Midterm Exam 1
Review

Solutions

Problem 1 Write each of the states below in the standard basis (**do not use the sum notation**).

(a) $\mathsf{H}\left(\frac{3|0\rangle - 4|1\rangle}{5}\right)$

Solution: $\mathsf{H}\left(\frac{3|0\rangle - 4|1\rangle}{5}\right) = \frac{3|+\rangle - 4|-\rangle}{5} = \frac{3|0\rangle + 3|1\rangle - 4|0\rangle + 4|1\rangle}{5\sqrt{2}} = \frac{-|0\rangle + 7|1\rangle}{5\sqrt{2}}$.

(b) $\mathsf{H}^{\otimes 5}(|+-++-\rangle)$

Solution: $\mathsf{H}^{\otimes 5}(|+-++-\rangle) = |01001\rangle$.

(c) $|0\rangle \otimes \mathsf{H}^{\otimes 3}(|011\rangle) \otimes |1\rangle$

Solution: $|0\rangle \otimes \mathsf{H}^{\otimes 3}(|011\rangle) \otimes |1\rangle = |0\rangle \otimes |+--\rangle \otimes |1\rangle = \frac{1}{\sqrt{8}}(|00001\rangle - |00011\rangle - |00101\rangle + |00111\rangle + |01001\rangle - |01011\rangle - |01101\rangle + |01111\rangle)$.

(d) $\mathsf{H}(|0\rangle) \otimes \mathsf{T}(|+\rangle)$[1]

Solution: $\mathsf{H}(|0\rangle) \otimes \mathsf{T}(|+\rangle) = |+\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{2}(|00\rangle + i|01\rangle + |10\rangle + i|11\rangle)$.

Problem 2 Check if the operator is unitary and find the matrix of the inverse operator if it is.

(a) $A = \begin{pmatrix} i & 0 & 0 \\ 0 & e^{2\pi i/7} & 0 \\ 0 & 0 & e^{2\pi i/2022} \end{pmatrix}$

Solution: we compute $A^\dagger = \begin{pmatrix} -i & 0 & 0 \\ 0 & e^{-2\pi i/7} & 0 \\ 0 & 0 & e^{-2\pi i/2022} \end{pmatrix}$ and $AA^\dagger = I$, hence, A is unitary with $A^{-1} = A^\dagger$.

(b) $B = \begin{pmatrix} i & 1 & 1+i \\ i-1 & e^{2\pi i/7} & 1-i \\ 0 & 3-i & e^{2\pi i/2022} \end{pmatrix}$

Solution: let $v_1 = (i, i-1, 0)$ be the vector in the first column of B. Recall that $v_1 = B(e_1)$, where $e_1 = (1,0,0)$. As $\langle v_1|v_1\rangle = \bar{i}\cdot i + (\overline{i-1})\cdot(i-1) = 1 + 2 = 3 \neq 1$, it follows that $\langle B(e_1)|B(e_1)\rangle \neq \langle e_1|e_1\rangle$ and B is not unitary.

Problem 3 Find all possible pairs of values of a and b for which the operator $C = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & a \\ 2i & b \end{pmatrix}$ is unitary.

Solution: the column vectors $v = \frac{1}{\sqrt{5}}(1, 2i)$ and $w = \frac{1}{\sqrt{5}}(a, b)$ need to be of unit norm and hermitian orthogonal. These conditions give rise to the following system of equations on parameters a and b: $\begin{cases} a - 2ib = 0 \\ \frac{|a|^2 + |b|^2}{5} = 1. \end{cases}$

[1] Recall that $\mathsf{T} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.

We get that $a = 2ib$ and $5|b|^2 = 5 \Leftrightarrow |b|^2 = 1$, so $b = e^{i\varphi}$ and $a = 2ie^{i\varphi} = 2e^{i\varphi+\pi/2}$ for any $\varphi \in [0, 2\pi)$.

Problem 4

(a) Write the matrix of the operator CH_1 in the standard basis of $(\mathbb{C}^2)^{\otimes 2}$, the controlled Hadamard operator with the first qubit being the controlling one. Draw the corresponding circuit.

Solution: we find the images of the standard basis vectors and record the corresponding coefficients in the columns of a matrix

$$|00\rangle \mapsto |00\rangle$$
$$|01\rangle \mapsto |01\rangle$$
$$|10\rangle \mapsto |1+\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$$
$$|11\rangle \mapsto |1-\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$$

$$CH_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

(b) Write the matrix of the operator CH_2 in the standard basis of $(\mathbb{C}^2)^{\otimes 2}$, the controlled Hadamard operator with the second qubit being the controlling one. Draw the corresponding circuit.

Solution: we find the images of the standard basis vectors and record the corresponding coefficients in the columns of a matrix

$$|00\rangle \mapsto |00\rangle$$
$$|01\rangle \mapsto |+1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$
$$|10\rangle \mapsto |10\rangle$$
$$|11\rangle \mapsto |-1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$$

$$CH_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

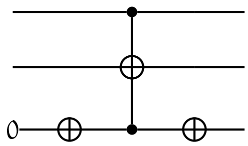
(c) Find a matrix σ with $CH_2 = \sigma CH_1 \sigma$.[2]

Solution: $\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Problem 5 Write the CNOT operator as a composition of negation $\neg$ and CCNOT gates using ancilla consisting of one auxiliary bit in state 0 (the bit must be returned to that state at the end of the circuit).

Solution: we make the auxiliary bit 'play the role' of the second control bit in CCNOT. As this bit is in state one, the activation of NOT is entirely contingent upon the value of the first bit:

[2] **Hint:** what is the matrix of the operator swapping the first and second qubit?



Problem 6 Let $U_1, U_2 \in U_n(\mathbb{C})$ be unitary operators and $\widetilde{U}_1, \widetilde{U}_2 \in U_n(\mathbb{C})$ approximations with precision ε_1 and ε_2, respectively. Show that $\widetilde{U}_1\widetilde{U}_2$ approximates U_1U_2 with precision $\varepsilon_1 + \varepsilon_2$. [3]

Solution: $\|U_1U_2 - \widetilde{U}_1\widetilde{U}_2\| = \|U_1U_2 - \widetilde{U}_1\widetilde{U}_2 + \widetilde{U}_1U_2 - \widetilde{U}_1U_2\| = \|\widetilde{U}_1(U_2 - \widetilde{U}_2) + (U_1 - \widetilde{U}_1)U_2\| \leq \|\widetilde{U}_1(U_2 - \widetilde{U}_2)\| + \|(U_1 - \widetilde{U}_1)U_2\| = \|\widetilde{U}_1\|\|(U_2 - \widetilde{U}_2)\| + \|(U_1 - \widetilde{U}_1)\|\|U_2\| = \|(U_2 - \widetilde{U}_2)\| + \|(U_1 - \widetilde{U}_1)\| \leq \varepsilon_1 + \varepsilon_2$.

Bernstein-Vazirani algorithm

Problem 7 Let $f_s : \mathbb{B}^4 \to \mathbb{B}$ be a function given by $f_s(x) := x \cdot s = x_1s_1 + x_2s_2 + x_3s_3 + x_4s_4 \pmod 2$ for any $x \in \mathbb{B}^4$ and a fixed string $s \in \mathbb{B}^4$.

(a) Find s given that

$$f_s(1111) = 1$$
$$f_s(1110) = 0$$
$$f_s(1100) = 0$$
$$f_s(1000) = 1.$$

Solution: given the above data, we get the system of linear equations on s:

$$\begin{cases} s_1 + s_2 + s_3 + s_4 = 1 \\ s_1 + s_2 + s_3 = 0 \\ s_1 + s_2 = 0 \\ s_1 = 1, \end{cases}$$

which has a unique solution $s = (1, 1, 0, 1)$.

(b) Using your answer in (a) show the steps of execution of Bernstein-Vazirani algorithm in this example.

Solution:

1. Applying $H^{\otimes 4}$:
   $H^{\otimes 4}(|0000\rangle) \otimes |-\rangle = \frac{1}{4}\sum_{i=0}^{15} |i\rangle \otimes |-\rangle$.
2. Applying the oracle $\mathcal{O}_f$:
   $\frac{1}{4}(|0000\rangle - |0001\rangle + |0010\rangle - |0011\rangle - |0100\rangle + |0101\rangle - |0110\rangle + |0111\rangle - |1000\rangle + |1001\rangle - |1010\rangle +$
   $+|1011\rangle + |1100\rangle - |1101\rangle + |1110\rangle - |1111\rangle)(= H^{\otimes 4}(|1101\rangle))$.
3. Applying $H^{\otimes 4}$ again:
   $H^{\otimes 4}(H^{\otimes 4}(|1101\rangle)) = |1101\rangle$.

[3] **Hint:** you need to show that $\|U_1U_2 - \widetilde{U}_1\widetilde{U}_2\| \leq \varepsilon_1 + \varepsilon_2$. Here is a helpful 'arithmetic trick': add $0 = \widetilde{U}_1U_2 - \widetilde{U}_1U_2$ to the expression and use various properties of the norm to deduce the result.

Simon's algorithm

Problem 8 Let $f : \mathbb{B}^3 \to \mathbb{B}^3$ be a map, promised to satisfy the condition

$$f(x) = f(y) \Leftrightarrow x = y \text{ or } x = y \oplus s$$

for some fixed string $s \in \mathbb{B}^3$. The function is given by

$$\begin{aligned} f(000) &= f(101) = 111 \\ f(001) &= f(100) = 010 \\ f(010) &= f(111) = 100 \\ f(110) &= f(011) = 001. \end{aligned}$$

(a) Find s in a straightforward way.

Solution: As $f(000) = f(101)$, we get $s = 101$.

(b) Now let's run Simon's algorithm. Write $\mathcal{O}_f(H^{\otimes 3}(|000\rangle) \otimes |000\rangle)$ in the standard basis.

Solution: $\frac{1}{\sqrt{2^3}}(|000\rangle|111\rangle + |001\rangle|010\rangle + |010\rangle|100\rangle + |100\rangle|010\rangle + |011\rangle|001\rangle + |101\rangle|111\rangle + |110\rangle|001\rangle + |111\rangle|100\rangle)$.

(c) Give the state after measuring the second register (suppose the measurement gave $|100\rangle$).

Solution: $\frac{1}{\sqrt{2}}(|010\rangle|100\rangle + |111\rangle|100\rangle)$.

(d) Apply $H^{\otimes 3}$ to the first three qubits of the state vector you obtained in (c) and measure the first register.

Solution:

$$\begin{aligned} &\frac{1}{\sqrt{2}}(|+-+\rangle + |---\rangle) = \\ &\frac{1}{4}(|000\rangle + |100\rangle - |010\rangle + |001\rangle - |011\rangle + |101\rangle - |110\rangle - |111\rangle) + \\ &\frac{1}{4}(|000\rangle - |100\rangle - |010\rangle - |001\rangle + |011\rangle + |101\rangle + |110\rangle - |111\rangle) = \\ &\frac{1}{2}(|000\rangle - |010\rangle + |101\rangle - |111\rangle). \end{aligned}$$

(e) Write the linear equation on s that you obtained from (d) given that the measurement produced any vector except $|000\rangle$.

Solution: say, we picked $|111\rangle$ and got

$$s_1 + s_2 + s_3 = 0.$$

(f) Repeat (c) − (e) with the assumption that this time the result of the first measurement was $|001\rangle$. Get one more linear equation on s. Use this equation together with the one you got in (e) and find s.

Solution: after the first measurement, we get the state $\frac{1}{\sqrt{2}}(|011\rangle|001\rangle + |110\rangle|001\rangle)$.

Applying $H^{\otimes 3}$ gives

$$\frac{1}{\sqrt{2}}(|+--\rangle+|--+\rangle)=$$
$$\frac{1}{4}(|000\rangle+|100\rangle-|010\rangle-|001\rangle+|011\rangle-|101\rangle-|110\rangle+|111\rangle)+$$
$$\frac{1}{4}(|000\rangle-|100\rangle-|010\rangle+|001\rangle-|011\rangle-|101\rangle+|110\rangle+|111\rangle)=$$
$$\frac{1}{2}(|000\rangle-|010\rangle-|101\rangle+|111\rangle).$$

Let's assume the second measurement produced the vector $|010\rangle$, hence, the equation $s_2 = 0$. Combining with the equation from (e), we get the system

$$\begin{cases} s_1 + s_2 + s_3 = 0 \\ s_2 = 0 \end{cases}$$

with $s_i \in \{0, 1\}$ and $(s_1, s_2, s_3) \neq (0, 0, 0)$. The only solution is $s = (1, 0, 1)$.