SHUFFLE ALGEBRA VS EHA

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Abstract. These are the notes for a talk at the MIT-Northeastern seminar for graduate students on Double Affine Hecke Algebras and Elliptic Hall Algebras, Spring 2017.

Contents

Date:

1. Introduction

The shuffle algebras were introduced by Feigin and Odesskii. These algebras are unital associative subalgebras of $\mathbb{C} \bigoplus_{n\in\mathbb{Z}_{>0}} \mathbb{C}(z_1,\dots,z_n)^{\mathfrak{S}_n}$ with multiplication defined by

$$f(z_1,\dots,z_n) * g(z_1,\dots,z_m) := \mathrm{Sym}\left(f(z_1,\dots,z_n) g(z_{n+1},\dots,z_{n+m}) \prod_{\substack{i\in\{1,\dots n\}\\ j\in\{n+1,\dots n+m\}}} \mu\left(\frac{z_i}{z_j}\right)\right),$$

for some function μ. In the work of Schiffmann and Vasserot [SV] it was shown that subalgebras in Hall algebras of vector bundles of smooth projective curves generated by $\mathbf{1}_{Pic^d(X)} := \sum_{\mathcal{L}\in Pic^d(X)} \mathcal{L}$, where $Pic^d(X)$ is the set of line bundles over X of degree d are isomorphic to subalgebras of S, generated by elements of degree 1. For a smooth projective curve of genus g, one takes $\mu_g(x) = x^{g-1}\frac{1-qx}{1-x^{-1}}\prod_{i=1}^{g}(1-\alpha_i x^{-1})(1-\overline{\alpha_i}x^{-1})$, where α_i are the roots of the numerator of the zeta function of the curve, i.e. $\zeta_X(t) = \exp(\sum_{d\geq 1} \#X(\mathbb{F}_{q^d})\frac{t^d}{d}) = \frac{\prod_{i=1}^{g}(x-\alpha_i)(x-\overline{\alpha_i})}{(1-t)(1-qt)}$. This isomorphism was made more explicit in case of elliptic curves (elliptic Hall algebras) in [Neg]. The latter paper will be the primary reference for this talk.

. The action of shuffle algebra on the sum of localized equivariant K-groups (with respect to the $T = \mathbb{C}^* \times \mathbb{C}^*$ induced from the action on $\mathbb{C}^2$) of Hilbert schemes of points on $\mathbb{C}^2$ was provided in [FT].

In Section 2 we show that the Hall algebra of locally free sheaves (vector bundles) on projective line is isomorphic to shuffle algebra with $\mu(x) = x^{-1}\frac{1-qx}{1-x^{-1}}$ and formulate the general isomorphism of Schiffmann and Vasserot.

Section 3 recalls the definition and basic properties of the elliptic Hall algebra (EHA) and Sections 4 and 5 are devoted to speculations on the isomorphism of EHA and the corresponding shuffle algebra and their Drinfeld doubles.

2. Shuffle Algebras vs Hall Algebras of Genus g Curves

2.1. **Hall Algebra of the Projective Line.** The goal of this section is to show that the Hall algebra of vector bundles on $\mathbb{P}^1(\mathbb{F}_q)$ is isomorphic to the shuffle algebra S_q with the function $\mu(x) = \frac{1-qx}{x-1}$.

Proposition 2.1. *The shuffle algebra S_q is generated by $\bigoplus_{d\in\mathbb{Z}} S_{d,1}$, the ideal of relations is generated by the following relations of degree 2 (in $\bigoplus_{d\in\mathbb{Z}} S_{d,2}$)*

$$z^{m+1} * z^n - qz^n * z^{m+1} = qz^m * z^{n+1} - z^{n+1} * z^m \tag{2.1}$$

Proof. We notice that the vector space generated by monomials $z^{i_1} * \ldots * z^{i_d}, i_j \in \mathbb{Z}$ forms an ideal inside $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{\mathfrak{S}_n}$. Indeed, $f(z_1, \ldots, z_n) z^{\alpha_1} * \ldots * z^{\alpha_n} = z^{\alpha_1 + i_1} * \ldots * z^{\alpha_n + i_n}$, where $f(z_1, \ldots, z_n) = Sym(z^{i_1} * \ldots * z^{i_n})$ is in $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{\mathfrak{S}_n}$.

The next step is to show that the ideal $\underbrace{\bigoplus_{d \in \mathbb{Z}} S_{d,1} * \ldots * \bigoplus_{d \in \mathbb{Z}} S_{d,1}}_{n}$ has no common zeros and therefore must coincide with S_n (we use that $S_n = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{\mathfrak{S}_n}$ is finitely generated over $\mathbb{C}$ and, therefore, any maximal ideal must be vanishing at a point). Suppose that all functions from the ideal vanished at a point with coordinates $(\alpha_1, \ldots, \alpha_n)$. It is not hard to show that this implies that there exist $\alpha_{j_1} = q\alpha_{j_2} = \ldots = q\alpha_{j_k} = q\alpha_{j_1}$, i.e. $\alpha_{j_1} = q^k \alpha_{j_1}$. But neither $\alpha_{j_1} = 0$ nor $q^k = 1$. This completes the proof of the first claim.

. **Exercise.** Alternatively, show that $\underbrace{1 * 1 * \ldots * 1}_{n} = c \prod_{i=1}^{n} \frac{q^i - 1}{q - 1} = c[n]_q!$, where $c \in \mathbb{C}$ is a constant. This implies that $1 \in \underbrace{\bigoplus_{d \in \mathbb{Z}} S_{d,1} * \ldots * \bigoplus_{d \in \mathbb{Z}} S_{d,1}}_{n}$, unless q is kth root of unity ($k \leq n$).[1]

The relations (2.1) can be checked directly. They allow to rewrite every monomial $z^{i_1} * \ldots * z^{i_k}$ in such a way that $i_{m+1} \leq i_m + 1 \ \forall m \in \{1, \ldots, k\}$. Indeed, if $i_{m+1} > i_m + 1$, then using (2.1) for $z^{i_m} * z^{i_{m+1}}$, we see that the other three summands have the difference $i'_{m+1} - i'_m - 1$ strictly less than $i_{m+1} - i_m - 1$. In case $i_{m+1} = i_m + 1$, (2.1) becomes

$$q z^{i_m + 1} * z^{i_m} - z^{i_m} * z^{i_m + 1} + q z^{i_m + 1} * z^{i_m} - z^{i_m} * z^{i_m + 1} = 0$$

and allows to swap the two factors. This implies that there no relations, other then those generated by (2.1), as otherwise taking the limit $q \to 1$ we would obtain relations between monomial symmetric Laurent polynomials, which are known to be independent. ☐

To describe the Hall algebra $\mathcal{H}_{lf}(\mathbb{P}^1)$ of vector bundles on $\mathbb{P}^1(\mathbb{F}_q)$, we first recall the Grothendieck theorem: every vector bundle V of rank k on $\mathbb{P}^1$ splits as a sum of line bundles $V = \bigoplus_{j=1}^{n} \mathcal{O}(i_j)$. The result holds over fields of $char = p$ as well. Therefore, the only indecomposable objects are the line bundles $\mathcal{O}(i)$.

Observation. If $m \leq n + 1$, then $\mathrm{Ext}^1(\mathcal{O}(m), \mathcal{O}(n)) = 0$.

Proof. Using that $\omega_{\mathbb{P}^1} = \mathcal{O}(-2)$ and Serre duality, we conclude $\mathrm{Ext}^1(\mathcal{O}(m), \mathcal{O}(n))^* = \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(m - 2))$, which is zero for $m \leq n + 1$. ☐

Definition. We introduce $\nu = \sqrt{q}$ and define the Euler form to be $\langle M, N \rangle = \dim(\mathrm{Hom}(M, N)) - \dim(\mathrm{Ext}^1(M, N))$. The product of two elements $[\mathcal{O}(n)]$ and $[\mathcal{O}(m)] \in \mathcal{H}_{lf}(\mathbb{P}^1)$ is defined to

[1] **Hint**: the degree of the polynomial is zero, hence, it is enough to evaluate it at a single point. One convenient choice is the point $(\xi, \xi^2, \ldots, \xi^{n-1}, 1)$, where $\xi = \sqrt[n]{1}$ is the primitive root of unity (use induction on n). Also, compare to formula (i) in Theorem 10 of [BK].

be $[\mathcal{O}(n)] * [\mathcal{O}(m)] := \nu^{\langle M,N\rangle} \sum_R P^R_{\mathcal{O}(n),\mathcal{O}(m)}[R]$, where R is a vector bundle of rank 2 and $P^R_{\mathcal{O}(n),\mathcal{O}(m)} = \frac{L^R_{\mathcal{O}(n),\mathcal{O}(m)}}{|\mathrm{Aut}\mathcal{O}(m)||\mathrm{Aut}\mathcal{O}(n)|}$ with $L^R_{\mathcal{O}(n),\mathcal{O}(m)}$ equal to the number of SES

$$0 \to \mathcal{O}(m) \to R \to \mathcal{O}(n) \to 0.$$

The next lemma shows computations of some structure constants in $\mathcal{H}_{lf}(\mathbb{P}^1)$ (see also **Theorem 10** in [BK]).

Lemma 2.2. *The following relations hold in $\mathcal{H}_{lf}(\mathbb{P}^1)$:*

(2.2)
$$[\mathcal{O}(n)]*[\mathcal{O}(m)] = \nu^{m-n}\left(q^{n-m+1}([\mathcal{O}(m)\oplus\mathcal{O}(n)] + \sum_{s=1}^{\lfloor\frac{n-m}{2}\rfloor}(q^2-1)q^{n-m-1}[\mathcal{O}(m+s)\oplus\mathcal{O}(n-s)])\right),\ n>m$$

Proof. We notice that for a nontrivial extension

$$0 \to \mathcal{O}(m) \to \mathcal{O}(p)\oplus\mathcal{O}(q) \to \mathcal{O}(n) \to 0$$

to exist, we must have $\min(p,q) > m$, $\max(p,q) < n$ and $p+q = m+n$, these are precisely the summands of the sum in the r.h.s of (2.2). To compute the coefficient with which $[\mathcal{O}(m+s)\oplus\mathcal{O}(n-s)]$ appears in the product, we notice that it is equal to the number of pairs of coprime polynomials of degrees s and $n-m-s$. Indeed a pair of such polynomials (φ_1,φ_2) defines a map $\psi : 0 \to \mathcal{O}(m) \to \mathcal{O}(m+s)\oplus\mathcal{O}(n-s)$ and $\mathrm{coker}(\psi)$ is locally free (has trivial support) whenever (φ_1,φ_2) are coprime. The number of such pairs of polynomials is computed in the next proposition, which completes the proof of the lemma (one also needs to use that $|\mathrm{Aut}\mathcal{O}(n)| = q-1\ \forall n\in\mathbb{Z}$). □

Proposition 2.3. *The number $\eta(a,b)$ of pairs (J,L), consisting of coprime homogeneous polynomials in $\mathbb{F}_q[x,y]$ of degrees a and b, respectively, is given by*

$$\begin{cases}\eta(a,b) = (q-1)(q^{a+b+1}-1),\ a=0 \text{ or } b=0\\ \eta(a,b) = (q-1)(q^2-1)q^{a+b-1},\ a\geq 1 \text{ and } b\geq 1\end{cases} \tag{2.3}$$

Proof. The first assertion follows from the fact that the space of homogeneous polynomials of degree s in two variables is of dimension $s+1$ (the number of nontrivial linear combinations of vectors from the basis is $q^{s+1}-1$ and the other polynomialal is a nonzero constant).

The second claim is verified by induction on $\min(a,b)$, using that the number of all possible pairs of polynomials in $\mathbb{F}_q[x,y]$ of degrees a and b can be expressed as

$$(q^{a+1}-1)(q^{b+1}-1) = \sum_{d=0}^{\min(a,b)} \frac{q^{d+1}-1}{q-1}\eta(a-d,b-d).$$

□

Corollary 2.4. *The following relations hold in $\mathcal{H}_{lf}(\mathbb{P}^1)$:*

$$\mathcal{O}(m+1)*\mathcal{O}(n) - q\mathcal{O}(n)*\mathcal{O}(m+1) = q\mathcal{O}(m)*\mathcal{O}(n+1) - \mathcal{O}(n+1)*\mathcal{O}(m). \tag{2.4}$$

Proof. We check (2.4) for $m > n$. Then

$$\mathcal{O}(n) * \mathcal{O}(m+1) = \nu^{m-n+2}\mathcal{O}(n) \oplus \mathcal{O}(m+1);$$

$$\mathcal{O}(n+1) * \mathcal{O}(m) = \nu^{m-n}\mathcal{O}(n+1) \oplus \mathcal{O}(m);$$

$$\mathcal{O}(m)*\mathcal{O}(n+1) = \nu^{n-m+2}(q^{m-n}\mathcal{O}(m)\oplus\mathcal{O}(n+1)+\sum_{s=1}^{\lfloor\frac{m-n-1}{2}\rfloor}(q^2-1)q^{m-n-2}[\mathcal{O}(n+s+1)\oplus\mathcal{O}(m-s)]);$$

$$\mathcal{O}(m+1)*\mathcal{O}(n) = \nu^{n-m}(q^{m-n+2}\mathcal{O}(m+1)\oplus\mathcal{O}(n)+\sum_{s=1}^{\lfloor\frac{m-n+1}{2}\rfloor}(q^2-1)q^{m-n}[\mathcal{O}(n+s)\oplus\mathcal{O}(m-s+1)])$$

and we get the desired equality.

□

The conclusion of the section is the following result.

Theorem 2.5. *The map $\Upsilon : z^i \to \mathcal{O}(i)$ extends to an isomorphism of algebras S_q and $\mathcal{H}_{lf}(\mathbb{P}^1)$.* □

2.2. **General Result.** Let X be a smooth connected projective curve of genus g over some finite field $\mathbb{F}_q$. Let $\zeta_X(t) \in \mathbb{C}(t)$ be its zeta function:

$$\zeta_X(t) = \exp(\sum_{d\geq 1}\#X(\mathbb{F}_{q^d})\frac{t^d}{d}).$$

Example. For a smooth elliptic curve E we have

$$\zeta_E(t) = \frac{1-a_q(E)t+qt^2}{(1-t)(1-qt)}.$$

. It is known that $\zeta_X(t) = \frac{\prod_{i=1}^{g}(1-\alpha_i t)(1-\overline{\alpha_i}t)}{(1-t)(1-qt)}$ is a rational function of t and the roots α_i of the polynomial in the numerator are such that $|\alpha_i| = q^{\frac{1}{2}}$, so $\alpha_i\overline{\alpha_i} = q$.

We choose $\mu_g(x) = x^{g-1}\frac{1-qx}{1-x^{-1}}\prod_{i=1}^{g}(1-\alpha_i x^{-1})(1-\overline{\alpha_i}x^{-1})$ and denote the corresponding shuffle algebra by S_g. For $d \in \mathbb{Z}$, let $Pic^d(X)$ be the set of line bundles over X of degree d, define

$$\mathbf{1}_{Pic^d(X)} := \sum_{\mathcal{L}\in Pic^d(X)}\mathcal{L}.$$

In [SV] it was shown that the map $\mathbf{1}_{Pic^d(X)} \mapsto z^i$ extends to an isomorphism

$$\Upsilon_X : U_X^{>} \to S_{\mathbf{1}},$$

where $U_X^{>}$ stands for the subalgebra of $\mathcal{H}_{lf}(X)$ generated by $\mathbf{1}_{Pic^d(X)}$ and $S_{\mathbf{1}}$ denotes the subalgebra of S_g, generated by $\mathbb{C}[z^{\pm 1}]$.

3. EHA: a Reminder

We recall the presentation of the elliptic Hall algebra $\mathcal{E}^+$ via generators and relations.

Definition. A triangle with vertices $X = (0,0)$, $Y = (k_2, d_2)$ and $Z = (k_1 + k_2, d_1 + d_2)$ on the lattice $\mathbb{Z}^2$ is said to be **quasi-empty**, if the following properties hold:

•$k_1, k_2 \in \mathbb{Z}_{>0}$;

•$\frac{d_1}{k_1} > \frac{d_2}{k_2}$;

• There are no lattice points inside the triangle and on at least one of the edges XY, YZ.

If the first two conditions hold and there are no lattice points on both XY and YZ, the triangle is called **empty**.

The positive half $\mathcal{E}^+$ is by definition generated by the elements $u_{k,d}$ (here $k \geq 1$ and $d \in \mathbb{Z}$), with relations:

$$[u_{k_1,d_1}, u_{k_2,d_2}] = 0, \tag{3.1}$$

whenever the points $(k_1, d_1), (k_2, d_2)$ are collinear, and:

$$[u_{k_1,d_1}, u_{k_2,d_2}] = \frac{\theta_{k_1+k_2,d_1+d_2}}{\alpha_1}, \tag{3.2}$$

whenever the triangle with vertices $(0,0)$, (k_2, d_2) and $(k_1 + k_2, d_1 + d_2)$ is quasi-empty. Here

$$\alpha_n = \frac{(q_1^n - 1)(q_2^n - 1)(q^{-n} - 1)}{n} \tag{3.3}$$

and θ is given by the generating function

$$\sum_{n=0}^{\infty} \theta_{na,nb} t^n = \exp(\sum_{n=0}^{\infty} \alpha_n u_{na,nb} t^n), \tag{3.4}$$

where $\gcd(a, b) = 1$.

The proof of the following result can be found in [SV]

Theorem 3.1. *The map $u_{1,d} \mapsto z^d$ extends to an isomorphism of algebras*

$$\Upsilon : \mathcal{E}^+ \to \tilde{S},$$

where $\tilde{S}$ is the subalgebra of S, generated by $\bigoplus_{d\in\mathbb{Z}} S_{d,1}$.

4. Shuffle Algebra and EHA

We consider the shuffle algebra depending on three parameters q_1, q_2, q, s.t. $q_1q_2 = q$. This is an associative graded unital subalgebra S of the graded space of symmetric rational functions in infinitely many variables endowed with the product

$$F(z_1,\ldots,z_n)*G(z_1,\ldots,z_m)=\frac{1}{n!m!}\mathrm{Sym}_{S_{n+m}}\left(F(z_1,\ldots,z_n)G(z_{n+1},\ldots,z_{n+m})\prod_{\substack{i\in\{1,\ldots n\}\\ j\in\{n+1,\ldots n+m\}}}\mu\left(\frac{z_i}{z_j}\right)\right),$$

where $\mu(x)=\frac{(x-1)(x-q)}{(x-q_1)(x-q_2)}$ and $\mathrm{Sym}_{S_k}\left(H(z_1,\ldots,z_k)\right)=\sum_{\sigma\in S_k} sgn(\sigma)H(z_{\sigma(1)},\ldots,z_{\sigma(k)})$. Each component S_n consists of rational functions

$$F(z_1,\ldots,z_n)=\frac{f(z_1,\ldots,z_n)\prod_{1\le i<j\le n}(z_i-z_j)^2}{\prod_{1\le i\neq j\le n}(z_i-q_1z_j)(z_i-q_2z_j)}, \tag{4.1}$$

with $f(z_1,\ldots,z_n)$ - a symmetric Laurent polynomial, satisfying the wheel condition.

Definition. A symmetric Laurent polynomial $f(z_1,\ldots,z_n)$ satisfies the **wheel condition** if

$$f(z_1,\ldots,z_n)=0 \text{ when } \frac{z_i}{z_j}=q_1, \frac{z_j}{z_k}=q_2 \text{ and } \frac{z_k}{z_i}=\frac{1}{q}. \tag{4.2}$$

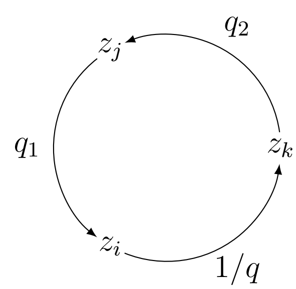

Figure 1. Wheel condition.

Next we verify that S is an algebra.

Proposition 4.1. *S is closed under the product* $(*)$.

Proof. The shuffle product of $F(z_1,\ldots,z_n)\in S_n$ and $G(z_1,\ldots,z_m)\in S_m$ can be written as

$$F(z_1,\ldots,z_n)*G(z_1,\ldots,z_m)=\frac{\prod_{1\le i<j\le n+m}(z_i-z_j)^2}{\prod_{1\le i\neq j\le n+m}(z_i-q_1z_j)(z_i-q_2z_j)}\cdot$$

$$\frac{1}{n!m!}\mathrm{Sym}_{S_{n+m}}\left(f(z_1,\ldots,z_n)g(z_{n+1},\ldots,z_{n+m})\prod_{1\le i\le n<j\le n+m}\frac{(z_i-qz_j)(z_i-q_1z_j)(z_i-q_2z_j)}{(z_i-z_j)}\right).$$

The rational function on the second line of the expression above does not have poles, as the only possible poles are at $z_i=z_j$ and those are simple. However, as the function is symmetric it cannot have poles of odd order and, therefore, is regular. It remains to check that the conditions (4.2) are satisfied. Indeed, if the indices of all three variables are in either $\{1,\ldots,n\}$

or $\{n+1,\dots,n+m\}$ simultaneously, this follows from the wheel conditions for f or g. Otherwise the product $\prod\limits_{1\le i\le n<j\le n+m} \frac{(z_i-qz_j)(z_i-q_1z_j)(z_i-q_2z_j)}{(z_i-z_j)}$ vanishes. $\square$

The shuffle algebra $S := \mathbb{C}\bigoplus\limits_{d\in\mathbb{Z},n>0} S_{d,n}$ is bigraded by the degree and the number of variables in f.

The next definition will be of great importance for the proof that S and $\mathcal{E}^+$ are isomorphic.

Definition. An element $F(z_1,\dots,z_n)\in S$ is said to have **slope** $\le \mu$ ($\mu\in\mathbb{R}$) if the limit $\lim\limits_{\xi\to\infty} \frac{F(\xi z_1,\dots,\xi z_i,z_{i+1},\dots,z_n)}{\xi^{\mu i}}$ exists and is finite for all $i\in\{1,\dots,n\}$.

The subspace of elements of $S_{d,n}$ with slope $\le \mu$ will be denoted by S^μ_n. Notice, that we have inclusions $S^\mu_{k,n}\subset S^{\mu'}_{d,n}$ for $\mu\le\mu'$ and $S_{d,n}=\bigcup\limits_{\mu\in\mathbb{R}} S^\mu_{d,n}$, thus, an increasing filtration on the infinite-dimensional vector space $S_{d,n}$. It is also true that $S^\mu=\mathbb{C}\bigoplus\limits_{k\in\mathbb{Z},n>0} S^\mu_{d,n}$ is a subalgebra. One advantage of considering the subspaces $S^\mu_{d,n}$ is that they are finite dimensional, the next proposition provides an upper bound on the dimension.

Proposition 4.2. *The dimension of $S^\mu_{d,n}$ does not exceed the number of unordered tuples (the order between pairs with $n_i=n_j$ is disregarded) of pairs $(n_1,d_1),\dots,(n_s,d_s)$, such that*

$$\begin{cases} n_1+\ldots+n_s=n\\ d_1+\ldots+d_s=d\\ d_i\le \mu n_i\ \forall\ i\in\{1,\dots,n\},\end{cases} \tag{4.3}$$

where $t,n_i\in\mathbb{N}$ and $d_i\in\mathbb{Z}$.

Proof. Let $\rho=(n_1,\dots,n_s)$ be a partition of n. We consider the map

$$\varphi_\rho : S^\mu_{d,n}\to\mathbb{C}[y_1^{\pm1},\dots,y_t^{\pm1}]$$

$$\varphi_\rho(F(z_1,\dots,z_n)) = f(qy_1,q^2y_1\dots,q^{k_1}y_1,qy_2,q^2y_2\dots,q^{k_2}y_2,\dots,qy_t,q^2y_t\dots,q^{k_t}y_t)$$

and define

$$S^{\mu,\rho}_{d,n} := \bigcap_{\rho'\succ\rho} \ker\varphi_{\rho'},$$

where $\succ$ stands for the usual dominance order on partitions, and set $S^{\mu,(n)}_{d,n}=S^\mu_{d,n}$. Then the subspaces $S^{\mu,\rho}_{d,n}$ form a filtration of $S^\mu_{d,n}$, i.e.

$$\rho\prec\rho'\Rightarrow S^{\mu,\rho}_{d,n}\subset S^{\mu,\rho'}_{d,n}.$$

We take $F\in S^{\mu,\rho}_{d,n}$ and notice that the wheel condition implies $\varphi_\rho(F)$, vanishes, if

$$y_j=q_2q^{a-b}y_i, a\in\{1,\dots,n_i-1\}, b\in\{1,\dots,n_j\} \text{ or} \tag{4.4}$$

$$y_j=q_1q^{a-b}y_i, a\in\{1,\dots,n_i-1\}, b\in\{1,\dots,n_j\}, \tag{4.5}$$

for $i<j$ as, for example, (4.4) guarantees that there are $z_i = q^{a+1}y_i$, $z_j = q_2q^a y_i$ and $z_k = q^a y_i$. As $\varphi_\rho(F)$ also belongs to $S^{\mu,\rho}_{d,n}$, it vanishes, whenever

$$y_j = q^{n_i-b+1}y_i \text{ or } y_j = q^{-b}y_i, \text{ where } b\in\{1,\dots,n_j\}, \tag{4.6}$$

with $i<j$, as well. We conclude that the Laurent polynomial $\varphi_\rho(F)$ is divisible by

$$D_F = \prod_{1\le i<j\le t}\left(\prod_{b=1}^{n_j}(y_j - q^{n_i-b+1}y_i)(y_j-q^{-b}y_i)\prod_{b=1}^{n_j}\prod_{a=1}^{n_i-1}(y_j-q_1q^{a-b}y_i)(y_j-q_2q^{a-b}y_i)\right),$$

a polynomial of degree

$$\deg(D_F) = \sum_{i<j}2n_j + 2(n_i-1)n_j = 2\sum_{i<j}n_in_j = n^2 - \sum_i n_i^2$$

and of degree in each variable y_i equal to

$$\deg_i(D_F) = \sum_{j\ne i}2n_in_j = 2n_i\sum_{j\ne i}n_j = 2n_i(n-n_i).$$

. As follows from the definition of elements of S in (4.1), $\deg(f) = \deg(F) + n(n-1) = d + n(n-1)$, and we obtain

$$\deg(\varphi_\rho(F)) = \deg(f) = d + n(n-1).$$

Next we use that the slope of $\varphi_\rho(F)$ is not greater than μ. This and (4.1) provide an upper bound on the degree of $\varphi_\rho(F)$:

$$\deg_i(\varphi_\rho(F)) - (n_i(n_i-1) + 2n_i(n-n_i)) \le \mu n_i$$
$$\deg_i(\varphi_\rho(F)) \le 2nn_i - n_i(n_i+1) + \mu n_i.$$

The above allows to conclude

$$\deg(\varphi_\rho(F)/D_F) = \sum_i n_i(n_i-1) + d$$

$$\deg_i(\varphi_\rho(F)/D_F) \le n_i(n_i-1) + \mu n_i.$$

The basis for such polynomials consists of monomials

$$y_1^{n_1(n_1-1)+d_1},\dots,y_t^{n_t(n_t-1)+d_t},$$

with $d_1+\ldots+d_t = d$ and $d_i\le \mu n_i$. To complete the proof it remains to notice that if $n_i = n_j$, then $\varphi_\rho(F)$ is invariant under the transposition (ij), so the respective order of d_i and d_j can be disregarded. ☐

Corollary 4.3. *The subspace of $S_{d,n}$, consisting of elements F, s.t.*

$$\lim_{\xi\to\infty}\frac{F(\xi z_1,\dots,\xi z_i,\dots,\xi z_n)}{\xi^{\frac{di}{n}}} = 0\ \forall i\in\{1,\dots,n\}$$

is at most one-dimensional.

Proof. In this case $\mu = \frac{d}{n}$, so $d_i \leq \frac{d}{n}n_i$ and $d_1 + \ldots + d_t \leq \frac{d}{n}n_1 + \ldots + \frac{d}{n}n_t = d$. Thus, we must have that each $d_i = \frac{d}{n}n_i$. On the other hand, for the limit above to be zero, the inequalities $d_i \leq \frac{d}{n}n_i$ must be strict. Therefore, the only possibility is $n_1 = n$ and $d_1 = d$. $\square$

In [SV] it was shown that the map $u_{1,d} \to z^d$ extends to an isomorphism

$$\Upsilon : \mathcal{E}^+ \to S_{\mathbf{1}},$$

where $S_{\mathbf{1}}$ is the subalgebra of S, generated by $\mathbb{C}[z, z^{-1}]$. This allows us to conclude that the map $\mathcal{E}^+ \to S$ (which we also denote by Υ) is also injective and the next proposition shows that it is surjective as well.

Proposition 4.4. *The map $\Upsilon : \mathcal{E}^+ \to S$ is surjective.*

Proof. We denote by $\mathcal{E}_{n,d} \subset \mathcal{E}^+$ the subspace of elements of bidegree (n, d) and

$$\mathcal{E}^\mu_{n,d} = \left\{ \text{sums of products of } u_{n',d'} \text{ with } \frac{d'}{n'} \leq \mu \right\} \subset \mathcal{E}_{n,d}.$$

The dimension of $\mathcal{E}^\mu_{n,d}$ equals to the number of convex paths in $\mathbf{Conv}^+$ with slope $\leq \mu$ and such paths are in bijection with the pairs of tuples from proposition 4.2 (see lemma 5.6 of [BS]). Therefore, the dimension of $S^\mu_{d,n}$ does not exceed the dimension of $\mathcal{E}^\mu_{n,d}$ and (due to invectivety of Υ) it is sufficient to show $\Upsilon(\mathcal{E}^\mu_{n,d}) \subset S^\mu_{d,n}$. The verification of this can be found in the proof of proposition 3.5 in [Neg]. $\square$

. We introduce $P_{k,d} := \Upsilon(u_{k,d})$.

Corollary 4.5. *S is generated by the first graded component, i.e. $\mathbb{C}[z^{\pm 1}]$.*

5. The Double of Shuffle Algebra

We start with the general construction. Suppose $(\mathcal{A}, *, \triangle)$ is a bialgebra (we assume that $\triangle$ is coassociative and the product and coproduct are compatible in the sense that $\triangle(a * b) = \triangle(a) * \triangle(b)$) with a symmetric non-degenerate form

$$(\cdot, \cdot) : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C},$$

satisfying

$$(a * b, c) = (a \otimes b, \triangle(c)) \ \forall\ a, b, c \in \mathcal{A}. \tag{5.1}$$

Definition. The **Drinfeld double** $D\mathcal{A} = \mathcal{A}^{coop} \otimes \mathcal{A}$ ($\mathcal{A}^{coop}$ has the same product as $\mathcal{A}$, but the coproduct is opposite) is a free product of algebras with both $\mathcal{A}^- = \mathcal{A}^{coop} \otimes 1$ and $\mathcal{A}^+ = 1 \otimes \mathcal{A}$ being subbialgebras, subject to the relations

$$\sum_{i,j} a_i^{(1)-} * b_j^{(2)+} (a_i^{(2)-}, b_j^{(1)+}) = \sum_{i,j} b_j^{(1)+} * a_i^{(2)-} (a_i^{(1)-}, b_j^{(2)+}), \text{ where}$$

$$\triangle(a)=\sum_i a_i^{(1)-}\otimes a_i^{(2)-} \text{ and } \triangle(b)=\sum_j b_j^{(1)+}\otimes b_j^{(2)+}\ \forall\ a\in\mathcal{A}^+,\ b\in\mathcal{A}^-.$$

The bialgebra structure on $D\mathcal{A}$ is determined uniquely.

Our next goal is to endow the Shuffle algebra S with a coproduct. For this we will need to consider a slightly larger algebra $\tilde{S}$. It is generated by S and the set of elements $h_i, i\in\mathbb{Z}_{\geq 0}$ with the following relations

$$[h_i,h_j]=0$$

$$F(z_1,\dots,z_n)*h(w)=h(w)*\left(F(z_1,\dots,z_n)\prod_{i=1}^n\Omega\left(\frac{w}{z_i}\right)\right), \tag{5.2}$$

where $h(w):=\sum\limits_{n\geq 0}h_nw^{-n}$ and

$$\Omega(x)=\frac{\mu\left(\frac{1}{x}\right)}{\mu(x)}=\frac{(x-q^{-1})(x-q_1)(x-q_2)}{(x-q)(x-q_1^{-1})(x-q_2^{-1})}=\exp\left(-\sum_{n\geq 1}\alpha_nx^{-n}\right). \tag{5.3}$$

We understand (5.2) by expanding the r.h.s. in negative powers of w and setting the corresponding terms equal. Now we can define the coproduct on $\tilde{S}$ (we skip the proof and refer to the Appendix in [Neg]).

Proposition 5.1. *The following formulas define a coassociative coproduct on $\tilde{S}$:*

$$\triangle(h(w)):=h(w)\otimes h(w)$$

$$\triangle(F(z_1,\dots,z_n))=\sum_{i=0}^n\frac{\prod\limits_{b>i}h(z_b)F(z_1,\dots,z_i\otimes z_{i+1},\dots,z_n)}{\prod\limits_{a\leq i<b}\mu\left(\frac{z_b}{z_a}\right)} \tag{5.4}$$

. The r.h.s of the second line above should be understood by expanding in nonnegative powers of $\frac{z_a}{z_b}$ for $a\leq i<b$, obtaining an infinite sum of monomials. Then in each summand all h_i's are moved to the left, followed by powers of $z_1,\dots,z_i$ to the left of $\otimes$ and powers of $z_{i+1},\dots,z_n$ - to the right. A typical summand looks like

$$h_{k_i+1}\dots h_{k_n}z_1^{s_1}\dots z_i^{s_i}\otimes z_{i+1}^{s_{i+1}}\dots z_n^{s_n},$$

here $\triangle(F(z_1,\dots,z_n))$ belongs to the completion $\tilde{S}\hat{\otimes}\tilde{S}$.

Finally, the bialgebra $\tilde{S}$ has a pairing, given by:

$$(h(v),h(w^{-1}))=\Omega\left(\frac{w}{v}\right)$$

$$(F,G)=\frac{1}{\alpha_1^k}:\int:\frac{F(z_1,\dots,z_n)G(\frac{1}{z_1},\dots,\frac{1}{z_n})}{\prod\limits_{1\leq i\neq j\leq n}\mu\left(\frac{z_i}{z_j}\right)}Dz_1\dots Dz_n \tag{5.5}$$

for $F, G \in S_{n,d}$ with $Dz := \frac{1}{2\pi i z}$ and we define the normal-ordered integral $:\int:$ by

(5.6)
$$\left(Sym\left(z_1^{k_1}\dots z_n^{k_n}\prod_{1\le i\neq j\le n}\mu\left(\frac{z_i}{z_j}\right)\right), F\right) = \frac{1}{\alpha_1^k}\int_{|z_1|<<|z_2|<<\dots<<|z_n|}\frac{z_1^{k_1}\dots z_n^{k_n}F(\frac{1}{z_1},\dots,\frac{1}{z_n})}{\prod_{1\le i\neq j\le n}\mu\left(\frac{z_i}{z_j}\right)}Dz_1\dots Dz_n.$$

This sufficient to define the pairing on $\tilde{S}$ since any element can be written as a linear combination of monomials $z^{i_1} * \dots * z^{i_k}$ by the corollary of proposition 4.4.

We refer to [Neg] for the proof of the next proposition.

Proposition 5.2. *The formulas 5.6 above produce a well-defined pairing on bialgebra $\tilde{S}$:*

$$\tilde{S} \otimes \tilde{S} \to \mathbb{C}(q_1, q_2).$$

We denote the Drinfeld double of $\tilde{S}$ with respect to the pairing 5.6 by $D\tilde{S}$.

Next we slightly expand the elliptic Hall algebra $\mathcal{E}^+$ by adding the commuting elements $\{u_{0,i}|i \in \mathbb{Z}\}$ and a central element c with relations

$$[u_{0,d}, u_{1,d'}] = u_{1,d+d'} \;\forall d \in \mathbb{Z}, d' \in \mathbb{Z}_{>0}$$

and denote the is algebra by $\tilde{\mathcal{E}}^+$.

The coproduct is given by

$$\triangle(u_{0,d}) = u_{0,d}\otimes 1 + 1\otimes u_{0,d},\;\; \triangle(u_{1,d}) = u_{1,d}\otimes 1 + c\sum_{n\ge 0}\theta_{0,n}\otimes u_{1,d-n},$$

where $\theta_{0,n}$ are computed according to 3.4. It remains to define a pairing on $\tilde{\mathcal{E}^+}$, which is done by setting

$$(u_{0,d}, u_{0,d}) = \frac{1}{\alpha_d}, (u_{1,d}, u_{1,d}) = \frac{1}{\alpha_1}.$$

Theorem 5.3. *The morphism of proposition 4.4 can be extended to $\Upsilon : \tilde{\mathcal{E}}^+ \to \tilde{S}$ by*

$$\Upsilon(c) = h_0, \;\text{and}\; \Upsilon(u_{0,d}) = p_d,$$

where $p_1, p_2, \dots$ are obtained from the series

$$h(w) = h_0\, exp\left(\sum_{n=1}^{\infty}\alpha_n p_n w^{-n}\right).$$

Thus extended Υ preserves the coproduct and bialgebra pairing and, therefore, induces the isomorphism of Drinfeld doubles:

$$\tilde{\Upsilon} : D\tilde{\mathcal{E}}^+ \to D\tilde{S}.$$

Proof. First one needs to show that the formulas above indeed extend the isomorphism defined in proposition 4.4, i.e. respect the relations between elements added to the algebras. Next, we need to check that Υ preserves the pairing. It is enough to show this for generators, provided it satisfies conditions 5.1 (this is shown on page 24 of [Neg]). For example,

$$(z_1^d, z_1^d) = \frac{1}{\alpha_1}\frac{1}{2\pi i}\int\limits_{|z_1|<1} \frac{z_1^d z_1^{-d}}{z_1} dz_1 = \frac{1}{\alpha_1} = (u_{1,d}, u_{1,d})$$

□

References

[BK] Baumann, P. and Kassel, C. The Hall algebra of the category of coherent sheaves on the projective line. *J. reine angew. Math*, (533):207–233, 2001.

[BS] Burban, I. and O. Shiffmann, O. On the Hall algebra of an elliptic curve. *Duke Math. J.*, 161(7):1171–1231, 2012.

[FT] Feigin, B. and Tsymbaliuk, A. Heisenberg action in the equivariant K-theory of Hilbert schemes via shuffle algebra. *Kyoto J. Math.*, 51:831–854, 2011.

[Neg] Negut, A. The shuffle algebra revisited. *Int.Math.Res.Not.*, 22: 6242–6275, 2014.

[SV] Schiffmann, O. and Vasserot, E. Hall algebras of curves, commuting varieties and Langlands duality. *Math. Ann.*, (353):1399–1451, 2012.

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