.

    (a) $x^2 + y^2 = 49$.

    Solution: $r^2 = 49$.

    (b) $(x-5)^2 + y^2 = 9$.

    Solution: $x^2 - 10x + 25 + y^2 = 9 \Leftrightarrow r^2 - 10r\cos(\varphi) = -16$.

    (c) $x^2 + (y+3)^2 = 25$.

    Solution: $x^2 + y^2 + 6y + 9 = 25 \Leftrightarrow r^2 + 6r\sin(\varphi) = 16$.

14. Rewrite the following equations in Cartesian coordinates.

    (a) $r = 3\cos(\theta)$.

    Solution: $r = 3\cos(\theta) \Leftrightarrow r^2 = 3r\cos(\theta) \Leftrightarrow x^2 + y^2 = 3x \Leftrightarrow (x-1.5)^2 + y^2 = 2.25$.

    (b) $r = 2\sin(\theta)$.

    Solution: $r = 2\sin(\theta) \Leftrightarrow r^2 = 2r\sin(\theta) \Leftrightarrow x^2 + y^2 = 2y \Leftrightarrow x^2 + (y-1)^2 = 1$.

    (c) $r = 5$.

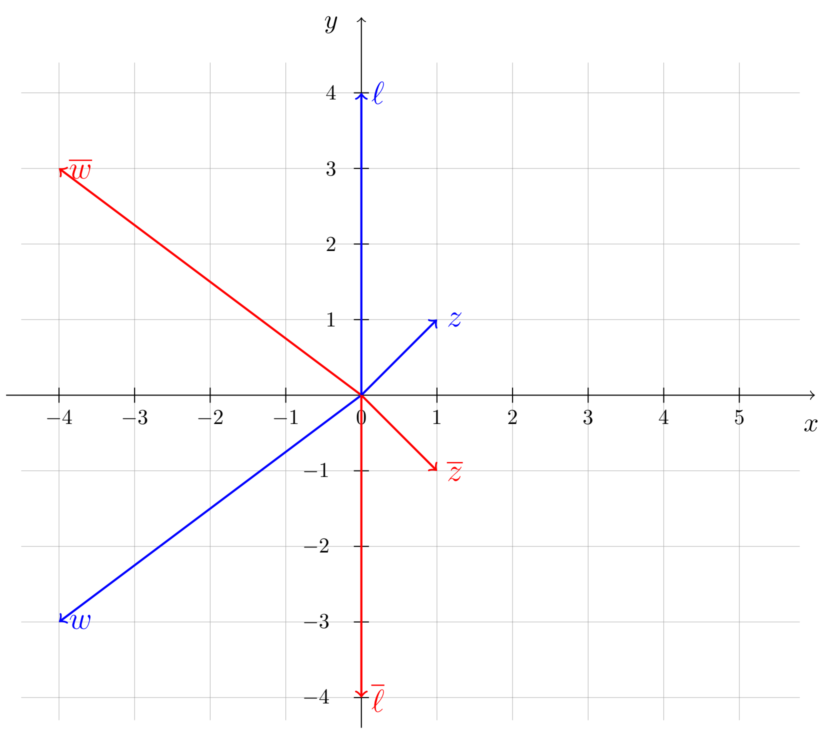
    Solution: $x^2 + y^2 = 25$.

15. Sketch the radius vectors corresponding to the following complex numbers and their conjugates.

    (a) $z = 1 + i$.

    (b) $w = -3i - 4$.

    (c) $\ell = 4i$.



16. Write the following complex numbers in the form $a + bi$.

(*a*) $1 + i - (\overline{i - 5})$.

Solution: $1 + i - \overline{i - 5} = 1 + i - (-i - 5) = 6 + 2i$.

(*b*) $(5i - 2)(3 - i)$.

Solution: $(5i - 2)(3 - i) = 15i - 5i^2 - 6 + 2i = -1 + 17i$.

(*c*) $(3 - i)^2$.

Solution: $(3 - i)^2 = 9 - 6i + i^2 = 8 - 6i$.

(*d*) $\dfrac{5 + 2i}{2 - i}$.

Solution: $\dfrac{5 + 2i}{2 - i} = \dfrac{(5 + 2i)(2 + i)}{(2 - i)(2 + i)} = \dfrac{10 + 5i + 4i + 2i^2}{5} = \dfrac{8 + 9i}{5} = \dfrac{8}{5} + \dfrac{9}{5}i$.

17. Write the following complex numbers in the form $z = r(\cos(\theta) + i\sin(\theta))$, where $r = |z|$ and θ is the angle that z forms with the x-axis.

(*a*) $z = \dfrac{1 + i}{\sqrt{2}}$.

Solution: $z = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$.

(*b*) $w = -5 - 5\sqrt{3}i$.

Solution: $r = \sqrt{(-5)^2 + (-5\sqrt{3})^2} = \sqrt{25 + 75} = 10$, hence $w = 10\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 10(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right))$.