Tensor products and unitary operators

Problem 1 (15 pts) Let $R_\varphi = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix of an operator performing counter-clockwise rotation through angle φ in the xy-plane.

(a) Show that R_φ preserves the dot product and conclude that the possible **real** eigenvalues of R_φ are ± 1.

(b) Find the values of $\varphi \in [0, 2\pi)$ for which R_φ is diagonalizable. What are the corresponding eigenvalues and eigenvectors?

Definition 1. Let V be an n-dimensional vector space. A vector $v = \sum_{1 \le i,j \le n} \alpha_{ij} e_i \otimes e_j \in V \otimes V$ is called a **symmetric tensor** provided $\sigma(v) = v$, where $\sigma(v) := \sum_{1 \le i,j \le n} \alpha_{ij} e_j \otimes e_i$ (swaps the components in each summand). In case $\sigma(v) = -v$, then we refer to v as **antisymmetric tensor**.

Example 2. The tensors $e_1 \otimes e_2 + e_2 \otimes e_1$ and $e_1 \otimes e_1$ are symmetric, while $e_1 \otimes e_2 - e_2 \otimes e_1$ is antisymmetric.

Problem 2 (15 pts)

(a) Check that symmetric tensors form a vector space and find its dimension.

(b) Check that antisymmetric tensors form a vector space and find its dimension.

(c) Let $S^2V \subset V \otimes V$ and $\Lambda^2 V \subset V \otimes V$ be the subspaces of symmetric and antisymmetric tensors. Construct an isomorphism $V \otimes V \simeq S^2V \oplus \Lambda^2 V$.[1]

Problem 3 (10 pts) Show that the space of quantum states is closed under the tensor product. [2]

[1]**Hint:** notice that $e_i \otimes e_j = \dfrac{(e_i \otimes e_j + e_j \otimes e_i) - (e_i \otimes e_j - e_j \otimes e_i)}{2}$ and construct a map $V \otimes V \to S^2V \oplus \Lambda^2 V$ using this observation. Then show that the map has no kernel and the vector spaces $V \otimes V$ and $S^2V \oplus \Lambda^2 V$ are of the same dimension (use the results in (a) and (b)). It follows that such a map must be an isomorphism.

[2]**Hint:** you need to verify that given an m-state $v = \sum_{i=1}^{m} \alpha_i e_i$ and an n-state $w = \sum_{j=1}^{n} \beta_j f_j$ (here $\sum_{i=1}^{m} |\alpha_i|^2 = \sum_{j=1}^{n} |\beta_i|^2 = 1$), one gets that $v \otimes w$ has the sum of the squares of absolute values of coefficients equal to 1 as well.

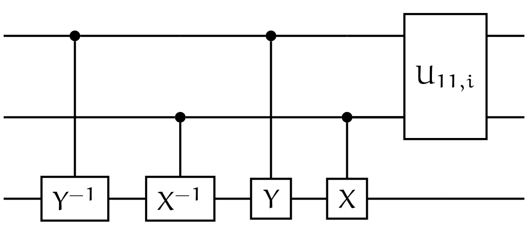
Problem 4 (5 pts) Show that a unitary operator cannot 'delete' information: there is no unitary operator $A \in U_2(\mathbb{C})$ that maps $|\varphi\rangle$ to $|0\rangle$ for every qubit $|\varphi\rangle$.[3]

Problem 5 (10 pts) Show that the eigenvalues values of a unitary operator have norm 1.[4]

Problem 6 (15 pts)

(a) Let $X = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & -1 \\ 1 & i \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Check that $X, Y, \sigma^x \in U_2(\mathbb{C})$ and the identity $\sigma^x = iXYX^{-1}Y^{-1}$ holds.

(b) Let $U_{11,i} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$ be the operator that maps $|11\rangle$ to $-i|11\rangle$ and acts as identity on the remaining standard basis vectors ($|00\rangle$, $|01\rangle$ and $|10\rangle$). Show that the circuit below computes the $CCNOT$ operator.



[3]**Hint:** can a unitary operator have a kernel?

[4]**Hint:** let v be an eigenvector of U with eigenvalue λ. Use that U preserves the inner product, in particular, $\langle Uv, Uv\rangle = \langle v, v\rangle$ to derive that $|\lambda|^2 = 1$.

Problem 7 Consider the operator $U = \frac{1}{2}\begin{pmatrix} 1 & -i & -1+i \\ i & 1 & 1+i \\ 1+i & -1+i & 0 \end{pmatrix}$.

(a) (5 pts) Check that U is unitary, i.e. $UU^\dagger = 1$.

$(b)^\star$ (15 pts) Write U as a product of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{11} & u_{12} \\ 0 & u_{21} & u_{22} \end{pmatrix}, \begin{pmatrix} v_{11} & 0 & v_{12} \\ 0 & 1 & 0 \\ v_{21} & 0 & v_{22} \end{pmatrix} \text{ and } \begin{pmatrix} w_{11} & w_{12} & 0 \\ w_{21} & w_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$ and $\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \in U_2(\mathbb{C})$.

Problem 8 (10 pts) Let $U = \begin{pmatrix} \lambda & u_{12} & u_{13} & \dots & u_{1m} \\ 0 & * & * & \dots & * \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & \dots & * \end{pmatrix} \in U_m(\mathbb{C})$ be a unitary operator.

Show that $u_{12} = u_{13} = \ldots = u_{1m} = 0$ and $|\lambda| = 1$.[5]

[5]**Hint:** you may want to use that $UU^\dagger = I$ and $U^\dagger U = I$, implying $(UU^\dagger)_{11} = 1$ and $(U^\dagger U)_{11} = 1$.