Classification of Solutions of CYBE

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1 Introduction

The goal of these notes is to explain the main steps in classifying and finding solutions of the classical Yang-Baxter equation (CYBE), which is $[X_{12}(u_1-u_2), X_{13}(u_1-u_3)] + [X_{12}(u_1-u_2), X_{23}(u_2-u_3)] + [X_{13}(u_1-u_3), X_{23}(u_2-u_3)] = 0$, where $X(u)$ takes values in $\mathfrak{g}\otimes\mathfrak{g}$, and $\mathfrak{g}$ is a simple Lie algebra. Our primary references are [1] and [4].

2 Constant Solutions of CYBE

We start with the classification of solutions of the system of equations

$$\begin{aligned} [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \\ r_{12} + r_{21} = t \end{aligned} \tag{1}$$

with values in $\mathfrak{g}\otimes\mathfrak{g}$, where $t \in \mathfrak{g}\otimes\mathfrak{g}$ is the Casimir element, i.e. if we choose an orthonormal basis $\{I_\nu\}$ of $\mathfrak{g}$ with respect to the Killing form (as $\mathfrak{g}$ is simple, any nondegenerate invariant bilinear form is proportional to it), then $t = \sum I_\nu \otimes I_\nu$. We can express $r = \sum r^{\mu\nu} I_\mu \otimes I_\nu$. We explain the notation r_{12}, other notations of this type should be understood accordingly. For this we fix an associative algebra A with unit (i.e. $A = U(\mathfrak{g})$), containing $\mathfrak{g}$ and consider the map $\phi_{12} : \mathfrak{g}\otimes\mathfrak{g} \to A\otimes A\otimes A$, given by $\phi_{12}(a\otimes b) = a\otimes b\otimes 1$. Thus, by r_{12} we will understand $\sum r^{\mu\nu} I_\mu \otimes I_\nu \otimes 1$, r_{13} stands for $\sum r^{\mu\nu} I_\mu \otimes 1 \otimes I_\nu$, etc. We notice that if r is a solution of (1) and $\sigma \in \mathrm{Aut}(\mathfrak{g})$, then $(\sigma\otimes\sigma)(r)$ is also a solution. In order to write down explicit formulas for the solutions, we need some notation. Namely, $\mathfrak{h} \subset \mathfrak{b}_+ \subset \mathfrak{g}$ are Cartan and Borel subalgebras of $\mathfrak{g}$, Γ is the set of simple roots. The solutions will depend on a discrete parameter - a triple $(\Gamma_1, \Gamma_2, \tau)$, where $\Gamma_1, \Gamma_2 \subset \Gamma$, $\tau : \Gamma_1 \to \Gamma_2$ is a bijection and satisfies

$$(a)\ (\alpha, \beta) = (\tau(\alpha), \tau(\beta)) \ \ \forall \alpha, \beta \in \Gamma$$
$$(b)\ \ \forall \alpha \in \Gamma_1 \ \ \exists k \in \mathbb{N} : \alpha, \tau(\alpha), \ldots, \tau^{k-1}(\alpha) \in \Gamma_1, \tau^k(\alpha) \notin \Gamma_1.$$

A triple satisfying the conditions above is called *admissible*. The solution also depends on a continuous parameter - an element $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$, which satisfies (below t_0 denotes the projection of t on $\mathfrak{h} \otimes \mathfrak{h}$)

$$\begin{array}{c} r_0^{12} + r_0^{21} = t_0 \\ (\tau\alpha \otimes 1)(r_0) + (1 \otimes \alpha)(r_0) = 0, \alpha \in \Gamma_1 \end{array} \tag{2}$$

If $r_0 = \sum_i h_i \otimes h'_i$, then $(\tau\alpha \otimes 1)(r_0) = \sum_i \tau\alpha(h_i) h'_i$ and $(1 \otimes \alpha)(r_0) = \sum_i \alpha(h'_i) h_i$.

We fix the system $\{X_\alpha, Y_\alpha, H_\alpha\}_{\alpha \in \Gamma}$ of Weyl generators of $\mathfrak{g}$ and denote by $\mathfrak{a}_i = \sum_{\alpha \in \Gamma_i} \mathbb{C} H_\alpha \oplus \sum_{\alpha \in \Gamma'_i} \mathfrak{g}^\alpha$, where Γ'_i stands for the roots which, whose expansion in terms of simple roots involves only roots from Γ_i. We notice that τ gives rise to an isomorphism $\phi : \mathfrak{a}_1 \to: \mathfrak{a}_2$, with $\phi(X_\alpha) = X_{\tau\alpha}, \phi(Y_\alpha) = Y_{\tau\alpha}, \phi(H_\alpha) = H_{\tau\alpha}$. In every root space $\mathfrak{g}^\alpha$, we choose e_α, s.t. $(e_\alpha, e_\alpha) = 1$ and set $\phi(e_\alpha) = e_{\tau(\alpha)}$ for $\alpha \in \Gamma'_1$. We write $\alpha < \beta$, if there is a $k > 0$, s.t. $\tau^k(\alpha) = \beta$.

Theorem 1 ([1], 6.1) Let r_0 satisfy the conditions above. The tensor

$$r = r_0 + \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha + \sum_{\alpha, \beta > 0, \alpha < \beta} e_{-\alpha} \otimes e_\beta - e_\beta \otimes e_{-\alpha}$$

is a solution of (1). Moreover, any solution of (1) is equivalent (under the action of $\mathrm{Aut}(\mathfrak{g})$) to a solution of this form.

Idea of the proof: first we write $r = \sum_\mu f(I_\mu) \otimes I_\mu$ for some $f : \mathfrak{g} \to \mathfrak{g}$. By a direct calculation, one can check that (1) is equivalent to

$$\begin{array}{c} f + f^* = 1 \\ (f-1)[f(x), f(y)] = f([(f-1)(x), (f-1)(y)]) \end{array} \tag{3}$$

The next step is to use the *Cayley transform* $\theta = \frac{f}{f-1}$. Then the system of equations (3) would imply $\theta\theta^* = \frac{f}{f-1}\frac{f^*}{f^*-1} = 1$ and $\theta[x, y] = [\theta x, \theta y]$. But, as will be shown later, $\det(\theta) = \det(\theta - 1) = 0$ and, therefore, also $\det(f) = \det(f-1) = 0$. This forces us to restrict the domain of θ to $\mathrm{im}(f-1)$. The precise definition of θ is that it is a map $\frac{\mathrm{im}(f-1)}{\ker(f)} \to \frac{\mathrm{im}(f)}{\ker(f-1)}$. We define $C_1 := \mathrm{im}(f-1)$ and $C_2 := \mathrm{im}(f)$. Then ([1], 6.3) we have $C_1^\perp = \ker(f)$ and $C_2^\perp = \ker(f-1)$, also, $\theta\theta^* = 1$ (θ is orthogonal), C_1 and C_2 are subalgebras and θ is a Lie algebra isomorphism. Conversely, if C_1 and C_2 are subalgebras and θ is a Lie algebra isomorphism, then the second equation of (3) holds. In ([1], pages 44-49) it is verified that the triples (C_1, C_2, θ) described above are derived from the triples $(\Gamma_1, \Gamma_2, \tau)$ constructed in the beginning of Section 2.

The detailed proof of this theorem can be found in Chapter 6 of [1]. We conclude this section with an example.

Example 1. [4] Let $\mathfrak{g} = \mathfrak{sl}_2 = \langle X, Y, H \rangle$ and the invariant product is given by the trace form. There is one simple root α and due to condition (b) above, we must set $\Gamma_1 = \Gamma_2 = \emptyset$. We notice that $r_0 = aH \otimes H$. As $\Gamma_1 = \emptyset$, the condition $(\tau\alpha \otimes 1)(r_0) + (1 \otimes \alpha)(r_0) = 0$ is vacuous, the second condition $r_0^{12} + r_0^{21} = t_0$ implies $2aH \otimes H = \frac{1}{2} H \otimes H$, therefore, $a = \frac{1}{4}$. So we come up with $r = \frac{1}{4} H \otimes H + Y \otimes X$

3 CYBE with Spectral Parameter

In this section we describe the classification of solutions of the system

$$[X_{12}(u_1 - u_2), X_{13}(u_1 - u_3)] + [X_{12}(u_1 - u_2), X^{23}(u_2 - u_3)] + [X_{13}(u_1 - u_3), X_{23}(u_2 - u_3)] = 0 \qquad (4)$$

$$X_{12}(u) + X_{21}(-u) = 0$$

(the second equality is called the unitarity condition and is usually imposed), where $X(u)$ takes values in $\mathfrak{g} \otimes \mathfrak{g}$ This system of equations is known as CYBE with spectral parameter. It will be convenient for us to use the expression $X(u) = \sum X^{\mu\nu}(u) I_\mu I_\nu$. We show the following result.

Definition 1 A solution $X(u)$ of (4) is called *nondegenerate* if one of the three equivalent conditions holds (the equivalence is shown in [1], Chapter 10):

(a) the determinant $X^{\mu\nu}(u)$ is not identically 0;

(b) the function $X(u)$ has at least one pole and there is no Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$, s.t. $X(u) \subset \mathfrak{g}' \otimes \mathfrak{g}'$ for all u;

(c) $X(u)$ has a first order pole at $u = 0$ and the residue is equal to λt.

Theorem 2 ([1], 2.0) Suppose $X(u)$ is nondegenerate, the function $X(u)$ satisfies the equation $[X_{12}(u_1 - u_2), X_{13}(u_1 - u_3)] + [X_{12}(u_1 - u_2), X_{23}(u_2 - u_3)] + [X_{13}(u_1 - u_3), X_{23}(u_2 - u_3)] = 0$ and has a first order pole with residue θ at the origin, then $\theta = \lambda t$.

Proof. We make the substitution $u = u_1 - u_2$ and $u = u_2 - u_3$, then the equation becomes

$$[X_{12}(u), X_{13}(u + v)] + [X_{12}(u), X_{23}(v)] + [X_{13}(u + v), X_{23}(v)] = 0 \qquad (5)$$

Multiplying the equation by u and letting u go to zero, we obtain $[\theta^{12}, X^{13}(v)] + [\theta^{12}, X^{23}(v)] = [\theta^{12}, \sum X^{\mu\nu}(v)(I_\mu \otimes 1 \otimes I_\nu + 1 \otimes I_\mu \otimes I_\nu)] = 0$. Now we choose a v with $\det X^{\mu\nu}(v) \neq 0$. Thus for every μ we must have $[\theta, I_\mu \otimes 1 + 1 \otimes I_\mu] = 0$. As t is in $Z(U\mathfrak{g})$, we see that $[t, I_\mu \otimes 1 + 1 \otimes I_\mu] = 0$. We will show that θ is proportional to t. For this we write θ as $\theta = \sum_\nu A(I_\nu) \otimes I_\nu = (A \otimes 1)(t)$ (there exists a linear operator A). Then $[A(I_\mu), I_\nu] = A[I_\mu, I_\nu]$ (this holds since $[t, I_\mu \otimes 1 + 1 \otimes I_\mu] = 0$, implies $(A \otimes 1)[t, I_\mu \otimes 1 + 1 \otimes I_\mu] = \sum_\nu (A([I_\nu, I_\mu] \otimes I_\nu) + A(I_\nu) \otimes [I_\nu, I_\mu]) = 0$ and also since $[\theta, I_\mu \otimes 1 + 1 \otimes I_\mu] = \sum_\nu ([A(I_\nu), I_\mu] \otimes I_\nu + A(I_\nu) \otimes [I_\mu, I_\nu]) = 0$), so for any $x, y \in \mathfrak{g}$

$$[A(x), y] = A([x, y]).$$

Let λ be a nonzero eigenvalue of A, then it follows from the equality above that the elements $\{x \in \mathfrak{g} | A(x) = \lambda x\}$ form an ideal of $\mathfrak{g}$, which must coincide with $\mathfrak{g}$ as it is a simple Lie algebra. □

The next result is as follows.

Theorem 3 ([1], 10.1) Assume that $X(u)$ is a solution of (4), defined on a small circle $U \subset \mathbb{C}$, s.t. $X(u)$ has at least one pole and there is no Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$, s.t. $X(u) \in \mathfrak{g}' \otimes \mathfrak{g}$ for any u. Then all the poles of $X(u)$ are simple, there is a pole at 0 with residue λt.

Proof. We assume that $X(u)$ has a pole of order k at γ, and set $\tau := \lim_{u \to \gamma} (u-\gamma)^k X(u)$. Multiplying both sides of (5) by $(v-\gamma)^k$ and taking v to γ, we arrive with

$$[X_{12}(u), \tau_{23}] + [X_{13}(u+\gamma), \tau_{23}(v)] = 0 \tag{6}$$

Similarly (multiplying both sides of (5) by $(u-\gamma)^k$ and taking u to γ), we obtain

$$[\tau_{12}, X_{13}(v+\gamma)] + [\tau_{12}, X_{23}(v)] = 0 \tag{7}$$

Expanding (7) around $v = 0$, we see that $X(v)$ must have a pole of order k at zero, as otherwise $[\tau_{12}, \tau_{13}] = 0$, which contradicts Lemma 1 (see the Appendix).

The next step is to show that the order of the pole at zero is at most 1 and $\lim_{u \to 0} uX(u) = \lambda t$. For these we write $X(u) = \frac{\theta}{u^l} + \frac{\mu}{u^{l-1}} + \sum_{i \geq 2-l} c_i x^i$, where $\theta \neq 0$. Now we take a closer look at the poles of $X(u)$. Fixing v, we find that the coefficient of u^{1-l} in the expansion of (4) around $u = 0$ is $[\mu_{12}, X_{13}(u) + X_{23}(v)] + [\theta^{12}, \frac{dX_{13}(v)}{dv}] = 0$ (here v is not a pole of $X(u)$). Considering the coefficient of v^{-l-1} in the expansion around $v = 0$, the equality becomes $[\theta^{12}, \theta^{23}] = 0$, which is impossible due to Lemma 1. Equations (6) and (7) imply $[X_{12}(u) + X_{13}(u), \theta_{23}] = 0$ and $[\theta_{12}, X_{13}(u) + X_{23}(u)] = 0$.

We introduce the Lie subalgebra $\{x \in \mathfrak{g} | [x \otimes 1 + 1 \otimes x, \theta] = 0\} =: \mathfrak{g}' \subset \mathfrak{g}$. It follows from (6) and (7) that $X(u) \in \mathfrak{g}' \otimes \mathfrak{g}'$ and, therefore, $\mathfrak{g}' = \mathfrak{g}$. So $[x \otimes 1 + 1 \otimes x, \theta] = 0$ for every $x \in \mathfrak{g}$. It follows that θ must be proportional to t. □

Theorem 4 ([1], 2.1) Let $X(u)$ be a nondegenerate solution of (4) defined in some disc $U \subset \mathbb{C}$ with $\lim_{u \to 0} uX(u) = t$. Then $X(u)$ satisfies the unitarity condition, i.e. $X_{12}(u) = -X_{21}(-u)$.

Proof. As $X(u)$ is a solution of CYBE, we have

$$[X_{12}(u_1-u_2), X_{13}(u_1-u_3)] + [X_{12}(u_1-u_2), X_{23}(u_2-u_3)] + [X_{13}(u_1-u_3), X_{23}(u_2-u_3)] = 0 \tag{8}$$

Interchanging u_1 with u_2 and the first two factors in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$, we also have

$$[X_{21}(u_2-u_1), X^{23}(u_2-u_3)] + [X_{21}(u_2-u_1), X_{13}(u_1-u_3)] + [X_{23}(u_2-u_3), X_{13}(u_1-u_3)] = 0 \tag{9}$$

and adding (9) to (8) gives

$$[X_{12}(u_1-u_2) + X_{21}(u_2-u_1), X_{13}(u_1-u_3) + X_{23}(u_2-u_3)] = 0.$$

Multiplying the equation above by $u_2 - u_3$ and considering $u_3 \to u_2$ with u_1 and u_2 fixed, we come up with

$$[X_{12}(u_1 - u_2) + X_{21}(u_2 - u_1), t_{23}] = 0.$$

As $t = \sum_\mu I_\mu \otimes I_\mu$, this implies

$$[X_{12}(u_1 - u_2) + X_{21}(u_2 - u_1), 1 \otimes I_\mu] = 0$$

for all I_μ.

We can write $X_{12}(u_1 - u_2) + X_{21}(u_2 - u_1) = \sum_\nu I_\nu \otimes X_\nu(u_1 - u_2)$, then the equality above gives for every ν, μ

$$[I_\nu \otimes X_\nu(u_1 - u_2), 1 \otimes I_\mu] = 0.$$

As $\mathfrak{g}$ is simple, it follows that each $X_\nu(u_1 - u_2) = 0$ therefore, $X_{12}(u_1 - u_2) + X_{21}(u_2 - u_1) = 0.$ □

We sketch the proof of the fact that $X(u+v)$ is a rational function of $X(u)$ and $X(v)$ (see [1], Theorem 2.2). This has an important corollary that $X(u)$ can be extended to a meromorphic function on $\mathbb{C}$.

Proof. We consider (5) as an inhomogeneous system of linear equations with $X(u)$ and $X(v)$ as coefficients. Then the corresponding homogeneous system is

$$[X_{12}(u) - X_{23}(v), X_{13}] = 0$$

and for the solution of the inhomogeneous system to be expressed as a rational function of coefficients, we need the homogeneous system to be nondegenerate (have only the trivial solution) for generic u, v in the neighborhood of 0. Considering $u = v \neq 0$, multiplying by u and letting $u \to 0$ turns the homogeneous system of equations above into $[uX_{12}(u) - uX_{23}(v), X_{13}] = [t_{12} - t_{23}, X_{13}] = 0$, which is equivalent to

$$[g \otimes 1 - 1 \otimes g, X] = 0 \quad \forall g \in \mathfrak{g}.$$

But then

$$[[g_1, g_2] \otimes 1 + 1 \otimes [g_1, g_2], X] = [[g_1 \otimes 1 - 1 \otimes g_1, g_2 \otimes 1 - 1 \otimes g_2], X] = 0,$$

where for the last equality we used that $[g \otimes 1 - 1 \otimes g, X] = 0 \;\; \forall g \in \mathfrak{g}$ and the Jacobi identity. As $\mathfrak{g}$ is simple (in particular, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$), the equalities above imply $[1 \otimes g, X] = 0, [g \otimes 1, X] = 0 \;\; \forall g \in \mathfrak{g}$, so $X = 0$.

To conclude the proof we use that the nondegeneracy of the homogeneous system of equations is equivalent to nonvanishing of certain minors, which are meromorphic functions in u and v. Thus, nondegeneracy is an open condition, and we can find a neighborhood of zero, where it holds. □

The set of poles of $X(u)$ will be denoted by Γ. As shown in Theorem 3 above, it consists of simple poles. The next result allows to enrich Γ with a group structure.

Theorem 5 ([1], 2.3) For every $\gamma \in \Gamma$ there exists an $A_\gamma \in \mathrm{Aut}(\mathfrak{g})$, s.t. $X(u+\gamma) = (A_\gamma \otimes 1)(X(u))$.

Proof. Again we set $\tau := \lim_{u\to\gamma}(u-\gamma)X(u)$ and define $A_\gamma : \mathfrak{g} \to \mathfrak{g}$ by

$$\tau = \sum_\mu A_\gamma(I_\mu) \otimes I_\mu = \sum_{\mu,\nu} (A_\gamma)_{\mu,\nu}(I_\nu) \otimes I_\mu.$$

If we multiply by $u-\gamma$ and let $u \to \gamma$, (5) becomes (7). From (7), using that $[t_{12}, r_{13} + r_{23}] = 0$ for any $r \in \mathfrak{g} \otimes \mathfrak{g}$, we derive

$$[\tau_{12}, X_{13}(v+\gamma)] = -(A_\gamma \otimes 1 \otimes 1)([t_{12}, X_{23}(v)]) = (A_\gamma \otimes 1 \otimes 1)([t_{12}, X_{13}(v)]) \tag{10}$$

The residues of both sides of (10) for $v = 0$, give the equality

$$[\tau_{12}, \tau_{13}] = (A_\gamma \otimes 1 \otimes 1)([t_{12}, t_{13}]),$$

which (using the definition of A_γ) can be rewritten as

$$\sum_{\mu,\nu} [A_\gamma(I_\mu), A_\gamma(I_\nu)] \otimes I_\mu \otimes I_\nu = \sum_{\mu,\nu} A_\gamma [I_\mu, I_\nu] \otimes I_\mu \otimes I_\nu.$$

It follows that A_γ is a Lie algebra homomorphism. As the kernel of A_γ would be an ideal of $\mathfrak{g}$, which is impossible, since the latter is simple. Therefore, A_γ is invertible, i.e. $A_\gamma \in \mathrm{Aut}(\mathfrak{g})$. Applying $(A_\gamma^{-1} \otimes 1 \otimes 1)$ to both sides of (10), we get

$$[t^{12}, (A_\gamma^{-1} \otimes 1)(X^{13}(v+\gamma)) - X^{13}(v)] = 0.$$

It follows that $(A_\gamma^{-1} \otimes 1)(X_{13}(v+\gamma)) = X_{13}(v)$, therefore, $X_{13}(v+\gamma) = (A_\gamma \otimes 1)X_{13}(v)$ and, finally,

$$X(v+\gamma) = (A_\gamma \otimes 1)X(v) \tag{11}$$

□

One immediate corollary of Theorem 5 is that if $\gamma, \gamma' \in \Gamma$ are poles of $X(u)$, so is $\gamma + \gamma'$. Indeed, the r.h.s of (11) has a pole at γ', so the l.h.s must have one as well. It is not hard to see that $A_{\gamma+\gamma'} = A_\gamma A_{\gamma'}$. Also, from unitarity of $X(u)$ we see that $\gamma \in \Gamma$ implies $-\gamma \in \Gamma$. So we have that $\Gamma \subset \mathbb{C}$ is a discrete subgroup. Such subgroups are lattices of rank $0, 1$ or 2. The next theorem shows, that in case the rank is equal to two, $X(u)$ is an elliptic function, i.e. double-periodic. Later, in Section 6, we will show that this happens only for $\mathfrak{g} = \mathfrak{sl}_n$. The other two cases ($\mathrm{rk}\,\Gamma = 0$ and $\mathrm{rk}\,\Gamma = 1$) correspond to rational and trigonometric solutions.

Theorem 6 ([1], 2.5) Let $\mathrm{rk}\,\Gamma = 2$, then there is no $a \in \mathfrak{g}$, s.t. $A_\gamma(a) = a$ for all $\gamma \in \Gamma$. Moreover, for any $\gamma \in \Gamma\ \ \exists n : A_\gamma^n = 1$.

Proof. Assume the first assertion does not hold, i.e. $\exists a \in \mathfrak{g}$, s.t. $A_\gamma(a) = a$ for all $\gamma \in \Gamma$. We define the meromorphic $\mathfrak{g}$-valued function $\phi(u) = \sum_{\mu,\nu} X^{\mu,\nu}(u)(I_\mu, a)I_\nu$ (here $(,)$ stands for the Killing form and $(A_\gamma v, w) = (v, A_\gamma w)$, for $A_\gamma \in \mathrm{Aut}(\mathfrak{g})$ and $v, w \in \mathfrak{g}$). It is easy to see that $\phi(u+\gamma) = \phi(u)$ for any $\gamma \in \Gamma$. Also, $\phi(u)$ has a simple pole at zero, as $X(u)$ does. We can choose the parallelogram P of periods in such a way that zero is the only pole of $\phi(u)$ in the closure of P. On the one hand $\frac{1}{2\pi i}\int_{\partial P} \phi(u)du = \mathrm{Res}_0\phi(u)$, on the other, it is zero, since the integrals over opposite sides of ∂P cancel each other - a contradiction.

The second assertion follows from the fact (see [1], Theorem 9.1) that if $H \subset \mathrm{Aut}(\mathfrak{g})$ is an infinite abelian subgroup, there exists $a \in \mathfrak{g} : ha = a \ \ \forall h \in H$. $\square$

4 Rational Solutions (Examples)

In this section we give examples of rational solutions, which correspond to r-matrices for the so-called *Yangian Lie bialgebras*. We follow Lecture 6 of [3].

We denote by $\mathfrak{g}_0$ a finite-dimensional Lie algebra with fixed nondegenerate invariant bilinear form $(\ ,\ )$ and set $\mathfrak{g} := \mathfrak{g}_0((v^{-1}))$, $\mathfrak{g}_- := v^{-1}\mathfrak{g}_0[[v^{-1}]]$ and $\mathfrak{g}_+ := \mathfrak{g}_0[v]$. Thus, $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Next, we equip $\mathfrak{g}$ with the nondegenerate invariant bilinear form, defined by

$$\langle a(v), b(v)\rangle := \mathrm{Res}_{v=0}(a(v), b(v)).$$

The subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic, moreover,

$$\mathfrak{g}_+^* = (\bigoplus_{n\geq 0}\mathfrak{g}_0 v^n)^* = \prod_{n\geq 0}\mathfrak{g}_0 v^{-n-1} = \mathfrak{g}_-.$$

This shows that $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple and, therefore ([4], Proposition 1.3.4) $\mathfrak{g}_0[v]$ is a Lie bialgebra, called the Yangian Lie bialgebra.

We choose an orthonormal basis (x_i) of $\mathfrak{g}_0$. The cocommutator is given by the formula

$$\delta(av^n) = \sum_{0\leq r\leq n-1}\sum_j [x_j, a]v^r \otimes x_j v^{n-1-r}.$$

We use different variables v and u to distinguish between the subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$. The r-matrix is given by

$$r_{\mathfrak{g}} = \sum_{i,n\geq 0} x_i v^n \otimes x_i u^{-n-1} = \frac{\sum_i x_i \otimes x_i}{u-v} = \frac{t}{u-v},$$

where we used the expansion $\frac{1}{u-v} = \sum_{n\geq 0} v^n u^{-n-1}$ in the region $|v| < |u|$.

Dually, we can start with the Manin triple $(\mathfrak{g} = \mathfrak{g}_0((v)), \mathfrak{g}_+ = v^{-1}\mathfrak{g}_0[v^{-1}], \mathfrak{g}_- = \mathfrak{g}_0[[v]])$. The corresponding cocommutator and r-matrix are

$$\delta(av^n) = \sum_{1\leq r\leq n}\sum_j [x_j, a]v^{-r} \otimes x_j v^{r-n-1};$$

$$r_{\mathfrak{g}} = \sum_{i,n\geq 0} x_i v^{-n-1} \otimes x_i u^n = \frac{t}{v-u},$$

this type we used the expansion $\frac{1}{v-u} = \sum_{n\geq 0} u^n v^{-n-1}$ in the region $|v| > |u|$.

Here $r_{\mathfrak{g}}(u_1 - u_2) = \frac{t}{u_1 - u_2}$ is a rational solution of CYBE with spectral parameter.

5 Trigonometric Solutions

Suppose that A is an automorphism of $\mathfrak{g}$, s.t. $(A \otimes 1)X(u) = X(u + 2\pi i)$. We denote by $\sigma \in \mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ the automorphism of the Dynkin diagram $\triangle$ of $\mathfrak{g}$, determined by A.

Definition 2 A finite order automorphism A of $\mathfrak{g}$ in the coset $\sigma\mathrm{Inn}(\mathfrak{g})$ is called a *Coxeter Automorphism* of the pair $(\mathfrak{g}, \sigma)$ if

(a) its fixed point subalgebra is abelian;

(b) A has the minimal order among the elements from the coset $\sigma\mathrm{Inn}(\mathfrak{g})$, which satisfy (a).

Definition 3 The order h_σ of the Coxeter element A is called the *Coxeter number* of the pair $(\mathfrak{g}, \sigma)$.

We denote $\epsilon = e^{\frac{2\pi i}{h_\sigma - 1}}$ and by $\mathfrak{g}_{\sigma,j}$ - the ϵ^j-eigenspace for the action of A on $\mathfrak{g}$. So, we have the direct sum decomposition:

$$\mathfrak{g} = \bigoplus_{j=0}^{h_\sigma - 1} \mathfrak{g}_{\sigma,j}.$$

The abelian subalgebra $\mathfrak{g}_{\sigma,0}$ should be thought of as an analogue of the Cartan subalgebra, so we denote it by $\mathfrak{h}_\sigma$, also, $t_{\sigma,j}$ stands for the projection of the Casimir element t on the $\mathfrak{g}_{\sigma,j} \otimes \mathfrak{g}_{\sigma,-j}$ - component of $\mathfrak{g} \otimes \mathfrak{g}$. So we can write $t = \sum_{j=0}^{h_\sigma - 1} t_{\sigma,j}$.

For any $\alpha \in \mathfrak{h}_\sigma^*$, we denote

$$\mathfrak{g}_{\sigma,j}^\alpha = \{x \in \mathfrak{g}_{\sigma,j} | [h, x] = \alpha(h)x \quad \forall h \in \mathfrak{h}_\sigma\}.$$

Then $\dim(\mathfrak{g}_{\sigma,j}^\alpha) \leq 1$ for all $\alpha \neq 0$ and we define $\prod_\sigma = \{\alpha \in \mathfrak{h}_\sigma^* | \mathfrak{g}_{\sigma,1}^\alpha \neq 0\}$ (in particular, $0 \notin \prod_\sigma$). The elements of $\prod_\sigma$ are called *simple weights*. They are not linearly independent, but satisfy a single linear relation with positive integer coefficients.

As in the theory of simple Lie algebras, we can associate a Dynkin diagram to the pair $(\mathfrak{g}, \sigma)$ - the vertices correspond to simple weights and the number of edges joining α and β is equal to $\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)}$, if $(\alpha, \alpha) > (\beta, \beta)$, then the edge is oriented from the longer root to the shorter one.

We define the linear operator $\widetilde{\theta} : \mathfrak{g} \to \mathfrak{g}$ by $\widetilde{\theta}(x) = \theta(P(x))$, where P is the unique projector $\mathfrak{g} \to \mathfrak{a}_1$, s.t. $P(\mathfrak{g}_{\sigma,j}^\alpha) = 0$, if $\mathfrak{g}_{\sigma,j}^\alpha \not\subset \mathfrak{a}_1$ and $\theta : \mathfrak{a}_1 \to \mathfrak{a}_2$ is the isomorphism, described in Section 2 and denoted by ϕ therein. It follows from the definition of an admissible triple that $\widetilde{\theta}$ is nilpotent, so it makes to define $\psi := \frac{\widetilde{\theta}}{1 - \widetilde{\theta}} = \widetilde{\theta} + \widetilde{\theta}^2 + \ldots$

Theorem 7 ([1], 7.3) Suppose that $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ satisfies the system of equations (2). Then the function

$$X(u) = r_0 + \frac{1}{e^u - 1} \sum_{j=0}^{h_\sigma - 1} e^{ju/h_\sigma - 1} t_j - \sum_{j=1}^{h_\sigma - 1} e^{ju/h_\sigma} (\psi \otimes 1) t_j + \sum_{j=1}^{h_\sigma - 1} e^{-ju/h_\sigma} (1 \otimes \psi) t_{-j} \tag{12}$$

is a solution of the triangle system (5) with the set of poles $\Gamma = 2\pi i \mathbb{Z}$ and residue t at the origin. Also,

$$X(u + 2\pi i) = (A \otimes 1) X(u).$$

Moreover, every trigonometric solution of (2) with the set of poles $\Gamma = 2\pi i \mathbb{Z}$, corresponding to an automorphism $\sigma \in \mathrm{Aut}(\triangle)$, and residue t at the origin is equivalent to a solution of the form (12).

Example 2. We consider the Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, which is $(\mathfrak{g}[v, v^{-1}] \oplus \mathfrak{h}, v\mathfrak{g}[v] \oplus \mathfrak{b}_+, v^{-1}\mathfrak{g}[v^{-1}] \oplus \mathfrak{b}_-)$ with $\mathfrak{g}$ a simple Lie algebra. As in the examples of Section 4, the inner product is given by $\langle a(v), b(v) \rangle$ the constant term of $(a(v), b(v))$. It is not hard to see, that in this case $r = \frac{t}{1 - \frac{v}{u}} = \frac{t}{1 - e^z}$, where $\frac{v}{u} = e^z$ (here to distinguish between $\mathfrak{g}_+$ and $\mathfrak{g}_-$, we use the variable u for $\mathfrak{g}_-$).

6 Elliptic Solutions

Theorem 8 ([1], 3.3). Let A_1, A_2 be commuting automorphisms of $\mathfrak{g}$ of finite order, s.t. there exists no $a \in \mathfrak{g}$ fixed by both A_1 and A_2. Then there exists an isomorphism $\mathfrak{g} \simeq \mathfrak{sl}_n$, under which A_1 and A_2 are inner automorphisms, corresponding to T_1 and T_2 defined below, i.e. $A_i(I_\mu) = T_i^{-1} I_\mu T_i$

Proof. We remind that A_1, A_2 are automorphisms of finite order. Now we show that any automorphism of finite order must fix some $x \in \mathfrak{g}$. Indeed, assume this is not the case and decompose $\mathfrak{g} = \bigoplus_{j=0}^{k-1} \mathfrak{g}_j$, where k is the order of the automorphism and $\mathfrak{g}_j$ is the $e^{\frac{2\pi i j}{k}}$-eigenspace. Then $[\mathfrak{g}_j, \mathfrak{g}_l] \subset \mathfrak{g}_{j+l}$ and it follows that the operator $ad(y)$ for $y \in \mathfrak{g}_j$ is nilpotent. Using the Jacobson-Morozov theorem, we complete y to an $\mathfrak{sl}_2$-triple, in particular, find $h \in \mathfrak{g}$, s.t. $[h, y] = 2y$, which implies that $[h, y] \in \mathfrak{g}_j$, thus, $h \in \mathfrak{g}_0$-a contradiction. So (slightly abusing notation) we set $g_0 = \{x \in \mathfrak{g} | A_1(x) = x\}$, and by the previous argument this is not empty. As A_1 and A_2 commute, A_2 preserves $\mathfrak{g}_o$ and there is no nonzero $a \in \mathfrak{g}_o$ fixed by A_2. Since any automorphism of finite order of a semisimple Lie algebra must have a fixed vector (this assertions can be proved using the argument above), it follows that $\mathfrak{g}_0$ is solvable. Also, Lemma 1 in [2] shows that $\mathfrak{g}_0$ is reductive. Being both, it must be abelian, as the adjoint representation is completely reducible (due to $\mathfrak{g}_0$ is reductive), but $[\mathfrak{g}_0, \mathfrak{g}_0]$ as $\mathfrak{g}_0$ is solvable.

It follows from the definition of the Dynkin diagram $\triangle$, associated to the pair $(\mathfrak{g}, A_1)$, that A_2 induces an automorphism of $\triangle$. Next, we show that the action of the cyclic subgroup $\langle A_2 \rangle \subset \mathrm{Aut}(\triangle)$ is transitive. Assume the contrary, so there are two subsets $S_1, S_2 \subset \mathrm{vertices}(\triangle), S_1 \cap S_2 = \emptyset, S_1, S_2 \neq \emptyset$, both preserved by A_2. The results of [2] imply that there is a single linear relation $\sum_{\delta_i \in \mathrm{vertices}(\triangle)} n_i h_{\delta_i} = 0$, but $A_2(\sum_{\delta_i \in S_1} l_i h_{\delta_i}) = \sum_{\delta_i \in S_1} l_i h_{\delta_i}$ and $A_2(\sum_{\delta_i \in S_2} m_i h_{\delta_i}) = \sum_{\delta_i \in S_2} m_i h_{\delta_i}$ implies $\sum_{\delta_i \in S_1} l_i h_{\delta_i} = \sum_{\delta_i \in S_2} m_i h_{\delta_i} = 0$ - two linear relations - a contradiction.

From the explicit classification of diagrams $\triangle$, associated to $(\mathfrak{g}, \sigma)$ given in [2], it follows that in our case $\triangle \simeq \widetilde{A_{n-1}}$ and $\mathfrak{g} \simeq \mathfrak{sl}_n$. Then one can show that A_1, A_2 are inner automorphisms and correspond to the matrices T_1, T_2 given below (see the discussion on pages 68-69, [1]):

$$T_1 = \begin{pmatrix} \xi & 0 & 0 & \dots & 0 \\ 0 & \xi^2 & 0 & \dots & 0 \\ 0 & 0 & \xi^3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \xi^{n-1} \end{pmatrix}, \; T_2 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

□

The classification of elliptic solutions of CYBE follows from the following theorem (the proof uses techniques, similar to those, appeared before and is skipped, [1], 3.2 being the reference).

Theorem 9 ([1], 3.2). Let A_1, A_2 be commuting automorphisms of $\mathfrak{g}$ with no common fixed nonzero eigenvectors. Then there is a unique meromorphic solution of (4) $X(u) : \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g}$, such that

(1) $\lim_{u \to 0} uX(u) = t$;

(2) $X(u + w_i) = (A_i \otimes 1)X(u), i = 1, 2$;

(3) $X(u)$ has no poles for $u \notin \Gamma$.

7 Appendix

Lemma 1 $[\tau^{12}, \tau^{13}] \neq 0$

Proof. We denote by $V \subset \mathfrak{g}$ the smallest vector subspace, s.t. $\tau \in V \otimes \mathfrak{g}$ and by $\mathfrak{g}' := \{x \in \mathfrak{g} | [x, V] \subset V\}$. It is clear that $\mathfrak{g}'$ is a subalgebra. As $[X^{13}(u + \gamma), \tau^{23}(v)] \in \mathfrak{g} \otimes V \otimes \mathfrak{g}$, from (6) it follows that $[X^{12}(u), \tau^{23}]$ is also in $\mathfrak{g} \otimes V \otimes \mathfrak{g}$, so $X(u) \in \mathfrak{g} \otimes \mathfrak{g}'$. Analogously, using (6), we verify that $X^{13}(v + \gamma) \in \mathfrak{g}' \otimes \mathfrak{g}$ for any v. Thus, $X(u) \in \mathfrak{g}' \otimes \mathfrak{g}'$ and $\mathfrak{g}'$ coincides with $\mathfrak{g}$, so $[\mathfrak{g}, V] \subset V$ and we must have $\mathfrak{g} = V$ as $\mathfrak{g}$ is simple. The assertion follows. □

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