On categories $\mathcal{O}$ of quiver varieties overlying the bouquet graphs

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Dedication

To my parents Tatiana and Dmitry, with love and admiration.

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Abstract of Dissertation

We study representation theory of quantizations of Nakajima quiver varieties associated to bouquet quivers. We show that there are no finite dimensional representations of the quantizations $\overline{\mathcal{A}}_\lambda(n, \ell)$ if both dim $V = n$ and the number of loops ℓ are greater than 1. We show that when $n \leq 3$ there is a Hamiltonian torus action with finitely many fixed points, provide the dimensions of Hom-spaces between standard objects in category $\mathcal{O}$ and compute the multiplicities of simples in standards for $n = 2$ in case of one-dimensional framing and generic one-parameter subgroups. We establish the abelian localization theorem and find the values of parameters, for which the quantizations have infinite homological dimension.

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Chapter 1

Introduction

Our primary goal is to study category $\mathcal{O}$ of quantizations of the Nakajima quiver variety with underlying quiver $Q = B_\ell$, which has one vertex, ℓ loops, where $\ell \in \mathbb{Z}_{\geq 0}$ and a one-dimensional framing. The notion of category $\mathcal{O}$ in the context of conical symplectic resolutions was introduced in [BLPW16]. In particular in [Los18] the author studies the properties of category $\mathcal{O}$ for the Gieseker varieties. These are the framed moduli spaces of torsion free sheaves on $\mathbb{P}^2$ with rank r and second chern class n. They admit a description as quiver varieties for the quiver with one vertex, one loop, n-dimensional space assigned to the vertex and an r-dimensional framing (see Chapter 2 of [Nak99] for details). The results and methods of [Los18] provide invaluable tools for our research. We start by recalling the setup.

1.1 Generalities on category $\mathcal{O}$ for conical symplectic resolutions

We fix the base field to be $\mathbb{C}$. Recall that an affine variety Y is Poisson provided it comes equipped with an algebraic Poisson bracket, i.e. a bilinear map

$$\{\cdot,\cdot\} : \Lambda^2\mathbb{C}[Y] \to \mathbb{C}[Y],$$

s.t. for any $f, g, h \in \mathbb{C}[Y]$

- $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$, the Jacobi identity;
- $\{fg,h\} = g\{f,h\} + h\{g,f\}$, the Leibnitz rule.

Let X_0 be a normal Poisson affine variety equipped with an action of the multiplicative group $\mathbb{S} := \mathbb{C}^*$, s.t. the Poisson bracket has a negative degree with respect to this action, i.e.

$$\{\mathbb{C}[X_0]_i, \mathbb{C}[X_0]_j\} \subseteq \mathbb{C}[X_0]_{i+j-d} \text{ with } d \in \mathbb{Z}_{>0}.$$

We assume that $\mathbb{C}[X_0] = \bigoplus_{i\geq 0}\mathbb{C}[X_0]_i$ with $\mathbb{C}[X_0]_0 = \mathbb{C}$ w.r.t. the grading coming from the $\mathbb{S}$-action (this action will be called the *contracting action*). Geometrically this means that there is a unique fixed point $o \in X_0$ and the entire variety is contracted to this point by the $\mathbb{S}$-action. Let (X, ω) be a symplectic variety and $\rho : X \to X_0$ a projective resolution of singularities, which is also a morphism of Poisson varieties. In addition, assume that the action of $\mathbb{S}$ admits a ρ-equivariant lift to X. A pair (X, ρ) as above is called a *conical symplectic resolution*.

Remark 1.1.1. There are sufficiently many $\mathbb{S}$-stable open affine subsets. Namely, due to a result of Sumihiro every point x of X_0 has an open affine neighborhood in the conical topology (see Section 3, Corollary 2 in [Sum75]).

Definition 1.1.2. Let (X, ρ) be a conical symplectic resolution. A *quantization* of the affine variety X_0 is an algebra $\mathcal{A}$ together with an isomorphism $\mathrm{gr}\mathcal{A} \xrightarrow{\sim} \mathbb{C}[X_0]$ of graded Poisson algebras. By a quantization of X we understand a sheaf (in the conical topology, i.e. open spaces are Zariski open and $\mathbb{S}$-stable) of filtered algebras $\widetilde{\mathcal{A}}$ (the filtration is complete and separated) together with an isomorphism $\mathrm{gr}\widetilde{\mathcal{A}} \xrightarrow{\sim} \mathcal{O}_X$ of sheaves of graded Poisson algebras.

Remark 1.1.3. We would like to point out that the algebra $A := \mathrm{gr}\mathcal{A}$ has a natural Poisson bracket. Let $a \in A_i$ and $b \in A_j$ with $\tilde{a} \in \mathcal{A}_{\leq i}$ and $\tilde{b} \in \mathcal{A}_{\leq j}$ any lifts, then the Poisson bracket is given by

$$\{a, b\} := [\tilde{a}, \tilde{b}] + \mathcal{A}_{i+j-2}.$$

Notice that $[\tilde{a}, \tilde{b}] \in \mathcal{A}_{i+j-1}$ since the algebra A is isomorphic to $\mathbb{C}[X_0]$ and hence commutative. It is this bracket that we want to match the original bracket on $\mathbb{C}[X_0]$ in Definition 1.1.2.

Remark 1.1.4. There is a map from the set of quantizations of X to the second de Rham cohomology $H^2_{DR}(X)$. This map is called the *period map* and is an isomorphism provided $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ (see [BK04]). If this is the case, the quantizations $\widetilde{\mathcal{A}}$ are parameterized (up to isomorphism) by the points of $H^2_{DR}(X)$. The quantization corresponding to the cohomology class λ will be denoted by $\widetilde{\mathcal{A}}_\lambda$.

Suppose, that X is equipped with a Hamiltonian action of a torus T with finitely many fixed points, i.e. $|X^T| < \infty$. Assume, in addition, that the action of T commutes with the contracting action of $\mathbb{S}$. A one-parametric subgroup $\nu : \mathbb{C}^* \to T$ is called *generic* if $X^T = X^{\nu(\mathbb{C}^*)}$. To a generic one-parametric subgroup $\nu : \mathbb{C}^* \to T$ one can associate a category of modules over the algebra $\mathcal{A}$ defined above, called category $\mathcal{O}_\nu(\mathcal{A})$. Namely, the action of ν lifts to $\mathcal{A}$ and induces a grading on it, i.e. $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}}\mathcal{A}_{i,\nu}$. We denote

$$\mathcal{A}^{\geq 0,\nu} = \bigoplus_{i\geq 0}\mathcal{A}_{i,\nu}, \mathcal{A}^{\leq 0,\nu} = \bigoplus_{i\leq 0}\mathcal{A}_{i,\nu} \text{ (similarly define } \mathcal{A}^{<0,\nu}, \mathcal{A}^{>0,\nu}\text{) and} \tag{1.1.1}$$

$$\mathcal{C}_\nu(\mathcal{A}) := \mathcal{A}^{\geq 0,\nu} / \left(\mathcal{A}^{\geq 0,\nu} \cap \mathcal{A}\mathcal{A}^{>0,\nu}\right) = \mathcal{A}_0 / \bigoplus_{i>0}\mathcal{A}_{-i}\mathcal{A}_i. \tag{1.1.2}$$

Let $\mathcal{A}$-mod be the category of finitely generated $\mathcal{A}$-modules.

Definition 1.1.5. The category $\mathcal{O}_\nu(\mathcal{A})$ is the full subcategory of $\mathcal{A}$-mod, on which $\mathcal{A}^{\geq 0,\nu}$ acts locally finitely.

Recall that if $\mathcal{R}$ is a commutative Noetherian ring and $X = Spec\mathcal{R}$, then one has an equivalence of abelian categories:

$$\mathcal{R}\text{-mod} \underset{\Gamma}{\overset{\mathrm{Loc}}{\rightleftarrows}} \mathrm{Coh}(X), \tag{1.1.3}$$

where Γ and Loc are the functor of global sections and localization respectively (see Chapter II, Corollary 5.5 in [Har77] for details).

Definition 1.1.6. An $\widetilde{\mathcal{A}}_\lambda$-module M is called *coherent* provided there is a global complete and separated filtration on M, s.t. $\mathrm{gr}M$ is a coherent $\mathcal{O}_X$-module. The category of coherent $\widetilde{\mathcal{A}}_\lambda$-modules will be denoted by $\mathrm{Coh}(\widetilde{\mathcal{A}}_\lambda)$.

The noncommutative analogue of equivalence (1.1.3) is

$$\mathcal{A}_\lambda\text{-mod} \underset{\Gamma_\lambda}{\overset{\mathrm{Loc}_\lambda}{\rightleftarrows}} \mathrm{Coh}(\widetilde{\mathcal{A}}_\lambda) \tag{1.1.4}$$

and has a weaker (derived form):

$$D^b(\mathcal{A}_\lambda\text{-mod}) \underset{R\Gamma_\lambda}{\overset{\mathrm{LLoc}_\lambda}{\rightleftarrows}} D^b(\mathrm{Coh}(\widetilde{\mathcal{A}}_\lambda)). \tag{1.1.5}$$

Definition 1.1.7. If the functors Γ_λ and Loc_λ are mutually inverse equivalences, we say that *abelian localization holds* for λ and if $R\Gamma_\lambda$ and LLoc_λ are quasi-inverse equivalences (between the bounded derived categories) that *derived localization holds*.

Example 1.1.8. Let $\mathfrak{g}$ be a simple Lie algebra with Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h}$. In order to fit the classical BGG category $\mathcal{O}$ in this framework, one needs to consider the Springer resolution $X = T^*(G/B) \to \mathcal{N} = X_0$ of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}^*$. Recall that an element $x \in \mathfrak{g}$ is called nilpotent if the operator $\mathrm{ad}_x^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is nilpotent and $\mathcal{N}$ is the set of all nilpotent elements of $\mathfrak{g}^*$. The nilcone $\mathcal{N}$ is a Poisson variety w.r.t. the Kirillov-Kostant-Souriau bracket and the symplectic leaves in $\mathcal{N}$ are the coadjoint orbits. The tori are the maximal torus $T \subset GL(V)$ and $\mathbb{S} := \mathbb{C}^*$ acting by inverse scaling on the cotangent fibers. Let $\mu : Z(\mathfrak{g}) \to \mathbb{C}$ be a central character, then the block $\mathcal{O}_\mu \subset \mathcal{O}$ consists of finitely generated $U(\mathfrak{g})$-modules for which $U(\mathfrak{b})$ acts locally finitely, $U(\mathfrak{h})$ semisimply and the center with generalized character μ. Pick a generic one-parameter subgroup $\nu(\mathbb{C}^*) \subset T$, s.t. $\mathfrak{b}$ is spanned by elements with positive $\nu(\mathbb{C}^*)$-weights. Let $U(\mathfrak{g})_\mu = U(\mathfrak{g})/\mathcal{I}_\mu$ with $\mathcal{I}_\mu$ the ideal generated by $z - \mu(z)$ for $z \in Z(\mathfrak{g})$ be the central reduction of $U(\mathfrak{g})$ w.r.t the central character μ.

We want to show that $U(\mathfrak{g})_\mu$ is a quantization of the nilcone $\mathcal{N}$. One can explicitly describe the Poisson bracket on $\mathbb{C}[\mathcal{N}]$ descending from $U(\mathfrak{g})_\mu$ (as explained in Remark 1.1.3). Recall that according to the PBW theorem $\mathrm{gr}U(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$. Moreover, the Harish Chandra theorem asserts that $Z(\mathfrak{g})$ is isomorphic to $S(\mathfrak{h})^W = \mathbb{C}[\mathfrak{h}^*]^W$. Here W is the Weyl group acting on $\mathfrak{h}^*$ via $w \cdot \mu = w(\mu+\rho) - \rho$, where ρ is half the sum of all positive roots. Combining these results allows to show the isomorphism of algebras $\mathrm{gr}U(\mathfrak{g})_\mu \simeq \mathbb{C}[\mathcal{N}]$. Let $x_1, \dots, x_n$ be a basis of $\mathfrak{g}$ and $c_{ij}^k \in \mathbb{C}$ the structure constants given by $[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k$. The Poisson bracket on $\mathcal{N} \subset \mathfrak{g}^*$ becomes the restriction of the bracket on $\mathfrak{g}^*$ given by

$$\{f, g\} = \sum_{k=1}^{n} c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \text{ for } f, g \in \mathbb{C}[\mathfrak{g}^*]$$

which can be more conveniently rewritten as

$$\{f, g\}(\xi) = \langle \xi, [d_\xi f, d_\xi g] \rangle,$$

where $\xi \in \mathfrak{g}^*$,$d_\xi f \in \mathfrak{g}^{**} \simeq \mathfrak{g}$ stands for the differential of f at ξ and $[,]$ denotes the Lie bracket on $\mathfrak{g}$ (see Proposition 1.3.18 in [CG10] for details). This is exactly the Kirillov-Kostant-Souriau bracket on the nilcone $\mathcal{N}$.

Next we want to compare the categories $\mathcal{O}_\nu(U(\mathfrak{g})_\mu)$ and $\mathcal{O}_\mu$. The difference in the requirements for an object $M \in U(\mathfrak{g})_\mu$ -mod to be in $\mathcal{O}_\nu(U(\mathfrak{g})_\mu)$ or $\mathcal{O}_\mu$ is that for the former containment $Z(\mathfrak{g})$ must act on M with an honest character μ, while for the latter the action of $U(\mathfrak{h})$ on M has to be semisimple. In case μ is regular ($\langle \mu+\rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ for all positive roots α) these conditions are interchangable, i.e. one gets an equivalent category by dropping one condition and adding the other (see Theorem 1 in [Soe86]), and, hence, the categories $\mathcal{O}_\nu(U(\mathfrak{g})_\mu)$ and $\mathcal{O}_\mu$ are equivalent.

Finally, let $\mathcal{D}_\mu(G/B)$ stand for the category of μ-twisted $\mathcal{D}$-modules on the flag variety G/B. Then one has an equivalence

$$U(\mathfrak{g})_\mu \text{-mod} \underset{\Gamma}{\overset{\mathrm{Loc}}{\rightleftarrows}} \mathcal{D}_\mu(G/B) \text{-mod}$$

for regular μ, this is the Beilinson-Bernstein theorem, see [BB81], while

$$D^b(U(\mathfrak{g})_\mu \text{-mod}) \underset{R\Gamma}{\overset{\mathrm{LLoc}}{\rightleftarrows}} D^b(\mathcal{D}_\mu(G/B) \text{-mod})$$

is an equivalence provided $\langle \mu+\rho, \alpha^\vee \rangle \neq 0$, see [BB93].

1.2 Questionnaire on quantizations

Let $\rho : X \to X_0$ be a conical symplectic resolution. Assume that X admits a Hamiltonian torus action with finitely many fixed points and the nonzero cohomology of the structure sheaf of X vanish. We list some typical questions that can be asked about quantizations and categories of modules thereof.

(1) For which λ does $\mathcal{A}_\lambda$ have finite homological dimension?

(2) What is the classification of finite dimensional irreducible modules?

(3) What are the supports of these modules?

(4) What are the two-sided ideals of $\mathcal{A}_\lambda$?

(5) For which $\lambda \in H^2_{DR}(X)$ do the abelian/derived localizations hold?

(6) What are the composition series of standard modules in category $\mathcal{O}$?

Remark 1.2.1. According to a result of McGerty and Nevins (see [MN14], Theorem 1.1) the 'derived equivalence locus' of (5) is the same as the locus, providing affirmative answer in (1).

Below we give examples of conical symplectic resolutions and properties of their quantizations and categories $\mathcal{O}$.

Example 1.2.2. Let $\Gamma \subset SL_2(\mathbb{C})$ be a nontrivial finite subgroup. Take $Y_0 = \mathbb{C}^2/\Gamma$ and the crepant resolution $\rho : Y \to Y_0$. The action of $\mathbb{S}$ is induced by the inverse of the diagonal action on $\mathbb{C}^2$, and has weight $d = 2$. In case $\Gamma = \mathbb{Z}/k\mathbb{Z}$, we can find a Hamiltonian $T \simeq \mathbb{C}^*$-action, where T is the group of symplectomorphisms of Y that commute with the action of $\mathbb{S}$.

Example 1.2.3. Next consider $X_0 := \mathrm{Sym}^n Y_0$ and the resolution $X := \mathrm{Hilb}^n \tilde{Y}$, where $\mathrm{Sym}^n Y_0$ is the symmetric variety of unordered n-tuples of points on the singular space Y_0 and $\mathrm{Hilb}^n Y$ is the Hilbert scheme of n points on the crepant resolution Y. Again, the action of $\mathbb{S}$ comes from the inverse diagonal action on $\mathbb{C}^2$. Partial answers to $(1)-(6)$ are provided by Remark 1.2.7.

Example 1.2.4. For a reductive algebraic group G and a Borel subgroup B, take X to be the cotangent bundle of flag variety $T^*(G/B)$ and X_0 to be the affinization of X. The map ρ is the Springer resolution. The action of $\mathbb{S}$ is via inverse scaling on the cotangent fibers, and $d = 1$, while $T \subset G$ is the maximal torus, with respect to which the algebra $\mathfrak{b} = \mathrm{Lie}(B)$ is spanned by positive roots. This example goes back to the celebrated paper of Bernstein, Gelfand and Gelfand (see [BGG76]) and served as one of the main inspirations for the development of the modern framework.

The description of finite dimensional representations is classical (see e.g. Chapters 1.6 and 2 in [Hum08] or Part III in [FH91]). Two-sided ideals of $\mathcal{U}_\lambda$ are closely related to primitive ideals in $\mathcal{U}_\lambda$ (annihilators of simple modules). The classification of the latter can be found in [Jos83]. Question (5) was answered by Beilinson and Bernstein (see Example

1.1.8), while the answer to (6) is known to be the value of the corresponding Kazhdan-Lusztig polynomial at 1. This fact is part of the Kazhdan-Lusztig conjecture (see [KL79]), independently proved by Beilinson, Bernstein ([BB81]) and Brylinski, Kashiwara ([BK81]). The answer to (3) can be found in the latter reference as well. We would also like to point the reader's attention to a more recent proof of the Kazhdan-Lusztig conjecture using Soergel bimodules due to Elias and Williamson ([EW14]).

Example 1.2.5. Hypertoric varieties associated to simple, unimodular hyperplane arrangements. These varieties admit an $\mathbb{S}$-action with $d = 1$ if and only if the arrangement has a bounded chamber; they always admit an action with weight $d = 2$. This was the first class of conical symplectic resolutions for which the unified definition of categories $\mathcal{O}$ in the context outlined in Section 1.1 was given. Most properties of quantization algebras and categories of modules thereof have explicit descriptions in terms of the underlying combinatorial data. In particular, the answers to $(1)-(3)$ and $(5), (6)$ are known (see [BLPW12]). For example, it is of interest that a simple object appears in the composition series of a standard with multiplicity either 0 or 1 (Section 4.4 of [BLPW12]). These varieties will be the main subject of Section 4.

Example 1.2.6. Nakajima quiver varieties (see Section 1.4). These varieties admit an action of $\mathbb{S}$ with $d = 1$ if and only if the quiver has no loops; they always admit an action with weight $d = 2$. The quantizations of varieties associated to quivers of finite and affine type were studied in [BL15] and [Los16]. There is a conjectural description for the locus of parameters on which (1) or (5) hold true in case the quiver is of finite or affine type (see Conjecture 9.2 in [BL15]). It is verified for some quivers including type A Dynkin quivers and the quiver with a single vertex and a loop ([Los18]). The answers to (2) and (3) were established for finite and affine quivers, (4) is known in some cases including the graph with a single vertex and a loop.

Remark 1.2.7. The last class of examples overlaps with the preceding ones. Namely, the first two examples are special cases of quiver varieties, where the underlying graph of the quiver is the extended Dynkin diagram corresponding to Q. The varieties that appear in the third example can be realized as quiver varieties if the group is of type A. Finally, a hypertoric variety is a quiver variety if and only if a certain technical condition on the hyperplane arrangement is satisfied. This correspondences will be outlined in Section 1.4.

1.3 A brief reminder on GIT

We start by recalling some basic facts on Geometric Invariant Theory, usually abbreviated as GIT (a more detailed exposition can be found e.g. in [Gin12]). Let Y be an affine algebraic variety, equipped with a reductive algebraic group G action. Given a character $\theta : G \to \mathbb{C}^*$, consider the action of G on $Y \times \mathbb{C}$ via $g \cdot (y, z) = (gy, \theta^{-1}(g)z)$. The algebra of invariants for the induced action on the coordinate ring $\mathcal{R}_\theta := \mathbb{C}[Y \times \mathbb{C}]^G = \mathbb{C}[Y][z]^G$ is finitely generated due to Hilbert's theorem on finite generation of algebras of invariants. Define $\mathbb{C}[Y]^{n,\theta} = \{f \in$

$\mathbb{C}[Y] \mid f(g^{-1}y) = \theta^n(g)f(y)\ \forall y \in Y, g \in G\}$ to be the space of θ^n semiinvariant functions. The algebra $\mathcal{R}_\theta$ has a grading given by the powers of z, i.e. $\mathcal{R}_\theta = \bigoplus_{n\geq 0} \mathbb{C}[Y]^{n,\theta}$.

Definition 1.3.1. The projective spectrum of the graded algebra $Y//_\theta G := \mathtt{Proj}(\mathcal{R}_\theta)$ is called a *GIT quotient of* Y *by the* G*-action*.

Notice that we have a canonical algebra embedding $\mathbb{C}[Y]^G \hookrightarrow \mathcal{R}_\theta$ as the degree zero subalgebra. This gives rise to a projective morphism of schemes $\pi : Y//_\theta G \to Y//G := \mathtt{Spec}\ \mathbb{C}[Y]^G$.

Definition 1.3.2. A point $y \in Y$ is called θ*-semistable*, provided there exists an $n \geq 1$ and $f \in \mathbb{C}[Y]^{n,\theta}$, s.t. $f(y) \neq 0$. Alternatively, one can consider the action of G on the trivial bundle $Y \times \mathbb{C}$ via $g(y,t) = (gy, \theta(g)t)$. Then y is θ-semistable provided the closure of the orbit $\overline{G.(y,1)}$ does not intersect the zero section $Y \times \{0\}$, i.e. does not contain a point of the form $(y', 0)$. The locus of θ-semistable points will be denoted by $Y^{\theta-ss}$.

Remark 1.3.3. The variety $Y//_\theta G$ parameterizes the closed orbits of θ-semistable points.

The following result is extremely useful for describing the locus of semistable points (the proof can be found in [Bir71]).

Theorem 1.3.4. *(Hilbert-Mumford Criterion). Let* G *be a reductive algebraic group acting rationally on a vector space* V*. Let* $x \in V$ *and* $y \in G.x$ *be such that* $G.y$ *is the unique closed orbit in* $\overline{G.x}$*. Then there exists a* 1*-parametric subgroup* $\lambda : \mathbb{C}^* \to G$ *such that* $\lim_{t\to 0} t.x$ *is contained in* $G.y$.

1.4 Nakajima quiver varieties

Let $Q = (Q_0, Q_1)$ be a finite quiver, i.e. a directed graph with finitely many vertices enumerated by the set Q_0 and finitely many edges enumerated by Q_1. Each edge is uniquely determined by the pair of vertices it connects, which we will denote by $t(a)$ and $h(a)$ standing for 'tail' and 'head'. Consider two dimension vectors $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geq 0}$, where n is the cardinality of Q_0 and form a vector space

$$R = \bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{ta}, V_{ha}) \oplus \bigoplus_{s\in Q_0} \mathrm{Hom}_{\mathbb{C}}(V_s, W_s).$$

Remark 1.4.1. The dimension vector w is often referred to as *framing*.

Notice that the space T^*R is symplectic and naturally identified with

$$\bigoplus_{a\in Q_1} \left(\mathrm{Hom}_{\mathbb{C}}(V_{ta}, V_{ha}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_{ha}, V_{ta})\right) \oplus \bigoplus_{s\in Q_0} \left(\mathrm{Hom}_{\mathbb{C}}(V_s, W_s) \oplus \mathrm{Hom}_{\mathbb{C}}(W_s, V_s)\right).$$

We will use the notation $(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j})$ to represent a point $p \in T^*R$, where

$$\mathbf{x} = (x_a \in \mathrm{Hom}_{\mathbb{C}}(V_{ta}, V_{ha}))_{a\in Q_1},$$
$$\bar{\mathbf{x}} = (x_{a^*} \in \mathrm{Hom}_{\mathbb{C}}(V_{ha}, V_{ta}))_{a\in Q_1},$$
$$\mathbf{i} = (i_s \in \mathrm{Hom}_{\mathbb{C}}(V_s, W_s))_{s\in Q_0} \text{ and}$$
$$\mathbf{j} = (j_s \in \mathrm{Hom}_{\mathbb{C}}(W_s, V_s))_{s\in Q_0}.$$

The reductive group $G := \prod_{i=1}^{n} GL(V_i)$ naturally acts on R. We are interested in the induced Hamiltonian action of G on T^*R. The corresponding moment map $\mu : T^*R \to \mathfrak{g}^*$ is given by

$$\mu(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j}) = \sum_{a\in Q_1} (x_a x_{a^*} - x_{a^*} x_a) - \sum_{s\in Q_0} j_s i_s. \tag{1.4.1}$$

To define the Nakajima quiver variety $\mathcal{M}^\theta(n, \ell)$, we need to choose some character θ of G. Such θ is uniquely determined by an n-tuple of integers, i.e. by $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{Z}^n$ we understand the character θ as a map $(g_1, \dots, g_n) \mapsto \prod_{s=1}^{n} \det(g_s)^{\theta_s}$, where $g_s \in GL(V_s)$.

Definition 1.4.2. The GIT quotient $\mathcal{M}^\theta_\lambda(Q, v, w) := \mu^{-1}(\lambda)^{\theta-ss}//^\theta G$ is called the *Nakajima quiver variety* with parameter θ, where by λ we understand the scalar matrix $\lambda \cdot \mathrm{Id} \in \mathfrak{g}^*$. In what follows $\theta(v)$ will denote the dot product $\sum_{s=1}^{n} \theta_s \cdot v_s$.

The variety $\mathcal{M}^0_\lambda(Q, v, w)$ is affine and there is a projective morphism $\rho : \mathcal{M}^\theta_\lambda(Q, v, w) \to \mathcal{M}^0_\lambda(Q, v, w)$, which is a symplectic resolution for generic θ.

Remark 1.4.3. In the setup of the current section an application of the Hilbert-Mumford criterion (see Theorem 1.3.4) shows that the θ-semistable locus admits the following natural description (see Lemma 3.8 in [Nak98] for details). A quadruple $(\mathbf{x}, \bar{\mathbf{x}}, \mathbf{i}, \mathbf{j}) \in \mu^{-1}(\lambda)$ is θ-semistable if and only if the following holds: for any collection of vector subspaces $S = (S_t)_{t\in Q_0} \subset V = (V_t)_{t\in Q_0}$, which is stable under the maps $\mathbf{x}, \bar{\mathbf{x}}$, we have

$$S_t \subset \mathrm{Ker}\, j_t \ \forall t \in Q_0 \Rightarrow \theta(\mathfrak{s}) \leq 0,$$
$$S_t \supset \mathrm{Im}\, i_t \ \forall t \in Q_0 \Rightarrow \theta(\mathfrak{s}) \leq \theta(v),$$

where $\mathfrak{s}$ is the dimension vector of S.

In particular, the above implies

$$S_t \subset \mathrm{Ker}\, j_t \ \forall t \in Q_0 \Rightarrow S = 0, \text{ if } \theta_t > 0 \ \forall t,$$

$$S_t \supset \mathrm{Im}\, i_t \ \forall t \in Q_0 \Rightarrow S = V, \text{ if } \theta_t < 0 \ \forall t.$$

Next we describe quiver varieties associated to some quivers (recall Remark 1.2.7).

Example 1.4.4. $Q = \bullet$.

First consider the quiver with one vertex and no arrows. In this case $R = \mathrm{Hom}_{\mathbb{C}}(V, W)$, $G = GL(V)$ and $T^*R = \mathrm{Hom}_{\mathbb{C}}(V, W) \oplus \mathrm{Hom}_{\mathbb{C}}(W, V)$. The moment map is $\mu(i, j) = -ji$.

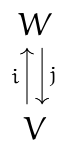


Figure 1.1: Quiver variety for $Q = \bullet$.

The set $\mu^{-1}(\lambda)$ consists of pairs $\{i, j \mid ji = -\lambda\}$, hence, $\mu^{-1}(\lambda) = \varnothing$ if and only if $\lambda \neq 0$ and $v > w$.

First we study the case $\lambda \neq 0$. It follows immediately from Remark 1.4.3 that any point of $\mu^{-1}(\lambda)$ is θ-semistable for any choice of θ. Hence, $\mu^{-1}(\lambda) = \mu^{-1}(\lambda)^{\theta-ss}$ and consists of pairs (i, j) which can be written in the block form as $i = \begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix}$ and $j = \begin{pmatrix} -\lambda \cdot \mathrm{Id} & 0 \end{pmatrix}$ up to the action of $GL(W)$. Notice that the product ji is invariant under the action of $GL(W)$, so $\mu^{-1}(\lambda)$ can be identified with the $GL(W)$-orbit of $ij = \begin{pmatrix} -\lambda \cdot \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}$, with the action via conjugation.

Let $\theta > 0$, then due to Remark 1.4.3, $\mu^{-1}(0)^{\theta-ss}$ is formed by $\{i, j \mid ji = 0\}$ with j injective. The choice of such a pair (i, j) is equivalent to a choice of a subspace $V \subset W$ and a map in $\mathrm{Hom}(W/V, V)$, which is naturally an element of the cotangent bundle $T^*Gr(v, w)$. We conclude that $\mathcal{M}_0^\theta(Q, v, w) \simeq T^*Gr(v, w)$.

In case $\theta < 0$ Remark 1.4.3 asserts the surjectivity of j. Since $ji = 0$, the image of i must be contained in the kernel of j, and the latter is isomorphic to W/V as j is surjective. So j is uniquely determined by its kernel, a $(w - v)$-dimensional subspace of W, while $i \in \mathrm{Hom}(V, W/V)$. This allows to identify $\mathcal{M}_0^\theta(Q, v, w)$ with $T^*Gr(w - v, w) \simeq T^*Gr(v, w)$.

It remains to see what happens when $\theta = \lambda = 0$. Notice that the product ij is G-invariant and $(ij)^2 = 0$, since ji vanishes. Construct the map $\phi : M_0^0(Q, v, w) \to \mathfrak{gl}(W)$ by setting $\phi(i, j) = ij$. Furthermore, the ring of invariants $\mathbb{C}[i, j]^G$ is generated by the matrix elements of the product ij. Writing $W = \ker j \oplus W'$ and observing that $ji = 0$ implies $\mathrm{im}\, ij \subset \ker j$, we conclude that $\mathrm{rk}(ij) \leq \min\left(v, \left\lfloor \frac{w}{2} \right\rfloor\right)$. Denote the number $\min\left(v, \left\lfloor \frac{w}{2} \right\rfloor\right)$ by k, then the image of ϕ consists of $A \in \mathfrak{gl}(W)$, s.t.

1. $A^2 = 0$ and

2. $\mathrm{rk}(A) \leq k$.

Consider a nilpotent matrix x in $\mathfrak{gl}(W)$, whose Jordan canonical form consists of k blocks of size 2×2 and $w - 2k$ blocks of size 1×1. Matrices A in the image of ϕ are in $\overline{\mathcal{O}}_x$, the closure of the orbit of x. We find that $M_0^0(Q, v, w)$ is isomorphic to $\overline{\mathcal{O}}_x$.

Remark 1.4.5. This example illustrates some basic tools for studying quiver varieties. These varieties provide a unified framework for working with many symplectic varieties and establish certain properties for all of them simultaneously.

Example 1.4.6. $Q = \underset{s}{\bullet} \longleftarrow \underset{s-1}{\bullet} \longleftarrow \underset{s-2}{\bullet} \quad \cdots \quad \underset{2}{\bullet} \longleftarrow \underset{1}{\bullet}$

We generalize the previous example and choose a Dynkin quiver of type A_s. Take an arbitrary dimension vector v and w with $w_1 = \ldots = w_{s-1} = 0$ and a stability condition $\theta = (\theta_1, \ldots, \theta_s)$ with all $\theta_t > 0$.

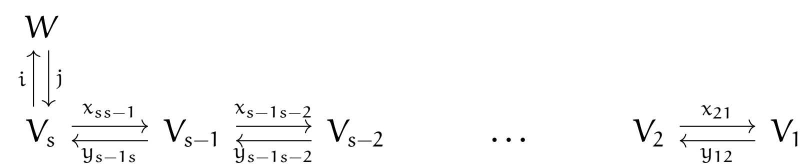
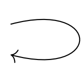


Figure 1.2: Quiver variety for $Q = A_s$.

Then $M_0^\theta(Q, v, w)$ is the cotangent bundle of the partial flag variety $\mathcal{F}\ell(v_1, \ldots, v_s; w)$ (or empty if $v_i > v_{i+1}$ for some i or $v_s > w_s$). On the other hand $M_0^0(Q, v, w) \simeq \overline{\mathcal{O}}_x$, where $x \in \mathfrak{gl}(W)$ is a nilpotent element, having blocks of sizes $v_1, v_2 - v_1, \ldots, v_s - v_{s-1}, w - v_s$ in its Jordan canonical form. The map

$$\rho : M_0^\theta(Q, v, w) = T^*(G/P) \to \overline{\mathcal{O}}_x = M_0^0(Q, v, w)$$

is the Springer resolution (see Lemma 15 in [Maf05] for details).

Example 1.4.7. $Q = \bullet$ 

The next quiver is $Q = \widetilde{A}_0$, the quiver with one vertex and a single loop. The moment equation (1.4.1) simplifies to

$$\mu(x, y, i, j) = [x, y] - ji \tag{1.4.2}$$

Let us start with the case $\lambda = 0$ and $w = 1$. Then one can deduce from equation (1.4.2) that $i = 0$, hence, x and y commute. If $\theta = 0$, by Definition 1.4.2 $M_0^0(Q, v, 1) =$

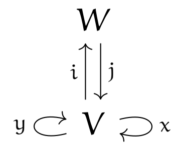


Figure 1.3: Quiver variety for $Q = \widetilde{A}_0$.

$\mathsf{Spec}(\mathbb{C}[\mu^{-1}(0)]^G)$. The latter can be seen to be $\mathbb{C}[\lambda_1, \dots, \lambda_\nu, \mu_1, \dots, \mu_\nu]^{\mathfrak{S}_\nu}$ with $\lambda_1, \dots, \lambda_\nu$ and $\mu_1, \dots, \mu_\nu$ the eigenvalues of x and y, respectively (invariants are taken for the diagonal action of $\mathfrak{S}_\nu$ on $\mathbb{C}^{2\nu}$). This allows to establish the isomorphism $M_0^0(Q, \nu, 1) \simeq \mathsf{Sym}^\nu(\mathbb{C}^2)$.

Next we show how to construct mutually inverse maps between $M_0^\theta(Q, \nu, 1)$ with $\theta > 0$ and the Hilbert scheme of ν points on $\mathbb{C}^2$. The choice of θ implies that any $0 \neq v \in \mathrm{im}\, j$ is a cyclic vector, i.e. generates V under the action of x and y. Consider a point $p = (x, y, i, j) \in M_0^\theta(Q, \nu, 1)$ and let $I_p \subset \mathbb{C}[x, y]$ be the ideal generated by polynomials $\{f(x, y) \mid f(x, y) \cdot v = 0\}$. As the dimension of the quotient $\mathbb{C}[x, y]/I_p$ is equal to ν, we see that $I_p \in \mathsf{Hilb}_\nu(\mathbb{C}^2)$. On the other hand, starting with an ideal $I \in \mathsf{Hilb}_\nu(\mathbb{C}^2)$, identify V with $\mathbb{C}[x, y]/I$. The action of operators x and y comes from multiplication by x and y on $\mathbb{C}[x, y]/I$, while $i = 0$ and j is uniquely defined by setting $j(1) = v$, where v corresponds to $\overline{1}$, the image of 1 in $\mathbb{C}[x, y]/I$.

The varieties $M_0^\theta(Q, \nu, 1)$ and $M_0^{-\theta}(Q, \nu, 1)$ are symplectomorphic via

$$\gamma : (x, y, i, j) \mapsto (y^*, x^*, j^*, -i^*).$$

Consequently, $M_0^{-\theta}(Q, \nu, 1)$ is a realization of the Hilbert scheme $\mathsf{Hilb}_\nu(\mathbb{C}^2)$ as well (see Chapter 1.4 of [Nak99] for a more detailed exposition).

Remark 1.4.8. The variety $M_\lambda^0(Q, \nu, 1)$ with $\lambda \neq 0$ is isomorphic to the Calogero-Moser space. This is the system of ν distinct points of unit mass with coordinates $x_1, \dots, x_\nu$ on the line. The pairwise interactions are governed by the potentials of the form $U(x_i, x_j) = \frac{1}{(x_i - x_j)^2}$. The total potential of the system is $U = \sum\limits_{1 \le i < j \le \nu} \frac{1}{(x_i - x_j)^2}$, while the kinetic energy is $K = \frac{1}{2} \sum\limits_{i=1}^{\nu} (\dot{x}_i)^2$ giving rise to the Hamiltonian $H = K + U = \frac{1}{2} \sum\limits_{i=1}^{\nu} (\dot{x}_i)^2 + \sum\limits_{1 \le i < j \le \nu} \frac{1}{(x_i - x_j)^2}$, where each $\dot{x}_i$ is the derivative of the corresponding coordinate x_i with respect to time and treated as an independent variable. Let $\triangle = \{(x_1, \dots, x_\nu) \mid x_i = x_j\}$ for some $i \neq j$ be the diagonal in $\mathbb{C}^\nu$. Since the points are indistinguishable, i.e. we consider unordered collections, the configuration space can be shown to be $C = (T^*(\mathbb{C}^\nu) \setminus \triangle)/\mathfrak{S}_\nu$. The idea of Kazhdan, Kostant and Sternberg (see [KKS78]) was to encode the coordinates of a point p in the phase space C by two matrices X_p, Y_p given by

$$X_p = \mathrm{diag}(x_1, \dots, x_\nu) \text{ and}$$

$$Y_p = (y_{ij})$$

with the diagonal entries $y_{ii} := \dot{x}_i$ and off-diagonal $y_{ij} := \frac{1}{2}(x_i - x_j)^2$. In particular, the Hamiltonian takes the form $H = \frac{1}{2}\mathrm{tr}(Y_p)^2$. Notice that the commutator $[X_p, Y_p] + I$ is the matrix with all entries equal to 1. Let $\mathcal{O}_L$ be the orbit of the matrix $L = \mathrm{diag}(-1, -1, \dots, -1, n-1)$, i.e. the set of traceless matrices A with $\mathrm{rk}(A + I) = 1$. Then the phase space C is $\mathcal{O}_L$, the space of conjugacy classes of pairs of $\nu \times \nu$ matrices (X, Y) such that the matrix $XY - YX + I$ is of rank 1. It is not hard to establish that the latter is isomorphic to $M^0_\lambda(Q, \nu, 1)$.

Remark 1.4.9. The variety $M^\theta_0(Q, \nu, w)$ with $w > 1$ is isomorphic to the Gieseker moduli space. This is the moduli space of rank w degree ν coherent torsion-free sheaves E on $\mathbb{P}^2$ with fixed trivialization on the line $\ell_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2\}$, i.e. $E_{\ell_\infty} \cong \mathcal{O}^{\oplus w}$ (see Chapter 2 of [Nak99]).

Example 1.4.10. We conclude this section with the description of quiver varieties associated to affine Dynkin diagrams. The extended vertex will have index 0. As in Examples 1.2.2 and 1.2.3 of Section 1.2, let $\Gamma \subset SL_2(\mathbb{C})$ be a finite subgroup and Q_Γ the corresponding McKay quiver. This is the graph with vertices corresponding to irreducible representations of Γ and the number of edges with $t(a) = i$ and $h(a) = j$ is equal to the dimension of $\mathrm{Hom}_\Gamma(S_i \otimes \mathbb{C}^2, S_j)$, where S_i and S_j are irreducibles corresponding to i and j, while $\mathbb{C}^2$ is the standard representation of Γ as a subgroup of $SL_2(\mathbb{C})$. It follows from the classification of finite subgroups of $SL_2(\mathbb{C})$ that the quiver Q_Γ is the double of an affine Dynkin diagram, which, slightly abusing notation, we will refer to as Q_Γ as well. Consider the dimension vectors $\nu = n\nu_0$ for ν_0 the minimal positive imaginary root of the affine root lattice corresponding to $\mathfrak{g}_{Q_\Gamma}$ and $w = (1, 0, \dots, 0)$. Let $Y_0 = \mathbb{C}^2/\Gamma$ and $\rho : Y \to Y_0$ be its crepant resolution. Consider $X_0 := \mathrm{Sym}^n Y_0$ and the resolution $X := \mathrm{Hilb}^n Y$, where $\mathrm{Sym}^n Y_0$ is the symmetric variety of unordered n-tuples of points on the singular space Y_0 and $\mathrm{Hilb}^n Y$ is the Hilbert scheme of n points on the crepant resolution Y. It can be shown that for $\theta = \theta^+$ or θ^- (any characters with all components positive or negative) the following diagram is commutative with the horizontal maps being isomorphisms (see [Kuz07] and Chapter 4 of [Nak99]):

$$\begin{array}{ccc} M^\theta_0(Q_\Gamma, \nu, w) & \xrightarrow{\eta_1} & \mathrm{Hilb}^n Y \\ \big\downarrow \tilde{\rho} & & \big\downarrow \rho \\ M^0_0(Q_\Gamma, \nu, w) & \xrightarrow{\eta_2} & \mathrm{Sym}^n Y_0. \end{array}$$

1.5 Category $\mathcal{O}$ for the quantizations of quiver varieties with $Q = B_\ell$

We study the Nakajima quiver variety with underlying quiver Q, which has one vertex, ℓ loops, where $\ell \in \mathbb{Z}_{\geq 0}$ and a one-dimensional framing. This variety admits the following description. One starts with a vector space V of dimension n and considers the space $R :=$

$\mathfrak{gl}(V)^{\oplus \ell} \oplus V^*$, which has a natural $G := GL(V)$ action. The identification of $\mathfrak{g} := \mathfrak{gl}(V)$ with $\mathfrak{g}^*$ via the trace form enables to identify the cotangent bundle T^*R with $\mathfrak{gl}(V)^{\oplus 2\ell} \oplus V^* \oplus V$. Next notice that T^*R is a symplectic vector space with a Hamiltonian action of G. The corresponding moment map is given by

$$\mu(X_1, \dots, X_\ell, Y_1, \dots, Y_\ell, i, j) = \sum_{k=0}^{\ell} [X_k, Y_k] - ji. \tag{1.5.1}$$

To define the Nakajima quiver variety $\mathcal{M}^\theta(n, \ell)$, we need to choose some character θ of G. It is known that θ an integral power of the determinant, i.e. $\theta = \det^k$ for some $k \in \mathbb{Z}$.

Definition 1.5.1. The Nakajima quiver variety $\mathcal{M}^\theta(n, \ell)$ is the GIT quotient $\mu^{-1}(0)^{\theta-ss}//^\theta G$. In particular, $\mathcal{M}(n, \ell) = \mu^{-1}(0)//G := \operatorname{Spec} \mathbb{C}[\mu^{-1}(0)]^G$.

The torus $T = (\mathbb{C}^*)^\ell$ acts on R by rescaling $X_1, \dots, X_\ell$. This naturally gives rise to an action on T^*R. This action is Hamiltonian and commutes with the action of G and, therefore, descends to $\mathcal{M}(n, \ell)$ and $\mathcal{M}^\theta(n, \ell)$. The action of $s \in \mathbb{S}$ is given by multiplication of all the components of $x \in T^*R$ by s^{-1}. Similarly, it commutes with the action of G and descends to $\mathcal{M}(n, \ell)$ and $\mathcal{M}^\theta(n, \ell)$.

For any $\theta \neq 0$ the action of G on $\mu^{-1}(0)^{\theta-ss}$ is free. This implies that the variety $\mathcal{M}^\theta(n, \ell)$ is smooth and symplectic and is known to be a symplectic resolution of the normal Poisson variety $\mathcal{M}(n, \ell)$. We denote by ρ the corresponding map $\rho : \mathcal{M}^\theta(n, \ell) \to \mathcal{M}(n, \ell)$. It is a conical symplectic resolution.

Set $\overline{R} = \mathfrak{sl}(V)^{\oplus \ell} \oplus V^*$ and let $\overline{\mathcal{M}}(n, \ell)$ be the affine variety $\mu^{-1}(0)//G$, where slightly abusing notation, we denote by μ the moment map for the Hamiltonian action of G on $T^*\overline{R}$. Similarly, we set $\overline{\mathcal{M}}^\theta(n, \ell) := \mu^{-1}(0)^{\theta-ss}//^\theta G$. Next we describe quantizations of $\overline{\mathcal{M}}(n, \ell)$. Denote the ring of differential operators on $\overline{R}$ by $D(\overline{R})$.

Definition 1.5.2. A G-equivariant linear map $\Phi : \mathfrak{g} \to D(\overline{R})$, satisfying $[\Phi(x), a] = x_{\overline{R}}(a)$ for any $x \in \mathfrak{g}$ and $a \in D(\overline{R})$ is called a *quantum comoment map*.

Remark 1.5.3. The quantum comoment map Φ is defined up to adding a character $\lambda : \mathfrak{g} \to \mathbb{C}$.

Notice that we can identify $D(\overline{R})$ with $D(\overline{R}^*)$ via the Fourier transform sending $\partial_r \in D(\overline{R})$ to the function $r \in D(\overline{R}^*)$ and $r^* \in D(\overline{R})$ to $-\partial_{r^*} \in D(\overline{R}^*)$. Thus defined isomorphism $D(\overline{R}) \to D(\overline{R}^*)$ allows to consider two quantum comoment maps $\Phi, \widetilde{\Phi} : \mathfrak{gl}(V) \to D(\overline{R})$ sending $x \in \mathfrak{g}$ to the corresponding vector field $x_{\overline{R}}$ or $x_{\overline{R}^*}$. Now define the *symmetrized quantum comoment map* to be $\Phi^{sym} := \frac{\Phi + \widetilde{\Phi}}{2}$. A direct computation shows that $\Phi^{sym}(x) = \Phi(x) - \zeta(x)$, where ζ is half the character of the action of G on $\Lambda^{top} R$. For our quiver Q with one-dimensional framing $\zeta(x) = \frac{1}{2}\operatorname{tr}(x)$.

Next we take a character λ of $\mathfrak{g}$ and consider the quantizations

$$\overline{\mathcal{A}}_\lambda(n,\ell) := (D(\overline{R})/[D(\overline{R})\{\Phi(x)-\lambda(x), x\in\mathfrak{g}\}])^G,$$

$$\overline{\mathcal{A}}_\lambda^{sym}(n,\ell) := (D(\overline{R})/[D(\overline{R})\{\Phi^{sym}(x)-\lambda(x), x\in\mathfrak{g}\}])^G.$$

The filtration on $\overline{\mathcal{A}}_\lambda(n,\ell)$ is induced from the Bernstein filtration on $D(\overline{R})$ (here $\deg \overline{R} = \deg \overline{R}^* = 1$). Recall that $\mathbb{C}[\overline{\mathcal{M}}(n,\ell)] = (\mathbb{C}[T^*\overline{R}]/I)^G$, where $I := \{\mu^*(\xi), \xi\in\mathfrak{g}\}$ is the ideal generated by the image of $\mathfrak{g}$ under the comoment map, and denote $\mathcal{I}_\lambda := \{\Phi(x)-\lambda(x), x\in \mathfrak{gl}(V)\}$. The surjectivity of the natural map $\mathbb{C}[\overline{\mathcal{M}}(n,\ell)] \to \operatorname{gr}\overline{\mathcal{A}}_\lambda(n,\ell)$ follows from the containment $I\subset \operatorname{gr}\mathcal{I}_\lambda$. The reverse containment of ideals follows from the regularity of the sequence $\mu^*(\xi_1),\ldots,\mu^*(\xi_{n^2})$, where $\xi_1,\ldots,\xi_{n^2}$ is some basis for $\mathfrak{g}$. The regularity of the sequence is equivalent to flatness of the moment map μ.

We notice that the difference between $\overline{\mathcal{A}}_\lambda(n,\ell)$ and the algebra $\mathcal{A}_\lambda(n,\ell)$ (constructed analogously for $R = \mathfrak{gl}(V)^{\oplus\ell}\oplus V^*$) is that $\mathcal{A}_\lambda(n,\ell) = D(\mathbb{C}^\ell)\otimes\overline{\mathcal{A}}_\lambda(n,\ell)$. Thus, some questions about representation theory of $\overline{\mathcal{A}}_\lambda(n,\ell)$ reduce to analogous ones for $\mathcal{A}_\lambda(n,\ell)$.

The quantizations $\overline{\mathcal{A}}^\theta$ of $\overline{\mathcal{M}}^\theta(n,\ell)$ are parameterized (up to isomorphism) by the points of $H^2(\overline{\mathcal{M}}^\theta(n,\ell))\simeq\mathbb{C}$ (see [BK04]). The quantization corresponding to λ will be denoted by $\overline{\mathcal{A}}^\theta_\lambda$.

In order to produce a quantization $\overline{\mathcal{A}}^\theta_\lambda$ of $\overline{\mathcal{M}}^\theta(n,\ell)$ corresponding to a character λ of $\mathfrak{gl}(V)$, one needs to describe the sections $\overline{\mathcal{A}}^\theta_\lambda(U)$ for any $U\subset\overline{\mathcal{M}}^\theta(n,\ell)$ open in the conical topology and check the sheaf axioms. The base of conical topology is formed by $V_f := (T^*\overline{R})_f//G$ with $\{f\in\mathbb{C}[T^*\overline{R}] | f(g^{-1}(x)) = \theta^n(g)f(x)\ \forall g\in G\}$ for some $n\in\mathbb{Z}_{\geq 0}$. As V_f is an affine variety, define the sections over V_f to be $\overline{\mathcal{A}}^\theta_\lambda(V_f) := (D(V_f)/[D(V_f)\{\Phi(x)-\lambda(x), x\in\mathfrak{g}\}])^G$. The restriction maps $\overline{\mathcal{A}}^\theta_\lambda(V_f)\to\overline{\mathcal{A}}^\theta_\lambda(V_{fg})$ are induced from the inclusions $D(V_f)\hookrightarrow D(V_{fg})$. In other words $\overline{\mathcal{A}}^\theta_\lambda = (D(\overline{R})/[D(\overline{R})\{\Phi(x)-\lambda(x), x\in\mathfrak{g}\}|_{(T^*\overline{R})^{\theta-ss}}])^G$.

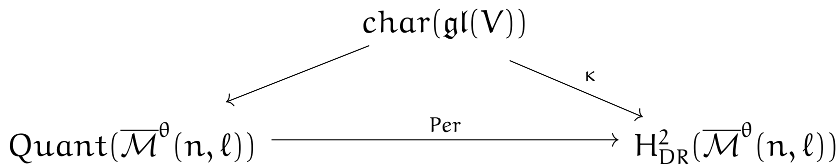
There is a natural period map Per from the set $\mathsf{Quant}(\overline{\mathcal{M}}^\theta(n,\ell))$ of isomorphism classes of quantizations of $\overline{\mathcal{M}}^\theta(n,\ell)$ to $H^2_{DR}(\overline{\mathcal{M}}^\theta(n,\ell))\simeq\mathbb{C}$ (the isomorphism will be established in Corollary 2.2.4), which we now describe explicitly. Starting with a character χ of G, one can produce a line bundle L_χ on $\overline{\mathcal{M}}^\theta(n,\ell)$. To describe this bundle, start with the trivial line bundle on $\mu^{-1}(0)^{\theta-ss}\subset T^*\overline{R}$ with the action of G on $\mu^{-1}(0)^{\theta-ss}\times\mathbb{C}$ given by $g\cdot(x,z) = (g\cdot x, \chi^{-1}(g)z)$. The total space of the line bundle L_χ is the quotient $(\mu^{-1}(0)^{\theta-ss}\times\mathbb{C})/G$ and the sections over an open subset $U\subset\overline{\mathcal{M}}^\theta(n,\ell)$ are

$$\mathbb{C}[\pi^{-1}(U)]^{G,\chi} := \{f\in\mathbb{C}[\pi^{-1}(U)] | f(g^{-1}(x)) = \chi(g)f(x)\},$$

where $\pi : \mu^{-1}(0)^{\theta-ss}\to\overline{\mathcal{M}^\theta(n,\ell)}$ is the quotient morphism.

Extending the map $\chi\mapsto c_1(L_\chi)$, where $c_1(L_\chi)$ is the first Chern class of the line bundle L_χ, by $\mathbb{C}$-linearity allows to produce the map $\kappa : \mathsf{char}(\mathfrak{g})\simeq\mathbb{C}\to\mathbb{C}\simeq H^2_{DR}(\overline{\mathcal{M}}^\theta(n,\ell))$.

The map $\mathsf{Per} : \mathsf{Quant}(\overline{\mathcal{M}}^{\theta}(n,\ell)) \to H^2_{DR}(\overline{\mathcal{M}}^{\theta}(n,\ell))$ sends $\overline{\mathcal{A}}^{\theta}_{\lambda}$ to $\kappa(\lambda + \frac{1}{2})$. As the map $\overline{\rho} : \overline{\mathcal{M}}^{\theta}(n,\ell) \to \overline{\mathcal{M}}(n,\ell)$ is a symplectic resolution of singularities, the Grauert-Riemenschneider theorem implies that $H^i(\overline{\mathcal{M}}^{\theta}(n,\ell), \mathcal{O}_{\overline{\mathcal{M}}^{\theta}(n,\ell)}) = 0$ for all $i > 0$. This, in turn, allows to conclude that the map $\mathsf{Per} : \mathsf{Quant}(\overline{\mathcal{M}}^{\theta}(n,\ell)) \to H^2_{DR}(\overline{\mathcal{M}}^{\theta}(n,\ell))$ is an isomorphism (see [BK04] and Section 2 of [Los12]).



Remark 1.5.4. The period of the quantization $\overline{\mathcal{A}}_{\lambda}(n,\ell)^{sym}$ is equal to λ.

Notation. We will denote by $\overline{\mathcal{A}}_{\lambda}$ -mod the category of finitely generated $\overline{\mathcal{A}}_{\lambda}$-modules and by $\overline{\mathcal{A}}^{\theta}_{\lambda}$ -mod - the category of coherent $\overline{\mathcal{A}}^{\theta}_{\lambda}$ - modules.

There are two basic functors between the categories of $\overline{\mathcal{A}_{\lambda}}$ -mod and $\overline{\mathcal{A}_{\lambda}}^{\theta}$ -mod and the corresponding derived categories:

$$\mathcal{A}_{\lambda}\text{-mod} \underset{\Gamma_{\lambda}}{\overset{\mathrm{Loc}^{\theta}_{\lambda}}{\rightleftarrows}} \mathcal{A}^{\theta}_{\lambda}\text{-mod}\,.$$

$$D^b(\overline{\mathcal{A}}_{\lambda}\text{-mod}) \underset{R\Gamma_{\lambda}}{\overset{\mathrm{LLoc}^{\theta}_{\lambda}}{\rightleftarrows}} D^b(\overline{\mathcal{A}}^{\theta}_{\lambda}\text{-mod}).$$

Definition 1.5.5. If the functors $\mathrm{Loc}^{\theta}_{\lambda}, \Gamma_{\lambda}$ $(\mathrm{LLoc}^{\theta}_{\lambda}, R\Gamma_{\lambda})$ are mutually inverse equivalences, we say that *abelian (derived) localization holds* for the pair (λ, θ).

Remark 1.5.6. The main Theorem of [MN14] asserts that the derived equivalence holds if and only if the homological dimension of the algebra $\overline{\mathcal{A}}_{\lambda}$ is finite.

Remark 1.5.7. The categories $\mathcal{O}_{\nu}(\overline{\mathcal{A}}_{\lambda}(n,\ell))$ and $\mathcal{O}_{\nu}(\mathcal{A}_{\lambda}(n,\ell))$ are, in fact, equivalent. Indeed, recall that $\mathcal{A}_{\lambda}(n,\ell) = D(\mathbb{C}^{\ell}) \otimes \overline{\mathcal{A}}_{\lambda}(n,\ell)$ and let $t_1, \ldots, t_{\ell}$ be the coordinates on $\mathbb{C}^{\ell}$. Then the functor $\mathcal{O}_{\nu}(\overline{\mathcal{A}}_{\lambda}(n,\ell)) \to \mathcal{O}_{\nu}(\mathcal{A}_{\lambda}(n,\ell))$ given by $M \mapsto \mathbb{C}[t_1, \ldots, t_{\ell}] \otimes M$ produces an equivalence of categories. It has a quasi-inverse functor which sends $N \in \mathcal{O}_{\nu}(\mathcal{A}_{\lambda}(n,\ell))$ to the annihilator of $\langle \partial t_1, \ldots, \partial t_{\ell} \rangle$.

Definition 1.5.8. We have the *standardization* and *costandardization* functors $\triangle_{\nu}$ and ∇_{ν} : $\mathcal{C}_{\nu}(\overline{\mathcal{A}}_{\lambda}(n,\ell))\text{-mod} \to \mathcal{O}_{\nu}(\overline{\mathcal{A}}_{\lambda}(n,\ell))$ given by

$$\triangle_{\nu}(N) := \overline{\mathcal{A}}_{\lambda}(n,\ell)/\overline{\mathcal{A}}_{\lambda}(n,\ell)\overline{\mathcal{A}}_{\lambda}^{>0}(n,\ell) \otimes_{\mathcal{C}_{\nu}(\overline{\mathcal{A}}_{\lambda}(n,\ell))} N$$

$$\nabla_{\nu}(N) := \mathrm{Hom}_{\mathcal{C}_{\nu}(\overline{\mathcal{A}}_{\lambda}(n,\ell))}(\overline{\mathcal{A}}_{\lambda}(n,\ell)/\overline{\mathcal{A}}_{\lambda}^{<0}(n,\ell)\overline{\mathcal{A}}_{\lambda}(n,\ell), N).$$

We consider the restricted Hom (w.r.t. the natural grading on $\overline{\mathcal{A}}_\lambda(n,\ell)/\overline{\mathcal{A}}_\lambda^{<0}(n,\ell)\overline{\mathcal{A}}_\lambda(n,\ell)$) in the definition of the operator ∇_ν above.

Definition 1.5.9. Let $\mathcal{C}$ be an abelian, artinian category enriched over $\mathbb{R}$ with simple objects $\{S_\alpha|\alpha \in \mathcal{I}\}$, projective covers $\{P_\alpha|\alpha \in \mathcal{I}\}$, and injective hulls $\{I_\alpha|\alpha \in \mathcal{I}\}$. Let $\preceq$ be a partial order on the index set $\mathcal{I}$. We call $\mathcal{C}$ *highest weight* with respect to this partial order if there is a collection of objects $\{\triangle_\alpha|\alpha \in \mathcal{I}\}$ and epimorphisms $P_\alpha \overset{\Pi_\alpha}{\to} \triangle_\alpha \overset{\pi_\alpha}{\to} S_\alpha$ such that for each $\alpha \in \mathcal{I}$, the following conditions hold:

1. the object $\ker \pi_\alpha$ has a filtration such that each subquotient is isomorphic to S_β for some $\beta \prec \alpha$;
2. the object $\ker \Pi_\alpha$ has a filtration such that each subquotient is isomorphic to $\triangle_\gamma$ for some $\gamma \succ \alpha$.

The objects $\triangle_\alpha$ are called *standard objects*.

The next result can be found in [Los16] (see Proposition 2.2).

Proposition 1.5.10. *Suppose that abelian localization holds and λ is generic (outside some finite set). Choose a generic one-parameter subgroup ν. Then the following is true:*

(1) the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(n,\ell))$ depends only on the chamber of ν;

(2) the natural functor $D^b(\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda((n,\ell))) \to D^b(\overline{\mathcal{A}}_\lambda(n,\ell)\text{-mod})$ is a full embedding;

(3) $\mathcal{C}_\nu(\overline{\mathcal{A}}_\lambda(n,\ell)) = \mathbb{C}[\overline{\mathcal{M}}^\theta(n,\ell)^T]$;

(4) Assume, in addition, that there are finitely many fixed points for the action of ν. The category $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(n,\ell))$ is highest weight with standard objects $\triangle_\nu(p_i)$ and costandard objects $\nabla_\nu(p_i)$ for $p_i \in \mathbb{C}[\overline{\mathcal{M}}^\theta(n,\ell)^T]$.

Remark 1.5.11. The order required for highest weight structure comes from the contraction order on the fixed points. This is the order, in which $p_i \preceq_\nu p_j$ iff $p_i \in \overline{X^\nu_{p_j}}$, where $X^\nu_{p_j} := \{x \in \overline{\mathcal{M}}^\theta(n,\ell) \mid \lim_{t\to 0}\nu(t)x = p_i\}$.

1.6 Main results and structure of the dissertation

We present the most important results of the paper in order of appearance. In Chapter 5 it is established that abelian localization holds for (λ,θ) with $\theta < 0$ and $\lambda < 1-\ell$ or $\theta > 0$ and $\lambda > \ell - 2$ (Theorem 5.0.3). It is shown that if $\lambda \in (-\infty; 1-\ell) \cup (\ell-2; +\infty)$, then the algebra $\overline{\mathcal{A}}_\lambda(2,\ell)$ has finite homological dimension (Corollary 5.0.4). In Chapter 6 we determine the Hom-spaces between standard objects in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))^{\triangle}$ (see Theorem 6.4.1) and compute the multiplicities of simples in standards (Corollary 6.4.2). We show that the algebra $\overline{\mathcal{A}}_\lambda(2,\ell)$ is not of finite homological dimension for $\lambda \in (-\ell;\ell-1)\cap\mathbb{Z}$ or $\lambda = -\frac{1}{2}$ (see Theorem 7.2.1). Finally, the complete form of abelian localization is established in Theorem 7.2.3.

The structure of the paper is as follows. Chapter 2 gives preliminary results on the varieties $\overline{\mathcal{M}}^\theta(n,\ell)$ and the category $\mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,\ell))$. It is shown that $\overline{\mathcal{M}}^\theta(n,\ell)$ has finitely many fixed points w.r.t. the Hamiltonian torus action for $n \leq 3$, the central fiber of the resolution $\bar{\rho} : \overline{\mathcal{M}}^\theta(n,\ell) \to \overline{\mathcal{M}}(n,\ell)$ is of dimension less than $\frac{1}{2}\overline{\mathcal{M}}^\theta(n,\ell)$ for $n,\ell > 1$. From this (using Gabber's theorem) one deduces that there are no finite dimensional $\overline{\mathcal{A}}_\lambda(n,\ell)$-modules with generic ν. Furthermore, the resolutions $\bar{\rho} : \overline{\mathcal{M}}^\theta(n,\ell) \to \overline{\mathcal{M}}(n,\ell)$ serve as counterexamples to Conjecture 1.3.1 in [ES14]. The explanation of this phenomenon concludes the chapter (see Remark 2.2.8 for details).

In Chapter 3, following the recipe of [Nak94], [Nak98] (see also Section 2 of [BL15]), the description of symplectic leaves of $\overline{\mathcal{M}}(n,\ell)$ and slices to points on them for $n = 2,3$ is obtained. One of the two nontrivial slices to $\overline{\mathcal{M}}(2,\ell)$ turns out to be a hypertoric variety. The description of T-fixed points on that slice is provided.

Following the lines of [BLPW12], we give an overview on generalities on hypertoric varieties and categories $\mathcal{O}$ associated to them and provide a description of category $\mathcal{O}$ for the slice (Proposition 4.3.6, Chapter 4).

The next chapter is devoted to the proof of Theorem 5.0.3 and the description of the locus of λ, for which the algebra $\overline{\mathcal{A}}_\lambda(2,\ell)$ has finite homological dimension (Corollary 5.0.4).

Then, using the construction of restriction functor introduced in [BE09] for rational Cherednik algebras (quantizations of the Hilbert scheme of points on $\mathbb{C}^2$) and its generalization for the Gieseker scheme in [Los18], we define a functor $\mathsf{Res} : \mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell)) \to \mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,\ell))$, where $\mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,\ell))$ stands for the category $\mathcal{O}$ for the slice. This functor is exact and faithful on standard objects. It serves as the main ingredient in the proof of Theorem 6.4.1, which appears in Chapter 6.

Chapter 7 is dedicated to the proof of Theorem 7.2.1. The main ingredients required here are the results of McGerty and Nevins from [MN16].

Chapter 2

First results on $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(n, \ell))$

In this chapter we collect some basic information on the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(n, \ell))$. Recall that $\overline{\mathcal{A}}_\lambda(n, \ell) := (D(\overline{R})/[D(\overline{R})\{\Phi(x) - \lambda(x), x \in \mathfrak{g}\}])^G$ stands for the quantization of $\overline{\mathcal{M}}^\theta(n, \ell)$. We fix our choice of character $\theta = \det^{-1}$. As can be inferred from the proof of the following lemma, this choice is generic.

Lemma 2.0.1. *There is an isomorphism* $\overline{\mathcal{A}}_\lambda(n, \ell) \cong \overline{\mathcal{A}}_{-\lambda-1}(n, \ell)$.

Proof. There is a symplectomorphism $\gamma : \overline{\mathcal{M}}^\theta(n, \ell) \simeq \overline{\mathcal{M}}^{-\theta}(n, \ell)$ produced by

$$(X_1, \dots, X_\ell, Y_1, \dots, Y_\ell, i, j) \mapsto (Y_1^*, \dots, Y_\ell^*, -X_1^*, \dots, -X_\ell^*, j^*, -i^*),$$

thus, inducing multiplication by -1 on $H^2(\overline{\mathcal{M}}^\theta(n, \ell), \mathbb{Z})$. As the image of λ under the period map is $\lambda + \frac{1}{2} \in H^2(\overline{\mathcal{M}}^\theta(n, \ell), \mathbb{Z})$, the result follows. $\square$

2.1 T-fixed points

To study the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(n, \ell))$, we first need to obtain some information on the torus fixed points. This is summarized in the theorem below.

Remark 2.1.1. Since the case $\ell = 1$ was studied in [Los18], henceforth we assume $\ell \geq 2$.

Theorem 2.1.2. *The variety* $\overline{\mathcal{M}}^\theta(n, \ell)$ *has finitely many* T*-fixed points if dim*$V \leq 3$.

Proof. Let $\tilde{p} = (X_1, \dots, X_\ell, Y_1, \dots, Y_\ell, i, j) \in \mu^{-1}(0)$ be a point in the preimage of a fixed point $p \in \overline{\mathcal{M}}^\theta(n, \ell)$, then there exists a homomorphism $\eta_p : T \to G$, s.t. the following system of equalities is satisfied ($t = (t_1, \dots t_\ell) \in T$):

$$\begin{cases} t_1 X_1 = \eta_p(t) X_1 \eta_p(t)^{-1} \\ \dots \\ t_\ell X_\ell = \eta_p(t) X_\ell \eta_p(t)^{-1} \\ t_1^{-1} Y_1 = \eta_p(t) Y_1 \eta_p(t)^{-1} \\ \dots \\ t_\ell^{-1} Y_\ell = \eta_p(t) Y_\ell \eta_p(t)^{-1} \\ i = \eta_p(t)^{-1} i \\ j = \eta_p(t) j. \end{cases} \tag{2.1.1}$$

Let $\{\varepsilon_1, \dots, \varepsilon_\ell\}$ be the set of coordinate characters of the torus T, i.e. $\varepsilon_i(t_1, \dots, t_\ell) = t_i$. The weight decomposition of V with respect to η_p is

$$V = \bigoplus_{\chi \in \mathrm{char}(T)} V_\chi,$$

with $V_\chi = \{v \in V \mid \eta_p(t) \cdot v = \chi(t) v\}$. It follows from the system of equations (2.1.1) that $X_i(V_\chi) \subset V_{\chi - \varepsilon_i}$ and, similarly, the Y_i's - to $Y_i(V_\chi) \subset V_{\chi + \varepsilon_i}$ (here multiplication of characters is written additively). As $\mathrm{im}\, j \neq 0$ due to the stability condition it follows from the last equation in (2.1.1) that im $j \in V_0$.

Below we provide a description of the fixed points when $\dim(V) \leq 3$.

Case 1. If $\dim(V) = 1$, the variety $\overline{\mathcal{M}}^\theta(1, \ell)$ is a single point.

Case 2. If $\dim(V) = 2$, we choose a cyclic vector $0 \neq v_0 \in \mathrm{im}\, j$ as the first vector in the basis. Then at least one of the X_k or Y_s must act nontrivially on v_0 and the image is v_1 inside some $V_{\pm \varepsilon_i}$. The vectors v_0 and v_1 already span V as they have different weights and cannot be collinear. We notice that $X_s v_0 = v_1$ or $Y_s v_0 = v_1$ immediately implies $X_{\neq s} v_0 = Y_{\neq s} v_0 = X_{\neq s} v_1 = Y_{\neq s} v_1 = 0$ as all these vectors would lie in weight spaces different from $V_{0,\dots,0}$ and $V_{0,\dots,0,\pm 1_s, 0,\dots,0}$. It remains to notice that equation (1.5.1) becomes $[X_s, Y_s] + ji = 0$, which shows that $X_s \neq 0$ implies $Y_s = 0$ and vice versa. Therefore, there are 2ℓ fixed points: $p_s = (X_{\neq s} = 0, X_s = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, Y_1 = 0, \dots, Y_\ell = 0, i = 0, j = \begin{pmatrix} 1 \\ 0 \end{pmatrix})$, $p_{s+\ell} = (X_1 = 0, \dots, X_\ell = 0, Y_{\neq s} = 0, Y_s = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, i = 0, j = \begin{pmatrix} 1 \\ 0 \end{pmatrix})$, where $s \in \{1, \dots, \ell\}$.

Case 3. Now $\dim(V) = 3$. Again let the cyclic vector $0 \neq v_0 \in \mathrm{im}\, j$ be the first vector in the basis. Now there are the following possibilities ($s, k \in \{1, \dots, \ell\}$):

- for some s, k: $X_s v_0 = v_1 \neq 0$ and $Y_k v_0 = v_2 \neq 0$;
- for some $s \neq k$: $X_s v_0 = v_1 \neq 0$ and $X_k v_0 = v_2 \neq 0$;

- for some $s \neq k$: $Y_s v_0 = v_1 \neq 0$ and $Y_k v_0 = v_2 \neq 0$;
- for some $s \neq k$: $X_s v_0 = v_1 \neq 0$ and $Y_k v_1 = v_2 \neq 0$;
- for some s, k: $X_s v_0 = v_1 \neq 0$ and $X_k v_1 = v_2 \neq 0$;
- for some s, k: $Y_s v_0 = v_1 \neq 0$ and $Y_k v_1 = v_2 \neq 0$;

In each of the cases above the vectors v_0, v_1 and v_2 are linearly independent and span V, while all the remaining X and Y coordinates of p are zero. We verify it when $X_s v_0 = v_1$ and $Y_k v_0 = v_2$, the remaining cases being similar.

First, $X_{\neq k}$ and $Y_{\neq s}$ must be zero, as otherwise there would be vectors with weights different from those of v_0, v_1 and v_2 and, therefore, linearly independent with them. For the same reason $X_k v_0 = X_k v_1 = X_s v_1 = X_s v_2 = Y_s v_0 = Y_s v_2 = Y_k v_1 = Y_k v_2 = 0$. To show $Y_s v_1 = 0$, we notice that equation (1.5.1) reduces to $[X_s, Y_s] + [X_k, Y_k] + ji = 0$. Applying to v_1, we get

$$X_s Y_s v_1 + ji v_1 = 0,$$

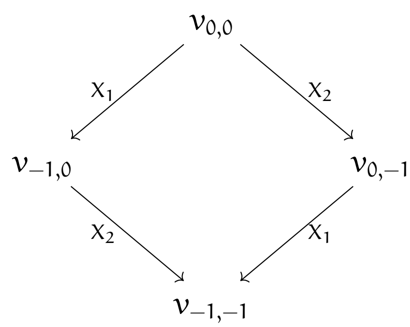
and notice that $X_s Y_s v_1 \in V_{0,\dots,-1_s,\dots,0}$, while $ji v_1 \in V_{0,\dots,0_s,\dots,0}$. Thus, $ji v_1 = 0$ and $X_s Y_s v_1 = 0$ separately, so $Y_s v_1 = 0$ and $Y_s = 0$. It is analogous to show that $X_k = 0$.

$$\begin{array}{ccc} v_0 & \xrightarrow{X_s} & v_1 \\ \big\downarrow{\scriptstyle Y_k} & & \\ v_2 & & \end{array}$$

□

Remark 2.1.3. Next we show that when $n = 4, \ell = 2$ the subvariety of fixed points contains a copy of the projective line $\mathbb{CP}^1 = \mathbb{C}[\mu_1 : \mu_2]$. The operators below are presented in a weight basis with the first vector of weight $(0, 0)$, the second $(-1, 0)$, the third $(0, -1)$ and the fourth $(-1, -1)$, the action of the subgroup of G, preserving the weight decomposition, can only simultaneously rescale μ_1 and μ_2. The subvariety is given by

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \end{pmatrix}, \; Y_1 = Y_2 = 0, \; i = 0, \; j = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$



Remark 2.1.4. Both varieties $\overline{\mathcal{M}}^{\theta}(1,\ell)$ and $\overline{\mathcal{M}}(1,\ell)$ consist of a single point, therefore, we proceed with the case $\dim V = 2$.

The following fact is a particular case of the result established in Section 5 of [Los17] and will be used in the proof of Theorem 6.4.1. Suppose $\tilde{\nu}$ lies in the face of a chamber containing ν. Then $\triangle_{\tilde{\nu}}$ restricts to an exact functor $\mathcal{O}_{\nu}(C_{\tilde{\nu}}(\overline{\mathcal{A}}_{\lambda}(2,\ell))) \to \mathcal{O}_{\nu}(\overline{\mathcal{A}}_{\lambda}(2,\ell))$. Moreover, there is an isomorphism of functors $\triangle_{\nu} = \triangle_{\tilde{\nu}} \circ \underline{\triangle}$, where $\triangle_{\tilde{\nu}} : C_{\tilde{\nu}}(\mathcal{A}_{\lambda})\text{-mod} \to \mathcal{A}_{\lambda}\text{-mod}$, $\underline{\triangle} : C_{\nu}(\mathcal{A}_{\lambda})\text{-mod} \to C_{\tilde{\nu}}(\mathcal{A}_{\lambda})\text{-mod}$ and $\triangle_{\nu}$ is the standardization functor given by Definition 1.5.8. This allows to study the functor $\triangle_{\tilde{\nu}}$ in stages.

We start by describing the fixed points loci $\overline{\mathcal{M}}^{\theta}(2,\ell)^{\nu(\mathbb{C}^*)}$ for certain one-parameter subgroups $\tilde{\nu} : \mathbb{C}^* \to T$ and the corresponding algebras $C_{\tilde{\nu}}(\mathcal{A}_{\lambda})$.

Theorem 2.1.5. *The fixed point set* $\overline{\mathcal{M}}^{\theta}(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)}$ *for* $\tilde{\nu} : \mathbb{C}^* \to T$ *with* $\tilde{\nu}(t) = (t^d, 1, \ldots, 1)$ *and* $d \in \mathbb{Z}_{>0}$ *is* $\overline{\mathcal{M}}^{\theta}(2,\ell-1) \amalg \mathbb{C}^{2\ell-2} \amalg \mathbb{C}^{2\ell-2}$.

Proof. The subset $\overline{\mathcal{M}}^{\theta}(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)}$ is formed by the points $p = (X_1, \ldots, X_\ell, Y_1, \ldots, Y_\ell, i, j)$ which satisfy the system of equations (2.1.2) below. These equations are obtained analogously to those in (2.1.1) with η_p standing for the composition $\mathbb{C}^* \xrightarrow{\tilde{\nu}} T \to G$, s.t.

$$\begin{cases} t^d X_1 = \eta_p(t) X_1 \eta_p(t)^{-1} \\ X_2 = \eta_p(t) X_2 \eta_p(t)^{-1} \\ \ldots \\ X_\ell = \eta_p(t) X_\ell \eta_p(t)^{-1} \\ t^{-d} Y_1 = \eta_p(t) Y_1 \eta_p(t)^{-1} \\ Y_2 = \eta_p(t) Y_2 \eta_p(t)^{-1} \\ \ldots \\ Y_\ell = \eta_p(t) Y_\ell \eta_p(t)^{-1} \\ i = \eta_p(t)^{-1} i \\ j = \eta_p(t) j. \end{cases} \tag{2.1.2}$$

and η_p is the same for points in the same connected component. Let $\tilde{\eta}_p$ be the one-parameter subgroup $(\tilde{\nu}, \eta_p) \subset T \times G$. The irreducible components of $\overline{\mathcal{M}}^{\theta}(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)}$ can be recovered as the Hamiltonian reductions of the vector space $(T^*\overline{R})^{\tilde{\eta}_p}$ with respect to the action of Z_{η_p} (the centralizer of η_p in G).

There are two possible cases. First, if $X_1 = Y_1 = 0$, it follows from (2.1.2) and our choice of the stability condition that the entire 2-dimensional vector space V is of weight 0 with respect to $\eta_p(t)$ and, hence, $\eta_p(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Such points form the fixed component $\overline{\mathcal{M}}^\theta(2, \ell-1) \subset \overline{\mathcal{M}}^\theta(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)}$. Indeed, $(T^*\overline{R})^{\tilde{\eta}_p} = \mathfrak{sl}_2^{\oplus 2\ell-2} \oplus V \oplus V^*$ and $Z_{\eta_p} = G$.

Next we treat the case when $(X_1, Y_1) \neq 0$. Let $v_0 \in \mathrm{im}\, j$ be a cyclic vector. Notice, that since $\dim V = 2$ and $X_1 v_0 \subset V_{-d\varepsilon_1}$ while $Y_1 v_0 \subset V_{d\varepsilon_1}$, we must have that at least one of the operators X_1, Y_1 is zero as well the remaining one squared. Therefore, the matrix of the nonzero operator is conjugate to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. One observes that $X_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $Y_1 = 0$ implies the weight basis of V consists of vectors with weights 0 and d, while $\eta_p(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^d \end{pmatrix}$ in this basis, similarly, $\eta_p(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^{-d} \end{pmatrix}$ if $Y_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $X_1 = 0$. In either of the two cases $(T^*\overline{R})^{\tilde{\eta}_p} = \{X_2, \ldots, X_\ell, Y_2, \ldots, Y_\ell \mid X_i, Y_j \in \mathfrak{h} \subset \mathfrak{sl}_2\}$ and the action of $Z_{\eta_p} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$ is trivial, hence the Hamiltonian reduction is isomorphic to $\mathbb{C}^{2\ell-2}$. □

Remark 2.1.6. The $T' \simeq (\mathbb{C}^*)^{\ell-1} := \{(t_1, \ldots, t_{\ell-1}, t_\ell) \subset T \mid t_\ell = 1\}$ fixed points on $\overline{\mathcal{M}}^\theta(2,\ell)$ are $\{p \in \overline{\mathcal{M}}^\theta(2,\ell) \mid X_1 = \ldots = X_{\ell-1} = Y_1 = \ldots = Y_{\ell-1} = 0\} \simeq T^*\mathbb{P}^1$ and $2\ell - 2$ copies of $\mathbb{C}^2$. Indeed, X_ℓ and Y_ℓ now preserve the weights of weight vectors. Therefore, there are two possibilities:

(i) the vector space $V = V_0$, so $X_{\neq \ell} = Y_{\neq \ell} = 0$ and we arrive at $T^*\mathbb{P}^1$ described above;

(ii) V is spanned by $v_0 \in V_0$ and $v_1 \in V_{\pm\varepsilon_s}$, in which case $X_\ell = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$, $Y_\ell = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}$, one of X_s or Y_s is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ (depending on the sign of the corresponding weight of v_1), the other X's and Y's as well as i are 0 and $j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since the remaining action of G is trivial and $s \in \{1, \ldots, \ell-1\}$, this gives rise to $2\ell - 2$ copies of $\mathbb{C}^2$.

Proposition 2.1.7. *Let ν_0 and ν' be the one-parameter subgroups from Theorem 2.1.5.*

(a) *We have an isomorphism of algebras $C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell)) \simeq \overline{\mathcal{A}}_\lambda(2, \ell-1) \oplus \mathcal{D}(\mathbb{C}^{2\ell-2}) \oplus \mathcal{D}(\mathbb{C}^{2\ell-2})$, where $\overline{\mathcal{A}}_\lambda(2,\ell-1)$ is a quantization of $Z = \overline{\mathcal{M}}^\theta(2,\ell-1)$.*

(b) *Similarly, $C_{\nu'}(\overline{\mathcal{A}}_\lambda(2,\ell)) \simeq \mathcal{A}^{Z_1}_{\lambda+1-\frac{\ell}{2}} \oplus \mathcal{A}^{Z_2}_{\lambda+\frac{\ell}{2}}$, where Z_1, Z_2 are the fixed components for ν' and $\mathcal{A}^{Z_i}_\mu$ stands for the quantization of Z_i with period μ.*

Proof. Proposition 2.2 [Los16] asserts that $C_{\nu_0}(\overline{\mathcal{A}}^{sym}_\lambda(2,\ell)) = \bigoplus_k \mathcal{A}^{Z_k}_{i^*_{Z_k}(\lambda)-\rho_{Z_k}}$, where Z_k's are the irreducible components of $\overline{\mathcal{M}}^\theta(2,\ell)^{\nu_0}$ and $\mathcal{A}^{Z_k}_{i^*_{Z_k}(\lambda)-\rho_{Z_k}}$ stands for the algebra of global sections of the filtered quantization of Z_k with period $i^*_{Z_k}(\lambda) - \rho_{Z_k}$. Here i^*_Z is the pull-back map $H^2(\overline{\mathcal{M}}^\theta(2,\ell),\mathbb{C}) \to H^2(Z,\mathbb{C})$ and ρ_{Z_k} equals half of the 1st Chern class of the contracting bundle of Z_k. We start with describing this bundle in our case. For the general description of tangent spaces to quiver varieties we refer to Lemma 3.10 and Corollary 3.12 in [Nak98]. The tangent bundle descends from the G-module ker β / im α, where α and β are in the following complex:

$$\mathrm{Hom}(V,V) \overset{\alpha}{\hookrightarrow} \mathfrak{sl}_2 \otimes \mathbb{C}^{2\ell} \oplus V \oplus V^* \overset{\beta}{\twoheadrightarrow} \mathrm{Hom}(V,V), \tag{2.1.3}$$

here α stands for the differential of the G-action and β is the differential of the moment map at that fixed point.

It is not hard to observe that the sequence (2.1.3) is equivariant with respect to the $(\mathbb{C}^*)^\ell$-action with β surjective and α injective.

We proceed with verifying the assertion of (a). As every bundle over the $\mathbb{C}^{2\ell-2}$ component of Z is trivial, we look at the restriction of the contracting bundle to $\overline{\mathcal{M}}^\theta(2,\ell-1)$.

It follows from the description of the tangent bundle as the middle cohomology of the complex (2.1.3) that the contracting bundle descends under G-action from $T^*\overline{R}^{\tilde{\eta}_p,>0}$ modulo two copies of $\mathfrak{g}^{\tilde{\eta}_p,>0}$. In our case $(T^*\overline{R})^{\tilde{\eta}_p,>0} = H$ is the three-dimensional space $\mathrm{Vec}(X_1)$, while $\mathfrak{g}$ is pointwise fixed under the action of $\tilde{\eta}_p$, hence, the contracting bundle descends from H.

The top exterior power of the vector bundle $\widetilde{H}$ descending from H under G-action is trivial, since G acts trivially on the top exterior power of H. By [Los12], Section 5, the period of a quantization $\overline{\mathcal{A}}_\lambda(2,\ell)$ is $\lambda - \zeta$, where ζ is half the character of the action of G on $\Lambda^{top}\overline{R}$. Thus the periods of the quantizations $\overline{\mathcal{A}}_\lambda(2,\ell)$ and $\overline{\mathcal{A}}_\lambda(2,\ell-1)$ are both equal to $\lambda + \frac{1}{2}$, the first claim of the proposition follows.

We verify the claim in (b) for Z_1. There is a line subbundle $L_{triv} \subset \tilde{V}$ with the fiber over a point $p \in Z_1$ being im j. It is trivial, since for a fixed $0 \neq w \in W$ one has a nowhere vanishing section $j(w)$ of $\tilde{V}$. Using the splitting principle, we write $\tilde{V} = L_{triv} \oplus L_1$ with $c_1(L_{triv}) = 0$ and $c_1(L_1) = c_{Z_1}$, where c_{Z_1} is the generator of $H^2(Z_1)$. In this case $V = \mathbb{C}\langle v_0, v_1\rangle$ with v_0 of weight 0 and v_1 of weight d, in other words, $\eta_p = \begin{pmatrix} 1 & 0 \\ 0 & t^d \end{pmatrix}$ in the basis $\langle v_0, v_1\rangle$. This implies that the bundle on Z_1 descending from $\mathfrak{sl}_2$ is $L_{triv} \oplus L_1 \oplus L_1^*$. Let $\tilde{\eta}_p(t) = (\nu'(t), \eta_p(t)) \subset T \times G$, then $U^{\tilde{\eta}_p,>0} = (z_1, \dots, z_\ell, v_1)$, where z_s is the 12-entry (first row and second column) of the matrix X_s, while $\mathfrak{g}^{\tilde{\eta}_p,>0}$ consists of the 12-entry of the corresponding matrix. Hence, the nontrivial part of the contracting bundle is $L_1 \otimes \mathbb{C}^{\ell-1}$. Thus we conclude that $\rho_{Z_1} = \frac{\ell-1}{2} c_{Z_1}$.

Analogously one can show that the nontrivial part of the contracting bundle on Z_2 is $L_1^* \otimes \mathbb{C}^{\ell-1}$ and $\rho_{Z_2} = \frac{1-\ell}{2} c_{Z_2}$. The maps $i_{Z_1}^*$ and $i_{Z_2}^*$ send $c_1(\tilde{V}) \in H^2(\overline{\mathcal{M}}^\theta(2,\ell))$ to the generators $c_{Z_1} \in H^2(Z_1)$ and $c_{Z_2} \in H^2(Z_2)$. The claim in (b) follows. □

Remark 2.1.8. The quantizations $\mathcal{A}^{Z_1}_{\lambda+1-\frac{\ell}{2}}$ and $\mathcal{A}^{Z_2}_{\lambda+\frac{\ell}{2}}$ are isomorphic to $\mathcal{D}^{\lambda-\ell+1}(\mathbb{P}^{\ell-1})$ and $\mathcal{D}^{\lambda}(\mathbb{P}^{\ell-1})$ (the algebras of twisted differential operators on projective spaces).

2.2 Central fibers

The next lemma provides some information on the preimages of zero under $\bar{\rho} : \overline{\mathcal{M}}^\theta(n,\ell) \to \overline{\mathcal{M}}(n,\ell)$ (central fibers) in $\overline{\mathcal{M}}^\theta(n,\ell)$.

Lemma 2.2.1. *(a) The preimage of 0 in $\overline{\mathcal{M}}^\theta(2,\ell)$ is $\bar{\rho}^{-1}(0) = \mathbb{P}^{2\ell-1}$.*

(b) Let $n, \ell > 1$, then $dim(\bar{\rho}^{-1}(0)) < \frac{1}{2} dim(\overline{\mathcal{M}}^\theta(n,\ell))$.

Proof. An application of the Hilbert-Mumford criterion shows (the argument is analogous to the one in Proposition 9.7.4. in [DW17]) that $p \in \overline{\mathcal{M}}^\theta(n,\ell)$ lies in $\bar{\rho}^{-1}(0)$ if and only if on the corresponding representation there exists a filtration $0 = L_0 \subset L_1 \subset L_2 \subset \ldots \subset L_n = r_p \in T^*\overline{R}$ by subrepresentations such that each quotient L_i/L_{i-1} for $i < n$ is isomorphic to a simple representation (of the framed quiver $\widehat{B}_{2\ell}$) with dimension vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and L_n/L_{n-1} is isomorphic to the simple representation with dimension vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (here the top coordinate corresponds to the dimension of framing and the bottom to the dimension of V). This implies that all the $\mathfrak{sl}_n$-components of p must be strictly upper-triangular matrices. It follows from equation (1.5.1) and our choice of stability condition, that $i = 0$.

(a) Pick a vector $0 \neq h \in \mathrm{im}\, j$. As h is a cyclic vector, it must have a nontrivial projection onto V/V_1. The action by matrices of the form $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ (conjugation by which does not change any of the 2×2 matrices of p) allows to assume that the component of h along the first vector is zero. Acting by $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \subset GL_2$, allows to pick a representative of p with $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the action by $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \subset GL_2$ to simultaneously rescale the 2×2 matrices of p. We conclude that $p = (X_1 = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \ldots, X_\ell = \begin{pmatrix} 0 & a_\ell \\ 0 & 0 \end{pmatrix}, Y_1 = \begin{pmatrix} 0 & a_{\ell+1} \\ 0 & 0 \end{pmatrix}, \ldots, Y_\ell = \begin{pmatrix} 0 & a_{2\ell} \\ 0 & 0 \end{pmatrix}, i = 0, j = \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ with at least one of X_k's and Y_s's, being nonzero due to the stability condition, up to simultaneous dilations of X_k's and Y_s's, which shows the claim, stated in (a).

Now we show the claim in (b). Acting by matrices of the form $\begin{pmatrix} 1 & 0 & \dots & 0 & * \\ 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$, we can assume that h is proportional to the last vector in the basis. The action by the subgroup

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & * \end{pmatrix} \subset GL_n$$

allows to assume $j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

Since $i = 0$, the moment equation (1.5.1) reduces to $\sum_{k=0}^{\ell} [X_k, Y_k] = 0$ and as each of the commutators is a matrix of the form

$$[X_k, Y_k] = \begin{pmatrix} 0 & 0 & * & * & \dots & * \\ 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & * \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

equation (1.5.1) imposes $\frac{(n-1)(n-2)}{2}$ independent conditions on the coordinates of $p \in \bar{\rho}^{-1}(0)$. The action of matrices of the form

$$\begin{pmatrix} * & * & \dots & * & 0 \\ 0 & * & \dots & * & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & * & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

preserves both j and the strictly upper-triangular matrices and reduces the dimension by $\frac{n(n-1)}{2}$. Therefore, we have established that

$$\dim(\bar{\rho}^{-1}(0)) \leq \frac{n(n-1)}{2} 2\ell - \frac{(n-1)(n-2)}{2} - \frac{n(n-1)}{2} = (n^2 - n)\ell - n^2 + 2n - 1$$

and a straightforward computation shows that $(n^2 - n)\ell - n^2 + 2n - 1 < (\ell - 1)n^2 - \ell + n = \frac{1}{2}\dim(\overline{\mathcal{M}}^{\theta}(n, \ell))$ provided $n, \ell > 1$. □

Corollary 2.2.2. *Assume $n, \ell > 1$, then the central fiber $\bar{\rho}^{-1}(0) \subset \overline{\mathcal{M}}^{\theta}(n, \ell)$ is an isotropic but not Lagrangian subvariety.*

Remark 2.2.3. The T - fixed points for the action on $\overline{\mathcal{M}}^{\theta}(2, \ell)$ (see Theorem 2.1) lie on $\bar{\rho}^{-1}(0) = \mathbb{P}^{2\ell-1}$.

Corollary 2.2.4. $H^2(\overline{\mathcal{M}}^{\theta}(2, \ell)) \simeq \mathbb{C}$.

Proof. This follows from the fact that $\overline{\mathcal{M}}^{\theta}(2, \ell)$ is homotopy equivalent to the central fiber, while the latter is isomorphic to $\mathbb{P}^{2\ell-1}$ as shown in Lemma 2.2.1 (a). □

Corollary 2.2.5. *There are no finite dimensional $\overline{\mathcal{A}}_{\lambda}(n, \ell)$-modules for $n, \ell > 1$ and generic ν.*

Proof. The support of a finite dimensional module M must be $0 \in \overline{\mathcal{M}}(n, \ell)$ (since 0 is the only fixed point of $\overline{\mathcal{M}}(n, \ell)$ for the $\mathbb{S}$-action, the support is $\mathbb{S}$-stable and the module is finite dimensional). Notice that the support of $\mathrm{Loc}(M)$ is contained in $\bar{\rho}^{-1}(0) \subset \overline{\mathcal{M}}^{\theta}(n, \ell)$. On the other hand, due to Gabber's involutivity theorem (see [Gab81]), the support of a coherent module must be a coisotropic subvariety of $\overline{\mathcal{M}}^{\theta}(n, \ell)$. However, this is impossible for dimension reasons. □

I would like to thank Pavel Etingof and Ivan Losev for bringing my attention to the following fact.

Let A be a Poisson algebra over $\mathbb{C}$, i.e. $A = \mathcal{O}(X)$, where X is an affine Poisson variety.

Definition 2.2.6. The zeroth Poisson homology, $HP_0(A)$ is the quotient $A/\{A, A\}$.

The following conjecture was formulated in [ES14] (see Conjecture 1.3.1 therein).

Conjecture 2.2.7. *Let $\rho : \tilde{X} \to X$ be a symplectic resolution with X affine, then $HP_0(\mathcal{O}(X)) = H^{\dim X}(\tilde{X})$.*

Conjecture 2.2.7 holds in many cases (see Examples $6.4 - 6.7$ in [ES18] for details):

1. Let Y be a smooth symplectic surface. Set $X = \mathrm{Sym}^n Y := Y^n/S_n$, the n-th symmetric power of Y and consider the resolution $\bar{\rho} : \tilde{X} = \mathrm{Hilb}^n Y \to X$.
2. Take $Y = \mathbb{C}^2/\Gamma$ and the crepant resolution $\tilde{Y} \to Y$ (here $\Gamma \subset SL(2, \mathbb{C})$ is a finite subgroup), consider $X := \mathrm{Sym}^n Y$ and the resolution $\rho_1 : \tilde{X} := \mathrm{Hilb}^n \tilde{Y} \to \mathrm{Sym}^n \tilde{Y}$. Now compose this with $\rho_2 : \mathrm{Sym}^n \tilde{Y} \to \mathrm{Sym}^n Y$ to obtain the resolution $\rho = \rho_2 \circ \rho_1 : \tilde{X} \to X$.

3. Let $\mathcal{N}$ be the cone of nilpotent elements in a complex semisimple Lie algebra $\mathfrak{g}$, and ρ the Springer resolution $T^*(G/B) \to \mathcal{N}$.

Remark 2.2.8. The resolutions $\overline{\rho} : \overline{\mathcal{M}}^{\theta}(n,\ell) \to \overline{\mathcal{M}}(n,\ell)$ serve as counterexamples to Conjecture 2.2.7. Indeed, $H^{top}(\overline{\mathcal{M}}^{\theta}(2,\ell),\mathbb{C}) = H^{3\ell-2}(\mathbb{P}^{2\ell-1},\mathbb{C}) = 0$ and, in general, $H^{top}(\overline{\mathcal{M}}^{\theta}(n,\ell),\mathbb{C}) = H^{\frac{1}{2}\dim(\overline{\mathcal{M}}^{\theta}(n,\ell))}(\overline{\mathcal{M}}^{\theta}(n,\ell),\mathbb{C}) = 0$, since the variety $\overline{\mathcal{M}}^{\theta}(n,\ell)$ is homotopy equivalent to $\overline{\rho}^{-1}(0)$ (via the contracting $\mathbb{C}^*$-action) and this variety has dimension strictly less than $\frac{1}{2}\dim(\overline{\mathcal{M}}^{\theta}(n,\ell))$ as shown in Lemma 2.2.1 (b). On the other hand, the point 0 is a symplectic leaf in affine Poisson varieties $\overline{\mathcal{M}}(n,\ell)$. This is true, since the Poisson bracket is of degree -2 and there are no invariant functions of degree one in $\mathbb{C}[T^*\bar{R}]^G$, hence, the maximal ideal of 0 is Poisson. From this it follows that the vector spaces $HP_0(\mathcal{O}(\overline{\mathcal{M}}(n,\ell)))$ are at least $1-$dimensional. Therefore, $H^{top}(\overline{\mathcal{M}}(n,\ell)) \neq HP_0(\overline{\mathcal{M}}(n,\ell))$, contradicting the claim of the conjecture.

Chapter 3

Symplectic leaves and slices

3.1 Symplectic leaves

First we describe the symplectic leaves and slices to them for the Poisson varieties $\overline{\mathcal{M}}(2,\ell)$ and $\overline{\mathcal{M}}(3,\ell)$. The general description was given by Nakajima, it can also be found in Section 2 of [BL15]. In particular (Section 6 of [Nak94] or Section 3 of [Nak98]), it was shown that

$$\overline{\mathcal{M}}(n,\ell) = \bigcup_{\hat{G}\subseteq G} \overline{\mathcal{M}}(n,\ell)_{\hat{G}},$$

where the strata are parametrized by reductive subgroups $\hat{G} \subseteq G$ and $\overline{\mathcal{M}}(n,\ell)_{\hat{G}}$ stands for the locus of isomorphism classes of semisimple representations, whose stabilizer is conjugate to $\hat{G}$. A semisimple representation $r \in T^*R$ is in $\overline{\mathcal{M}}(n,\ell)_{\hat{G}}$, if it can be decomposed as $r = r^0 \bigoplus_{i=1}^{n} r^i \otimes U_i$, where r_i's are simple and pairwise nonisomorphic with zero-dimensional framing and U_i's are their multiplicity spaces, and $\hat{G}$ is conjugate to $\prod GL(U_i)$. Moreover, according to Theorem 1.3 of [CB01], the stratum $\overline{\mathcal{M}}(n,\ell)_{\hat{G}}$ is an irreducible locally closed subset of $\overline{\mathcal{M}}(n,\ell)$. Each stratum $\overline{\mathcal{M}}(n,\ell)_{\hat{G}}$, being irreducible, must be a symplectic leaf. The information about the symplectic leaves of $\overline{\mathcal{M}}(2,\ell)$ and $\overline{\mathcal{M}}(3,\ell)$ is summarized in the tables below.

Remark 3.1.1. We would like to notice that there are no irreducible representations with dimension vector $(1,1)$, as each summand $[X_k, Y_k]$ in equation (1.5.1) equals zero and, therefore, $ji = 0$ as well, forcing $i = 0$ or $j = 0$ (or $i = j = 0$) and making the representation with dimension vector $(1,0)$ (zero-dimensional framing) in the former case and with dimension vector $(0,1)$ in the latter a subrepresentation.

The third leaf in Table 3.1 corresponds to representations $r = r^0 \oplus r^1 \oplus r^2$, while the fourth $r = r^0 \oplus r^1 \otimes \mathbb{C}^2$, the multiplicities in Table 3.2 are indicated in the second column therein.

Remark 3.1.2. Since $\overline{\mathcal{M}}(2,\ell)$ has a unique symplectic leaf of codimension 2, the slice to which is an A_1 singularity the Namikawa Weyl group (see [Nam10]) of $\overline{\mathcal{M}}(2,\ell)$ is $\mathbb{Z}/2\mathbb{Z}$. As there are no symplectic leaves of codimension 2 in $\overline{\mathcal{M}}(3,\ell)$, the corresponding Namikawa Weyl group is trivial.

type	dim vector	dim of leaf	stabilizer (in GL_2)
1	(2,1)	$6\ell-4$	$\{id\}$
2	$(2,0)\oplus(0,1)$	$6\ell-6$	$\mathbb{C}^*\cdot id$
3	$(1,0)\oplus(1,0)\oplus(0,1)$	2ℓ	$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}, \lambda,\mu\in\mathbb{C}^*$
4	$(1,0)^{\oplus 2}\oplus(0,1)$	0	GL_2

Table 3.1: Symplectic leaves of $\overline{\mathcal{M}}(2,\ell)$

type	dim vector	dim of leaf	stabilizer (in GL_3)
1	(3,1)	$16\ell-12$	$\{id\}$
2	$(3,0)\oplus(0,1)$	$16\ell-16$	$\mathbb{C}^*\cdot id$
3	$(2,1)\oplus(1,0)$	$6\ell-4$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \nu\end{pmatrix}, \nu\in\mathbb{C}^*$
4	$(2,0)\oplus(1,0)\oplus(0,1)$	$6\ell-6$	$\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{pmatrix}, \lambda,\mu\in\mathbb{C}^*$
5	$(1,0)\oplus(1,0)\oplus(1,0)\oplus(0,1)$	4ℓ	$\begin{pmatrix}\lambda & 0 & 0\\ 0 & \nu & 0\\ 0 & 0 & \mu\end{pmatrix}, \lambda,\nu,\mu\in\mathbb{C}^*$
6	$(1,0)^{\oplus 2}\oplus(1,0)\oplus(0,1)$	2ℓ	$\begin{pmatrix}* & * & 0\\ * & * & 0\\ 0 & 0 & \mu\end{pmatrix}, \mu\in\mathbb{C}^*$
7	$(1,0)^{\oplus 3}\oplus(0,1)$	0	GL_3

Table 3.2: Symplectic leaves of $\overline{\mathcal{M}}(3,\ell)$

3.2 Fixed points on the slice

Next we study the slice taken at some point of the leaf of type 3 in Table 3.1 above. This slice is the quiver variety on the picture below with $k,s\in\{1,\dots,\ell-1,\ell+1,\dots,2\ell-1\}$. The dimension vector is $(1,1)$ and the framing is also one-dimensional.

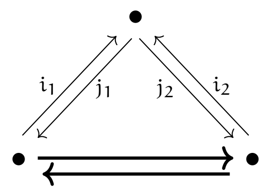


Figure 3.1: Slice quiver with the maps corresponding to thick edges from left to right being $x_1, \ldots, x_{\ell-1}, y_\ell, \ldots, y_{2\ell-2}$ and from right to left $y_1, \ldots, y_{\ell-1}, x_\ell, \ldots, x_{2\ell-2}$.

We consider the point $p = \left(X_\ell = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_{\neq \ell} = 0, Y_k = 0, i = 0, j = 0\right)$. As the representation is semisimple, the G orbit through p in $T^*\overline{R}$ is closed and slightly abusing notation we will refer to the corresponding point in $\overline{\mathcal{M}}(2, \ell)$ as p as well. The slice to the symplectic leaf at p will be denoted by $\mathcal{SL}_p$. The description of slices as quiver varieties can be found in Section 2 of [BL15]. In our case the slice $\mathcal{SL}_p$ is the hypertoric variety obtained from the $(\mathbb{C}^*)^2$-action on $\mathbb{C}^{2\ell}$. In the basis $\langle x_1, x_2, \ldots, x_{2\ell-2}, i_1, i_2 \rangle$ the weights are $(t_1^{-1}t_2, \ldots, t_1^{-1}t_2, t_1t_2^{-1}, \ldots, t_1t_2^{-1}, t_1^{-1}, t_2^{-1})$. It is the quiver variety for the underlying graph depicted on Figure 3.1 with one-dimensional vector spaces assigned to the vertices and one-dimensional framing. We denote by ρ_s the map $\mathcal{SL}_p^\theta \to \mathcal{SL}_p$ and fix $\theta = (-1, -1)$. The preimage of zero $\rho_s^{-1}(0)$ and the fixed points for the $T' \simeq (\mathbb{C}^*)^{\ell-1}$-action on $\mathcal{SL}_p^\theta$ are described in the proposition below.

Proposition 3.2.1. (a) $\rho_s^{-1}(0) \cong \mathbb{CP}^{2\ell-2} \cup \mathbb{CP}^{2\ell-2}$ *consists of two irreducible components, intersecting in a single point* $(x_s = y_k = i_1 = i_2 = 0, j_1 = j_2 = 1)$.

(b) *There are* $4\ell - 3$ *fixed points on* $\mathcal{SL}_p^\theta$ *for the* T'*-action. These points are (the* $(\mathbb{C}^*)^2$*-orbits of)* $(x_i = 1, x_{\neq i} = y_j = i_1 = i_2 = j_2 = 0, j_1 = 1)$, $(y_j = 1, x_i = y_{\neq j} = i_1 = i_2 = j_1 = 0, j_2 = 1)$ *and* $(x_s = y_k = i_1 = i_2 = 0, j_1 = j_2 = 1)$.

Proof. To see that (a) is true, we first notice that for $(\mathbf{x}, \mathbf{y}, i, j) \in \rho_s^{-1}(0)$ we have either all $x_k = 0$ or all $y_s = 0$ (use the Hilbert-Mumford criterion in a similar way to the proof of Lemma 2.2). In the former case the stability condition guarantees $j_2 \neq 0$ and j_1 or at least one of y_s's is nonzero. Therefore, the first equation in (3.2.1) below immediately implies that $i_2 = 0$. To see that $i_1 = 0$ as well, notice that the one-dimensional torus, acting on the vector space assigned to the left vertex, acts on i_1 and y_s with j_1 with opposite weights. We look at the space $\mathbb{C}^{2\ell-1} \setminus \{0\}$, formed by y_s's and j_1. The $\mathbb{C}^*$-action on the one-dimensional framing attached to the right vertex allows to assume $j_2 = 1$. Observing that the action of the remaining $\mathbb{C}^*$ simultaneously rescales the vectors in $\mathbb{C}^{2\ell-1} \setminus \{0\}$, we recover the first $\mathbb{CP}^{2\ell-2}$ component in $\rho_s^{-1}(0)$. Similarly, if all $y_s = 0$, one comes up with $\mathbb{CP}^{2\ell-2}$ with coordinates x_k and j_2. It remains to notice that the projective spaces have exactly one point of intersection, $(x_s = y_k = i_1 = i_2 = 0, j_1 = j_2 = 1)$.

Next we verify the assertion of (b). The moment map equations considered separately for the two vertices are equivalent to

$$\begin{cases} \sum_{i=1}^{\ell-1}(x_i y_{\ell+i} + x_{\ell+i} y_i) + j_1 i_1 = 0 \\ j_1 i_1 = j_2 i_2. \end{cases} \tag{3.2.1}$$

Recall that $\theta = (-1, -1)$. Then the θ-semistable locus consists of all representations for which at least one of j_1, j_2 is not equal to zero and

- if $j_1 \neq 0$ and $j_2 = 0$ there exists an i such that $x_i \neq 0$;
- if $j_2 \neq 0$ and $j_1 = 0$ there exists a j such that $y_j \neq 0$.

The formulas for the torus action below are derived from the fact that $x_i \in \mathrm{Hom}(r_1, r_2)$ and $y_i \in \mathrm{Hom}(r_2, r_1)$ are the elements above and below diagonal in the ith matrix of our quiver variety, where $r = r_0 \oplus r_1 \oplus r_2$ is the decomposition of the representation into simples.

$$\begin{cases} t'_1 x_1 = t_1 x_1 t_2^{-1} \\ \dots \\ t'^{-1}_{\ell-1} x_{2\ell-1} = t_1 x_{2\ell-1} t_2^{-1} \\ t'_1 y_1 = t_1^{-1} y_1 t_2 \\ \dots \\ t'^{-1}_{\ell-1} y_{2\ell-1} = t_1^{-1} y_{2\ell-1} t_2 \\ j_1 = t_1^{-1} j_1 \\ j_2 = t_2^{-1} j_2 \\ i_1 = t_1 i_1 \\ i_2 = t_2 i_2, \end{cases} \tag{3.2.2}$$

here $(t'_1, \dots, t'_{\ell-1}) \in T'$ and $(t_1, t_2) \in (\mathbb{C}^*)^2$. We first notice that it is not possible for both i_s and j_s to be nonzero ($s \in \{1, 2\}$), as otherwise the second equation of (3.2.1) would imply all i_s, j_s ($s \in \{1, 2\}$) were nonzero and consequently $t_1 = t_2 = 1$, implying all $x_k = y_c = 0$, hence, contradicting the first equation of (3.2.1). It follows from (a) that $i_1 = i_2 = 0$. From the system of equalities (3.2.2) it also follows that we must have one of the following

- $x_i \neq 0, y_{l+i} \neq 0$ and $y_i \neq 0$ with $i \in \{2, \dots, \ell\}$;
- $x_{l+i} \neq 0$ and $y_i \neq 0$ with $i \in \{2, \dots, \ell\}$;
- all x_i and all y_j are zero with $j_1 = j_2 = 1$ and $i_1 = i_2 = 0$.

In each of the former two cases (3.2.1) reduces to either $x_i y_{\ell+i} = 0$ or $x_{\ell+i} y_i = 0$, then the claim of the proposition easily follows from the description of semistable points.

□

Remark 3.2.2. The slice $\mathcal{SL}_p \subset \overline{\mathcal{M}}^\theta(2,\ell)$ is a formal subscheme (formal neighborhood of the point p). We describe the intersection of the fixed point loci $\mathcal{SL}_p^{T'} \cap \overline{\mathcal{M}}^\theta(2,\ell)^{T'}$ (the latter was found in Remark 2.1.6). Each fixed point $(x_s = 1, x_{\neq s} = y_j = i_1 = i_2 = j_2 = 0, j_1 = 1)$ on the slice with $s \in \{1,\dots,\ell-1\}$ is the fixed point $(X_\ell = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y_\ell = 0,$ $X_s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_{\neq s} = Y_k = 0, i = 0, j = \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ on $\overline{\mathcal{M}}^\theta(2,\ell)^{T'}$; $(y_s = 1, x_i = y_{\neq s} = i_1 = i_2 = j_1 = 0, j_2 = 1)$ is $(X_\ell = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, Y_\ell = 0, X_s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_{\neq s} = Y_k = 0, i = 0, j = \begin{pmatrix} 1 \\ 0 \end{pmatrix})$. Notice that these points are respectively the points $(1,0)$ and $(-1,0)$ on $\mathbb{C}^2_s \subset \overline{\mathcal{M}}^\theta(2,\ell)^{T'}$ (see Remark 2.1.6). In case $s \in \{\ell+1,\dots,2\ell\}$ the fixed points on the slice are $(X_\ell = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y_\ell = 0, Y_s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_{\neq s} = Y_k = 0, i = 0, j = \begin{pmatrix} 1 \\ 0 \end{pmatrix})$ and $(X_\ell = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, Y_\ell = 0, Y_s = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_{\neq s} = Y_k = 0, i = 0, j = \begin{pmatrix} 1 \\ 0 \end{pmatrix})$, finally, $(x_s = y_k = i_1 = i_2 = 0, j_1 = j_2 = 1)$ becomes $(X_\ell = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_{\neq \ell} = Y_k = 0, i = 0, j = \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \in T^*\mathbb{P}^1$.

Chapter 4

Category $\mathcal{O}_\xi(\overline{\mathcal{S}}_\lambda(2,\ell))$ for the slice $\mathcal{SL}_p$

The main goal of this chapter is to provide a description of the category $\mathcal{O}_\nu(\overline{\mathcal{SL}}_\lambda(2,\ell))$ for the slice $\mathcal{SL}_p$. These results will be used in the next Chapter 6 for the study of the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$. As $\mathcal{SL}_p$ is a hypertoric variety, we use the results of [BLPW10] and [BLPW12], where analogous categories were explicitly described in a more general setting.

We start by briefly recalling the basic definitions, notions and results (for a more detailed exposition see [BLPW10] and [BLPW12]).

4.1 Hypertoric varieties (a brief overview)

Consider the moment map for the action of the torus $K \subset \tilde{T} = (\mathbb{C}^*)^n$ on the variety $T^*\mathbb{C}^n$, i.e.

$$\mu : T^*\mathbb{C}^n \to \mathfrak{k}^*.$$

Fix a direct summand $\Lambda_0 \subset W_\mathbb{Z}$, let $W_\mathbb{R} := W_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{R}, V_{0,\mathbb{R}} = \mathbb{R}\Lambda_0, V_0 := \mathbb{C}\Lambda_0 \subset W \cong t^*$, $\mathfrak{k} = V_0^\perp$ and $K \subset \widetilde{T}$ be the connected subtorus with Lie algebra $\mathfrak{k}$. Thus Λ_0 may be identified with the character lattice of $\widetilde{T}/K$ and $W_\mathbb{Z}/\Lambda_0$ may be identified with the character lattice of K.

Definition 4.1.1. The *hypertoric variety* associated to the triple $X = (\Lambda_0, \eta, \xi)$ with η a Λ_0-orbit in $W_\mathbb{Z}$ is $\mathfrak{M}(X) := \mu^{-1}(0)^{\eta-ss}//K$. Also define $\mathfrak{M}_0(X) := \mu^{-1}(0)//K$. We consider the categorical quotient in both cases. The projective map $\mathfrak{M}(X) \to \mathfrak{M}_0(X)$ will be denoted by κ. We will denote the subspace $\eta + V_{0,\mathbb{R}} \subset W_\mathbb{R}$ by V_η. The triple $X = (\Lambda_0, \eta, \xi)$ is called a *polarized arrangement*.

For a sign vector $\alpha \in \{+,-\}^n$ define the chamber $P_{\alpha,0}$ to be the subset of the affine space

$V_\nu := \{\nu + v \mid v \in V_{0,\mathbb{R}}\}$ cut out by the inequalities

$$h_i \geq 0 \text{ for all } i \in I_\Lambda \text{ with } \alpha(i) = + \text{ and } h_i \leq 0 \text{ for all } i \in I_\Lambda \text{ with } \alpha(i) = -.$$

If $P_{\alpha,0} \neq \varnothing$ we say that α is *feasible* for X and let $\mathcal{F}_\eta$ be the set of feasible sign vectors.

Remark 4.1.2. The hypertoric variety $\mathfrak{M}_0(X)$ is affine, and for any central character λ of the hypertoric enveloping algebra U there is a natural isomorphism $\mathrm{gr}U_\lambda \simeq \mathbb{C}[\mathfrak{M}_0] \simeq \mathbb{C}[\mathfrak{M}]$ (Proposition 5.2 in [BLPW12]).

Let $\mathbb{S} := \mathbb{C}^*$ act on $T^*\mathbb{C}^n$ by inverse scalar multiplication i.e. $s \cdot (z, w) := (s^{-1}z, s^{-1}w)$. This induces an $\mathbb{S}$-action on both $\mathfrak{M}(X)$ and $\mathfrak{M}_0(X)$, and the map κ is $\mathbb{S}$-equivariant. We have that $\kappa : \mathfrak{M}(X) \to \mathfrak{M}_0(X)$ is a conical symplecic resolution. The symplectic form ω has weight 2 w.r.t. the aforemented $\mathbb{S}$-action.

4.2 Hypertoric category $\mathcal{O}$

Let $\mathbb{D}$ be the Weyl algebra of polynomial differential operators on $\mathbb{C}^n$, i.e.

$$\mathbb{D} = \mathbb{C}\langle x_1, \partial_1, \dots x_n, \partial_n \rangle,$$

with $[x_i, x_j] = [\partial_i, \partial_j] = 0$ and $[\partial_i, x_j] = \delta_{ij}$. The action of the torus $\widetilde{T} = (\mathbb{C}^*)^n$ on $\mathbb{C}^n$ induces an action on $\mathbb{D}$. This provides the $\mathbb{Z}^n$- grading

$$\mathbb{D} = \bigoplus_{z \in W_\mathbb{Z}} \mathbb{D}_z,$$

where $W_\mathbb{Z}$ is the character lattice of $\widetilde{T}$, $\deg(x_i) = -\deg(\partial_i) = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$ and $\mathbb{D}_z := \{a \in \mathbb{D} \mid t \cdot a = t_1^{z_1} \dots t_n^{z_n} a \;\forall\, t \in \widetilde{T}\}$.

Observe that the 0^{th} graded piece is $\mathbb{D}^{\widetilde{T}} = \mathbb{C}[x_1\partial_1, \dots, x_n\partial_n]$ and define $h_i^- := \partial_i x_i$ and $h_i^+ := x_i \partial_i$ with $h_i^- - h_i^+ = 1$. We consider the Bernstein filtration on $\mathbb{D}$ (here $\deg(x_i) = \deg(\partial_j) = 1$) and let $H := \mathrm{gr}(\mathbb{D}_0) = \mathbb{C}[h_1, \dots, h_n]$, where $h_i := h_i^+ + F_0(\mathbb{D}_0) = h_i^- + F_0(\mathbb{D}_0)$.

Definition 4.2.1. *The hypertoric enveloping algebra associated to* Λ_0 *is the ring of* K-*invariants* $U := \mathbb{D}^K = \bigoplus_{z \in \Lambda_0} \mathbb{D}_z$.

Consider a module $M \in U$-mod. For a point $\nu \in W$, let $\mathcal{J}_\nu$ denote the corresponding maximal ideal. Then the generalized ν-weight space of M is defined as

$$M_\nu := \{m \in M \mid \mathcal{J}_\nu^k m = 0 \text{ for } k \gg 0\}.$$

The support of M is defined by

$$\text{Supp } M := \{\nu \in W \mid M_\nu \neq 0\}.$$

We will use the notation $U\text{-mod}_\Lambda$ for $M \in U\text{-mod}$ with Supp $M \subset \Lambda$.

Let $Z(U)$ denote the center of U. It is not hard to show that $Z(U)$ is the subalgebra isomorphic to the image of $S[\mathfrak{k}]$ under the quantum comoment map (Section 3.2 of [BLPW12]). Let $\lambda : Z(U) \to \mathbb{C}$ be a central character. Notice that the isomorphism $Z(U) \simeq S[\mathfrak{k}]$ allows to think of λ as an element of $\mathfrak{k}^*$. We will denote by $U_\lambda := U/\langle \ker(\lambda)\rangle U$ the corresponding central quotient. Set $V_\lambda := \lambda + V_0 = \lambda + \mathbb{C}\Lambda_0$, $V_{\lambda,\mathbb{R}} := \lambda + \mathbb{R}\Lambda_0$ and let Λ be a Λ_0-orbit.

Choose a generic element $\xi \in \Lambda_0^* \simeq (\mathfrak{t}/\mathfrak{k})^*$, the action of ξ lifts to U and produces a grading given by

$$U := \bigoplus_{\xi(z)=k} U_z.$$

Set

$$U^+ := \bigoplus_{k\geq 0} U^k \text{ and } U^- := \bigoplus_{k\leq 0} U^k,$$

similarly, U_λ^+ and U_λ^- are the images of U^+ and U^- under the quotient map $U \to U_\lambda$.

Definition 4.2.2. The *hypertoric category* $\mathcal{O}$ is the full subcategory of U-mod consisting of modules that are U^+ - locally finite and semisimple over the center $Z(U)$. Define $\mathcal{O}_\lambda$ to be the full subcategory of $\mathcal{O}$ consisting of modules on which U acts with central character λ. Equivalently, it is as the full subcategory of U_λ-mod consisting of modules that are U_λ^+ - locally finite. Finally, define $\mathcal{O}(\Lambda_0, \Lambda, \xi)$ to be the full subcategory of $\mathcal{O}_\lambda$ consisting of modules supported in Λ; equivalently, the full subcategory of $U_\lambda\text{-mod}_\Lambda$ consisting of modules that are U_λ^+ - locally finite. The triple $\mathbf{X} := (\Lambda_0, \Lambda, \xi)$ is called a *quantized polarized arrangement*.

Similarly to category $\mathcal{O}$ of a semisimple Lie algebra, we have the direct sum decompositions

$$\begin{aligned} \mathcal{O} &= \bigoplus_{\Lambda \in W/\Lambda_0} \mathcal{O}(\Lambda_0, \Lambda, \xi) \text{ and} \\ \mathcal{O}_\lambda &= \bigoplus_{\Lambda' \in V_\lambda/\Lambda_0} \mathcal{O}(\Lambda_0, \Lambda', \xi). \end{aligned} \tag{4.2.1}$$

The summands in the decompositions above are blocks, i.e. they are the smallest possible direct summands (see Section 4.1 of [BLPW12] for details).

Let I_Λ be the set of indices $i \in \{1, \ldots, n\}$ for which $h_i^+(\Lambda) \subset \mathbb{Z}$ (or equivalently $h_i^-(\Lambda) \subset \mathbb{Z}$). For a sign vector $\alpha \in \{+, -\}^n$ define the chamber P_α to be the subset of the affine space $V_\lambda := \{\nu + \lambda \mid \nu \in V\}$ cut out by the inequalities

$$h_i^+ \geq 0 \text{ for all } i \in I_\Lambda \text{ with } \alpha(i) = + \text{ and } h_i^- \leq 0 \text{ for all } i \in I_\Lambda \text{ with } \alpha(i) = -.$$

If $P_\alpha \cap \Lambda$ is nonempty, we say that α is *feasible* for Λ. We call α *bounded for* ξ if the restriction of ξ is proper and bounded above on P_α. The set of feasible sign vectors will be denoted by $\mathcal{F}_\Lambda$, the set of bounded vectors by $\mathcal{B}_\xi$ and the set of bounded feasible vectors by $\mathcal{P}_{\Lambda,\xi} := \mathcal{F}_\Lambda \cap \mathcal{B}_\xi$.

Example 4.2.3. In case $\ell = 2$, the slice $\mathcal{SL}_p$ is the hypertoric variety obtained from the $K = (\mathbb{C}^*)^2$-action on $\mathbb{C}^4$ via

$$t \cdot (x_1, x_2, i_1, i_2) = (t_1^{-1} t_2 x_1, t_1 t_2^{-1} x_2, t_1^{-1} i_1, t_2^{-1} i_2).$$

Notice that $\mathfrak{k} \hookrightarrow \mathrm{Lie}(\widetilde{T})$ and the image is $\mathrm{span}((-1,1,-1,0),(1,-1,0,-1))$, set $L := \mathrm{span}_{\mathbb{R}}((-1,1,-1,0),(1,-1,0,-1))$. Then $V_{0,\mathbb{R}} = \mathrm{span}_{\mathbb{R}}((1,1,0,0),(0,1,1,-1))$ (the subspace of $W_{\mathbb{R}}$ orthogonal to L) and we consider $\Lambda_0 = V_{0,\mathbb{R}} \cap W_{\mathbb{Z}}$ and the central character $\lambda : S[\mathfrak{k}] \to \mathbb{C}$ defined by $\lambda(t_1, t_2) = (\tilde{\lambda}, \tilde{\lambda})$ for $\tilde{\lambda} \in \mathbb{C}$. We take $\eta = (-1,-1)$ to be the restriction of the character θ of G.

Then V_λ is cut out in W (or $V_{\lambda,\mathbb{R}}$ inside $W_{\mathbb{R}}$) by the following equations:

$$\begin{cases} -x_1 + x_2 - i_1 = \tilde{\lambda} \\ \quad x_1 - x_2 - i_2 = \tilde{\lambda} \end{cases},$$

equivalently,

$$\begin{cases} i_1 + i_2 = -2\tilde{\lambda} \\ x_1 - x_2 = i_2 + \tilde{\lambda} \end{cases}.$$

This is a 2-dimensional affine subspace of W. We identify V_λ with $\mathbb{C}^2$ (or $V_{\lambda,\mathbb{R}}$ with $\mathbb{R}^2$) by choosing the origin of V_λ to be the point $(0,0,-\tilde{\lambda},-\tilde{\lambda})$ and the basis $u_1 := (1,1,0,0)$, $u_2 := (0,1,1,-1)$. Next we pick a one-parameter subgroup $\xi = (2,1)$. In case $\tilde{\lambda} \in \mathbb{Z}$, we have $\mathcal{P}_{\lambda,\xi} = \{+---, -+--, ----, ---+, --+-\}$ (see Figure 4.2).

Remark 4.2.4. If $\alpha \in \mathcal{P}_{\Lambda,\xi}$ and ξ is a generic character then the differential of ξ attains its maximal value at a single point of P_α. This point will be denoted by a_α. It is the intersection of $\dim(V_\lambda)$ hyperplanes from

$$h_i^+ = 0 \text{ for all } i \in I_\Lambda \text{ with } \alpha(i) = + \text{ and } h_i^- = 0 \text{ for all } i \in I_\Lambda \text{ with } \alpha(i) = -.$$

Let C_α be the unique polyhedral cone cut out in V_λ by $\dim V_\lambda$ inequalities

$$h_i^+ \geq 0 \text{ for all } i \in I_\Lambda, h_i^+(a_\alpha) = 0 \text{ with } \alpha(i) = + \text{ and}$$

$$h_i^- \leq 0 \text{ for all } i \in I_\Lambda, h_i^-(a_\alpha) = 0 \text{ with } \alpha(i) = -.$$

Notice that $P_\alpha \subset C_\alpha$ and the differential of ξ is negative on the extremal rays of C_α.

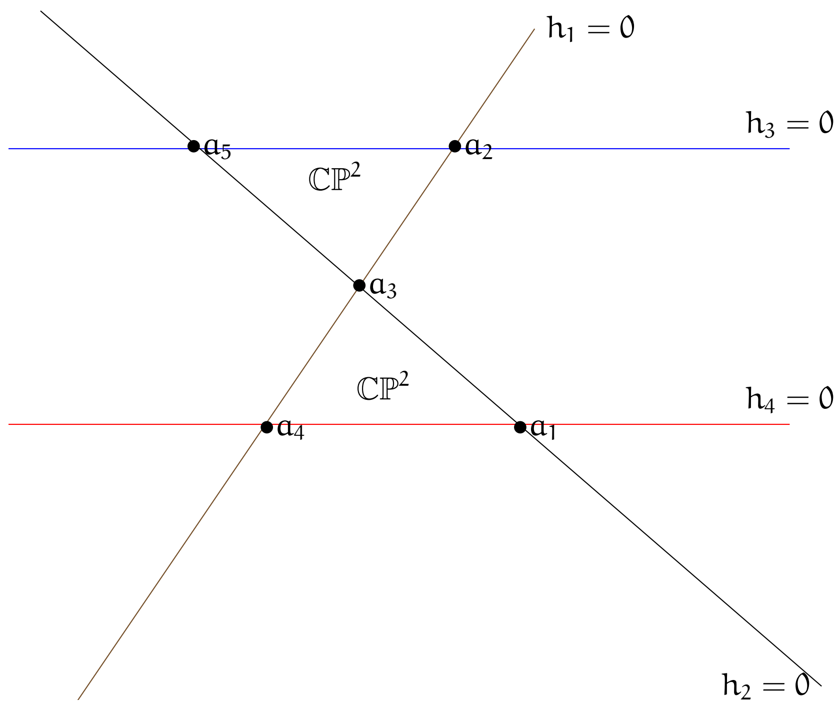


Figure 4.1: Polarized arrangement for $\ell = 2$

Next we describe the standard objects of $\mathcal{O}(\mathbf{X})$. For any sign vector $\alpha \in \{+,-\}^{I_\Lambda}$, consider the $\mathbb{D}$-module

$$\triangle_\alpha := \mathbb{D}/I_\alpha,$$

where I_α is the left ideal generated by the elements

- $\partial_i, h_i^+(a_\alpha) = 0,$
- $x_i, h_i^-(a_\alpha) = 0,$
- $h_i^+ - h_i^+(a_\alpha), i \notin I_\Lambda.$

Define $\triangle_\alpha^\Lambda := \bigoplus_{\nu \in \Lambda} (\triangle_\alpha)_\nu$, then the standard objects of $\mathcal{O}(\Lambda_0, \Lambda, \xi)$ are $\triangle_\alpha^\Lambda$ for $\alpha \in \mathcal{P}_{\Lambda,\xi}$ (see Section 4.4 in [BLPW12]). Let S_α^Λ denote the unique simple quotient of $\triangle_\alpha^\Lambda$.

We will need one more definition.

Definition 4.2.5. The quantized polarized arrangement $\mathbf{X} = (\Lambda_0, \Lambda, \xi)$ and polarized arrangement $X = (\Lambda_0, \eta, \xi)$ are said to be *linked* if $\pi(\mathcal{F}_\Lambda) = \mathcal{F}_\eta$ for the projection $\pi : \{1, \ldots, n\} \to I_\Lambda$.

Remark 4.2.6. The hypertoric category $\mathcal{O}_\lambda$ is a category $\mathcal{O}_\xi$ for U_λ in the sense of Definition 1.1.5.

Remark 4.2.7. If $\mathbf{X} = (\Lambda_0, \Lambda, \xi)$ is regular, then the category $\mathcal{O}(\mathbf{X})$ is highest weight and Koszul (see Definition 2.10 and Corollary 4.10 in [BLPW12]).

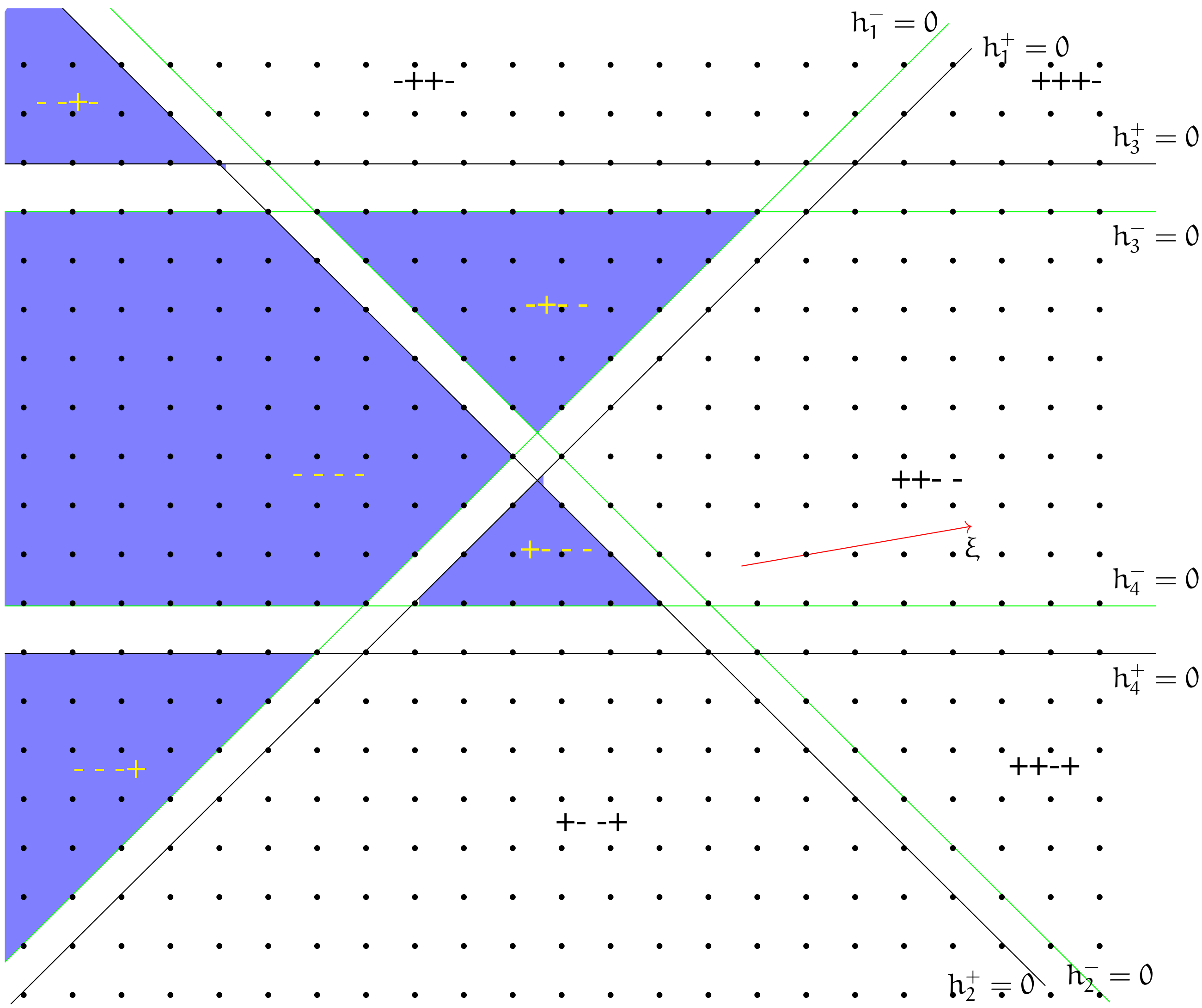


Figure 4.2: Chambers and sign vectors for $\ell = 2$

4.3 Hypertoric category $\mathcal{O}$ for the slice $\mathcal{SL}_p$

The slice $\mathcal{SL}_p$ is the hypertoric variety obtained from the $K = (\mathbb{C}^*)^2$-action on $\mathbb{C}^{2\ell}$ via

$t{\cdot}(x_1, \dots, x_{\ell-1}, x_\ell, \dots, x_{2\ell-2}, i_1, i_2) = (t_1^{-1}t_2x_1, \dots, t_1^{-1}t_2x_{\ell-1}, t_1t_2^{-1}x_\ell, \dots, t_1t_2^{-1}x_{2\ell-2}t_1^{-1}i_1, t_2^{-1}i_2),$

it can be also viewed as a quiver variety (see Figure 3.1). This is the toric variety $\mathfrak{M}(X)$ for the polarized arrangement $X = (\Lambda_0, \eta, \xi)$. Let $\Lambda_0 = V_{0,\mathbb{R}} \cap W_{\mathbb{Z}}$ for $V_{0,\mathbb{R}} = \operatorname{span}_{\mathbb{R}}(u_1, \dots, u_{2\ell-2})$ (where the vectors $u_1, \dots, u_{2\ell-2}$ are defined below) and the same character λ and same η as for $\ell = 2$ above, then V_λ is the affine subset of W given by

$$\begin{cases} -\sum_{k=1}^{\ell-1} x_k + \sum_{k=1}^{\ell-1} x_{\ell-1+k} - i_1 = \tilde{\lambda} \\ \sum_{k=1}^{\ell-1} x_k - \sum_{k=1}^{\ell-1} x_{\ell-1+k} - i_2 = \tilde{\lambda} \end{cases},$$

equivalently,

$$\begin{cases} i_1 + i_2 = -2\tilde{\lambda} \\ \sum_{k=1}^{\ell-1} x_k - \sum_{k=1}^{\ell-1} x_{\ell-1+k} = i_2 + \tilde{\lambda} \end{cases}.$$

This is a $2\ell - 2$-dimensional affine subspace of W. We choose the origin to be the point $(0, \dots, 0, 0, -\tilde{\lambda}, -\tilde{\lambda})$ and the basis

$$\begin{aligned} u_1 &:= (1, -1, 0, \dots, 0, 0) \\ u_2 &:= (1, 0, -1, \dots, 0, 0) \\ &\dots \\ u_{\ell-1} &:= (1, 0, \dots, -1, 0, \dots, 0, 0) \\ u_\ell &:= (1, 0, \dots, 0, 1, 0, \dots, 0, 0) \\ u_{\ell+1} &:= (1, 0, \dots, 0, 0, 1, 0, \dots, 0, 0) \\ u_{2\ell-3} &:= (1, 0, \dots, 0, 1, 0, 0) \\ u_{2\ell-2} &:= (0, \dots, 0, 1, 1, -1). \end{aligned}$$

One convenient choice of the character is $\xi = (1, \dots, \ell-2, \ell, \dots, 2(\ell-1), \ell-1)$.

Definition 4.3.1. The algebra $\overline{\mathcal{S}}_\lambda(2, \ell)$ will stand for the quantization of the slice $\mathcal{SL}_p$ with period $(\lambda + \frac{1}{2}, \lambda + \frac{1}{2})$.

Remark 4.3.2. The restriction of the quantization $\overline{\mathcal{A}}_\lambda(2, \ell)$ to $\mathcal{SL}_p$ is $\overline{\mathcal{S}}_\lambda(2, \ell)$. This is true since the map $\hat{r}$ from Section 5.4 of [BL15] sends λ to (λ, λ). Indeed, $\hat{r}(\lambda) = r(\lambda - \zeta) + \tilde{\zeta}$, where $\zeta = -\frac{1}{2}$ is the character for the action of G on $\Lambda^{\mathrm{top}}\bar{R}$, $\tilde{\zeta} = (-\frac{1}{2}, -\frac{1}{2})$ is the character for the action of K on $\Lambda^{\mathrm{top}}\mathbb{C}^{2\ell}$ and $r(\nu) = (\nu, \nu)$ is the restriction.

Proposition 4.3.3. *Pick a central character $\lambda : Z(U) \to \mathbb{C}$ with $\tilde{\lambda} \in (-\infty; 1-\ell) \cup (\ell-2; \infty)$, let $\tilde{\Lambda} := \{\nu \in W_{\mathbb{Z}} \mid h^+_{2\ell-1}(\nu) + h^+_{2\ell}(\nu) = -2\tilde{\lambda},\ \sum_{k=1}^{\ell-1} h^+_k(\nu) - \sum_{k=1}^{\ell-1} h^+_{\ell-1+k}(\nu) = h^+_{2\ell}(\nu) + \tilde{\lambda}\}$. The quantized polarized arrangement $\mathbf{X} = (\Lambda_0, \tilde{\Lambda}, \xi)$ is regular.*

Remark 4.3.4. Henceforth, unless stated explicitly otherwise, we work with λ with corresponding $\tilde{\lambda}$ regular.

Remark 4.3.5. Abelian localization holds for the algebra $\overline{\mathcal{S}}_\lambda(2,\ell)$ for $\lambda < 1-\ell$ (it is easy to see that $\mathcal{F}_\Lambda = \mathcal{F}_{\Lambda + r\eta}$ with $r \in \mathbb{Z}_{\geq 0}$, so the conditions of Theorem 6.1 in [BLPW12] are met).

Proposition 4.3.6. *Pick a central character $\lambda : Z(U) \to \mathbb{C}$ with $\tilde{\lambda} \in \mathbb{Z}_{<0} + 1 - \ell$, let $\mathbf{X} = (\Lambda_0, \tilde{\Lambda}, \xi)$ be the quantized polarized arrangement. Assume, in addition, $\mathbf{X}$ is linked to X (see Definition 4.2.5).*

(a) There is an equivalence of categories $\mathcal{O}_\xi(\overline{\mathcal{S}}_\lambda(2,\ell)) = \mathcal{O}(\mathbf{X})$.

(b) The set of feasible bounded vectors $\mathcal{P}_{\Lambda,\xi}$ consists of the following $4\ell-3$ sign vectors (notice that the sign vector $\alpha_{\mathrm{mid}} = \underbrace{- - - \ldots -}_{\ell-1}\underbrace{- - - - \ldots -}_{\ell-1} - -$ appears in both sets below for convenience but is counted once only)

$$2\ell-1 \begin{cases} \alpha_{2\ell-1} = \underbrace{- \ldots - - -}_{\ell-1}\underbrace{+ \ldots + + -}_{\ell-1} - + \\ \alpha_{2\ell-2} = \underbrace{- \ldots - - -}_{\ell-1}\underbrace{+ \ldots + - -}_{\ell-1} - + \\ \ldots \\ \alpha_{\ell+1} = \underbrace{- \ldots - - -}_{\ell-1}\underbrace{- \ldots - - -}_{\ell-1} - + \\ \alpha_{\mathrm{mid}} = \underbrace{- \ldots - - -}_{\ell-1}\underbrace{- \ldots - - -}_{\ell-1} - - \\ \alpha_{\ell-1} = \underbrace{- \ldots - - +}_{\ell-1}\underbrace{- \ldots - - -}_{\ell-1} - - \\ \alpha_{\ell-2} = \underbrace{- \ldots - + +}_{\ell-1}\underbrace{- \ldots - - -}_{\ell-1} - - \\ \ldots \\ \alpha_1 = \underbrace{+ \ldots + + +}_{\ell-1}\underbrace{- \ldots - - -}_{\ell-1} - - \end{cases},$$

$$2\ell-1\begin{cases}\beta_{2\ell-1}=\underbrace{-++\ldots+}_{\ell-1}\underbrace{---\ldots-}_{\ell-1}+-\\ \beta_{2\ell-2}=\underbrace{--+\ldots+}_{\ell-1}\underbrace{---\ldots-}_{\ell-1}+-\\ \ldots\\ \beta_{\ell+1}=\underbrace{---\ldots-}_{\ell-1}\underbrace{---\ldots-}_{\ell-1}+-\\ \alpha_{\mathrm{mid}}=\underbrace{---\ldots-}_{\ell-1}\underbrace{---\ldots-}_{\ell-1}--\\ \beta_{\ell-1}=\underbrace{---\ldots-}_{\ell-1}\underbrace{+--\ldots-}_{\ell-1}--\\ \beta_{\ell-2}=\underbrace{---\ldots-}_{\ell-1}\underbrace{++-\ldots-}_{\ell-1}--\\ \ldots\\ \beta_{1}=\underbrace{---\ldots-}_{\ell-1}\underbrace{+++\ldots+}_{\ell-1}--\end{cases}.$$

(c) *The simple and standard objects in the category $\mathcal{O}_\xi(\overline{\mathcal{S}}_\lambda(2,\ell))$ are indexed by the sign vectors in (b). We have the short exact sequences $0\to S^\wedge_{\alpha_{i+1}}\to\Delta^\wedge_{\alpha_i}\to S^\wedge_{\alpha_i}\to 0$ (resp. $0\to S^\wedge_{\beta_{i+1}}\to\Delta^\wedge_{\beta_i}\to S^\wedge_{\beta_i}\to 0$). The socle filtration of $\Delta^\wedge_{\alpha_{\mathrm{mid}}}$ has subquotients $S^\wedge_{\alpha_{\mathrm{mid}}}, S^\wedge_{\alpha_{\ell+1}}$ and $S^\wedge_{\beta_{\ell+1}}$. Finally, if $1\le i<\ell$, we have that $\Delta^\wedge_{\alpha_i}$ (resp. $\Delta^\wedge_{\beta_i}$) have a socle filtration with subquotients $S^\wedge_{\alpha_i}, S^\wedge_{\alpha_{i+1}}$, $S^\wedge_{\beta_{2\ell-i}}$ and $S^\wedge_{\beta_{2\ell-i+1}}$ (resp. $S^\wedge_{\beta_i}, S^\wedge_{\beta_{i+1}}$, $S^\wedge_{\alpha_{2\ell-i}}$ and $S^\wedge_{\alpha_{2\ell-i+1}}$).*

(d) *We have dim(Hom($\Delta^\wedge_\gamma,\Delta^\wedge_\alpha$)) $=1$, if $S^\wedge_\gamma$ appears as a subquotient in filtration of $\Delta^\wedge_\alpha$ and dim(Hom($\Delta^\wedge_\beta,\Delta^\wedge_\alpha$)) $=0$ otherwise as determined in (c).*

Proof. Since Λ_0 is unimodular and **X** is integral, i.e. $\Lambda\subset W_{\mathbb{Z}}$, (a) follows from Remark 4.2 of [BLPW12]. To determine the sign vector α of each chamber, we first notice that ξ is maximized at one of the vertices. The vertex is formed by the intersection of $2\ell-2$ hyperplanes in the arrangement (see Table 4.3). The corresponding $2\ell-2$ coordinates of α are derived from the decomposition of ξ in terms of the normal vectors to the $2\ell-2$ hyperplanes (in Table 4.2 the direction of each normal vector η_i is chosen so that the corresponding coordinate x_i increases along η_i). There is a unique way to choose the polyhedral cone C_α so that the dot product of any vector inside the cone with ξ is negative. The remaining two coordinates are determined by the coordinates of the vertex itself (see Tables 4.2 and 4.3).

We proceed with verifying the assertions in (c) and (d). The appearance of $S^\wedge_\gamma$ in the composition series of $\triangle^\wedge_\alpha$ is equivalent to the containment $P_\gamma\subset C_\alpha$ (see Proposition 4.15 in [BLPW12]). This, in turn, means that the $2\ell-2$ coordinates of the sign vectors γ and α corresponding to the defining hyperplanes of a_α coincide (here a_α is the point of maximum of ξ on the chamber P_α). The result follows from the explicit description provided in Table 4.3.

□

Proposition 4.3.7. *If* $\tilde{\lambda} \in \mathbb{Z}_{<0} + \frac{3}{2} - \ell$, *we have* $\mathcal{O}_{\xi}(\overline{\mathcal{S}}_{\lambda}(2,\ell)) = \bigoplus_{i=1}^{\ell-1}(\mathcal{O}_{\{\alpha_i,\beta_{2\ell-i}\}} \oplus \mathcal{O}_{\{\beta_i,\alpha_{2\ell-i}\}}) \oplus \mathcal{O}_{\alpha_{mid}}$. *Each block of the form* $\mathcal{O}_{\{a,b\}}$ *is equivalent to the principal block* $\mathcal{O}_0$ *in the BGG category* $\mathcal{O}$ *for* $\mathfrak{sl}_2$. *In case* $\tilde{\lambda} \notin \frac{\mathbb{Z}}{2}$, *the category* $\mathcal{O}_{\xi}(\overline{\mathcal{S}}_{\lambda}(2,\ell))$ *is semisimple.*

Proof. According to the general result on block decompositions of hypertoric categories $\mathcal{O}$ (see (4.2.1)), it is sufficient to notice that the partition of points a_{γ} corresponding to sign vectors γ according to Λ_0-orbits in which they lie, is the same as for corresponding sign vectors in the proposed block decompositions (see Table 4.2). □

Example 4.3.8. We illustrate the results for $\ell = 2$ (see also Figure 4.2). In case $\tilde{\lambda} \in \mathbb{Z}$ we have $\mathcal{P}_{\lambda,\xi} = \{1 = +---, 2 = -+--, 3 = ----, 4 = ---+ \text{ and } 5 = --+-\}$. The standards in $\mathcal{O}_{\nu}(\overline{\mathcal{S}}_{\lambda}(2,2))$ are filtered as shown in the table below. In case $\tilde{\lambda} \in \mathbb{Z}+\frac{1}{2}$, we have

Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
S_1	S_2	S_3	S_4	S_5
S_3	S_3	S_4		
S_5	S_4	S_5		

Table 4.1: Multiplicities of simples in standards for $\ell = 2$

$\mathcal{O}_{\xi}(\overline{\mathcal{S}}_{\lambda}(2,2)) = \mathcal{O}_{\{1,5\}} \oplus \mathcal{O}_{\{2,4\}} \oplus \mathcal{O}_3$. Finally, if $\tilde{\lambda} \notin \frac{\mathbb{Z}}{2}$, the category $\mathcal{O}_{\xi}(\overline{\mathcal{S}}_{\lambda}(2,2)) = \bigoplus_{i=1}^{5}\mathcal{O}_i$ is semisimple.

Hyperplane	$h_i = 0 \cap V_\lambda$	Normal vector
$h_1 = 0$	$\sum_{i=1}^{2\ell-3} u_i = 0$	$\eta_1 = (1,\dots,1,0)$
$h_2 = 0$	$u_1 = 0$	$\eta_2 = (-1,0,\dots,0)$
$h_3 = 0$	$u_2 = 0$	$\eta_3 = (0,-1,0\dots,0)$
$\dots$	$\dots$	$\dots$
$h_{2\ell-3} = 0$	$u_{2\ell-4} = 0$	$\eta_{2\ell-3} = (0,\dots,0,-1,0,0)$
$h_{2\ell-2} = 0$	$u_{2\ell-3} + u_{2\ell-2} = 0$	$\eta_{2\ell-2} = (0,\dots,0,-1,-1)$
$h_{2\ell-1} = 0$	$u_{2\ell-2} = -a$	$\eta_{2\ell-1} = (0,\dots,0,1)$
$h_{2\ell} = 0$	$u_{2\ell-2} = a$	$\eta_{2\ell} = (0,\dots,0,-1)$

Table 4.2: Collection of hyperplanes and normal vectors

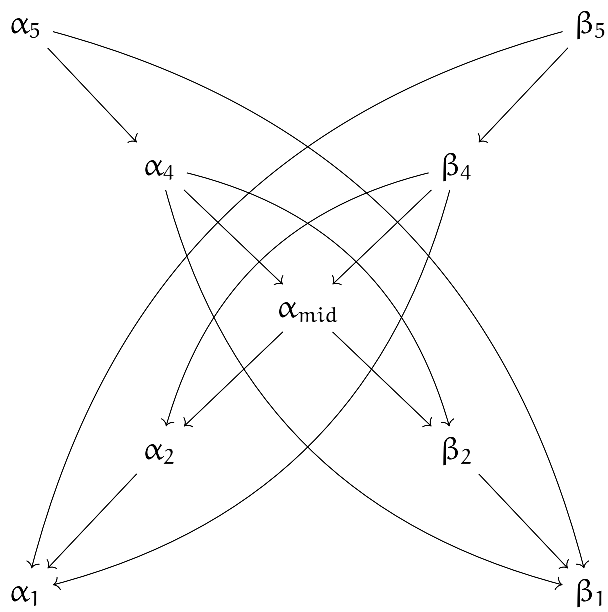


Figure 4.3: Homs between standards in $\mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,3))$ for $\lambda \in -2 + \mathbb{Z}_{<0} \cup \mathbb{Z}_{>0} + 1$

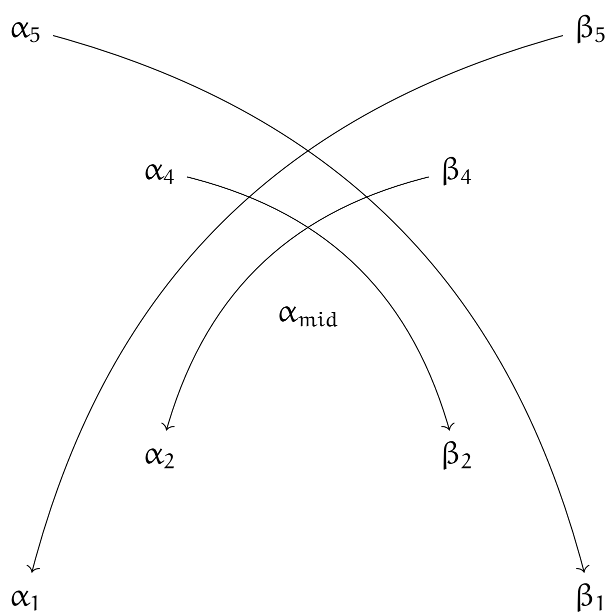


Figure 4.4: Homs between standards in $\mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,3))$ for $\lambda \in -\frac{5}{2} + \mathbb{Z}_{<0} \cup \mathbb{Z}_{>0} + \frac{3}{2}$

Sign vector $\gamma \in \mathcal{P}_{\Lambda,\xi}$	Coordinates of a_γ in W	Hyperplanes H, s.t. $a_\alpha \notin H$
α_1	$(\lambda, 0, \dots, 0, -2\lambda, 0)$	$h_1 = 0, h_{2\ell-1} = 0$
β_1	$(0, 0, \dots, 0, \lambda, 0, -2\lambda)$	$h_{2\ell-2} = 0, h_{2\ell} = 0$
$\dots$	$\dots$	$\dots$
α_{mid}	$(0, 0, \dots, 0, -\lambda, -\lambda)$	$h_{2\ell-1} = 0, h_{2\ell} = 0$
$\dots$	$\dots$	$\dots$
$\alpha_{2\ell-1}$	$(0, \dots, 0, \lambda, -2\lambda, 0)$	$h_{2\ell-2} = 0, h_{2\ell-1} = 0$
$\beta_{2\ell-1}$	$(\lambda, 0, \dots, 0, -2\lambda)$	$h_1 = 0, h_{2\ell} = 0$

Table 4.3: Sign vectors, walls of chambers and points of maximum of ξ

Chapter 5

Harish-Chandra bimodules, ideals and localization theorems

In this chapter we recall the definition of Harish-Chandra bimodules and the restriction functor between the bimodules on the variety and the slice. We show how using this functor allows to obtain results on two-sided ideals and abelian localization.

Definition 5.0.1. Let $\overline{\mathcal{A}}_\lambda(n,\ell), \overline{\mathcal{A}}_{\lambda'}(n,\ell)$ be two quantizations of $A = \mathbb{C}[\overline{\mathcal{M}}(n,\ell)]$. An $\overline{\mathcal{A}}_\lambda(n,\ell) - \overline{\mathcal{A}}_{\lambda'}(n,\ell)$-bimodule $\mathcal{B}$ is *Harish-Chandra* (HC) provided there exists a filtration on $\mathcal{B}$, s.t. the induced left and right actions of A on $\operatorname{gr}\mathcal{B}$ coincide and $\operatorname{gr}\mathcal{B}$ is a finitely generated A-module. Such filtrations will be referred to as *good*.

Pick a point $x \in \overline{\mathcal{M}}(2,\ell)$ on a symplectic leaf $\mathcal{L}$. Then for the slice $\mathcal{SL}_x$ at x we have a restriction functor (see Section 5.4 of [BL15])

$$\mathrm{Res}_{\dagger,x} : \mathrm{HC}(\overline{\mathcal{A}}_\lambda(2,\ell) - \overline{\mathcal{A}}_{\lambda'}(2,\ell)) \to \mathrm{HC}(\mathcal{SL}_{x,\tilde{\lambda}} - \mathcal{SL}_{x,\tilde{\lambda'}}).$$

This functor is exact.

Theorem 5.0.2. *If* $\lambda \in (-\infty; 1-\ell) \cup (\ell-2; +\infty)$ *is not an integer or half-integer, then the algebra* $\overline{\mathcal{A}}_\lambda(2,\ell)$ *has no proper two-sided ideals.*

Proof. Assume $\mathcal{I}$ is a proper two-sided ideal in $\overline{\mathcal{A}}_\lambda(2,\ell)$. Then pick a point x in an open symplectic leaf in $V(\overline{\mathcal{A}}_\lambda(2,\ell)/\mathcal{I})$, so $\mathrm{Res}_{\dagger,x}(\mathcal{I})$ is an ideal in the algebra $\mathcal{SL}_{x,\tilde{\lambda}}$. Since for λ as in the statement of the theorem there no finite-dimensional representations neither in the category $\mathcal{S}_\lambda$-mod nor the category of finitely generated modules over the corresponding quantization of the 2-dimensional slice, the type A_1 Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_2$ (see Remark 3.1.2), the argument is concluded by contradiction. □

Similarly, we prove the following.

Theorem 5.0.3. *Abelian localization holds for* (λ, θ) *with* $\theta < 0$ *and* $\lambda < -\ell$ *or* $\theta > 0$ *and* $\lambda > \ell - 1$.

Proof. The argument is completely anagolous to the one in the proof of Lemma 5.3 in [Los18], which can be briefly summarized as follows. The abelian localization holds for λ if and only if the natural homomorphisms for $m \gg 0$ and $\chi = \det$

$$\begin{aligned} \overline{\mathcal{A}}_{\lambda+(m+1)\chi,-\chi}(2,\ell) \otimes_{\overline{\mathcal{A}}_{\lambda+(m+1)\chi}} \overline{\mathcal{A}}_{\lambda+m\chi,\chi}(2,\ell) \to \overline{\mathcal{A}}_{\lambda+m\chi}(2,\ell) \\ \overline{\mathcal{A}}_{\lambda+m\chi,\chi}(2,\ell) \otimes_{\overline{\mathcal{A}}_{\lambda+m\chi}} \overline{\mathcal{A}}_{\lambda+(m+1)\chi,-\chi}(2,\ell) \to \overline{\mathcal{A}}_{\lambda+(m+1)\chi}(2,\ell), \end{aligned} \tag{5.0.1}$$

with $\overline{\mathcal{A}}_{\lambda+m\chi,\chi}(2,\ell) := (D(\overline{R})/[D(\overline{R})\{\Phi(x) - (\lambda + m\chi)(x), x \in \mathfrak{g}\}])^{G,\chi}$ the $\overline{\mathcal{A}}_{\lambda+(m+1)\chi} - \overline{\mathcal{A}}_{\chi}$-bimodule, are isomorphisms. Assuming that this is not the case, there must be a nontrivial module M in the kernel or cokernel of the first or the second map. Then the support of M must be $\overline{\mathcal{L}}$, the closure of a symplectic leaf $\mathcal{L}$. Applying the functor $\mathrm{Res}_{\dagger,x}$ to (5.0.1) with $x \in \mathcal{L}$, we again get natural homomorphisms. Furthermore, since the order on the leaves is linear and $\mathcal{L} \neq o$ (otherwise M would be of finite dimension, which is impossible due to Corollary 2.2.2), we can pick x to be on the 2ℓ-dimensional leaf (the one with number 3 in Table 3.1). Since the slice $\mathcal{SL}_x$ is the hypertoric variety $\mathcal{SL}_p$ and abelian localization holds for the algebra $\overline{\mathcal{S}}_\lambda(2,\ell)$ for $\lambda < -\ell$ (see Remark 4.3.5), the restricted homomorphisms must be isomorphisms. As the module $\mathrm{Res}_{\dagger,x}(M)$ is nonzero, we obtain a contradiction.

The assertion for $\theta > 0$ and $\lambda > \ell - 1$ follows from the isomorphism $\overline{\mathcal{A}}_\lambda(n,\ell) \cong \overline{\mathcal{A}}_{-\lambda-1}(n,\ell)$ (see Lemma 2.0.1). □

Corollary 5.0.4. *If* $\lambda \in (-\infty; -\ell) \cup (\ell - 1; +\infty)$*, then the algebra* $\overline{\mathcal{A}}_\lambda(2,\ell)$ *has finite homological dimension.*

Proof. Theorem 1.1 of [MN14] asserts that the derived localization holds for λ if and only if $\overline{\mathcal{A}}_\lambda(2,\ell)$ is of finite homological dimension. □

Chapter 6

Structure of the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$

The main goal of the present chapter is to present a proof of Theorem 6.4.1 and Corollary 6.4.2, which provide a complete description of homomorphisms between standard objects in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$ and multiplicities of simple objects in the standard ones. In order to accomplish this task we make an extensive use of methods and results introduced in [BLPW16] and [Los17]. A brief overview of these techniques will be given in Sections 6.1 through 6.3, after which the chapter concludes with the proof of Theorem 6.4.1.

6.1 Parabolic induction functor

Let $\rho : X \to X_0$ be a conical symplectic resolution equipped with a Hamiltonian action of a torus T. Following Section 5.5 of [Los17], introduce a pre-order $\prec^\lambda$ on $\mathrm{Hom}(\mathbb{C}^*, T)$, the one-parameter subgroups of T, via $\nu' \prec^\lambda \nu$, if

- $\mathcal{A}_\lambda(\mathcal{A}_\lambda^{>0,\nu'} + (\mathcal{A}_\lambda^{\nu'})^{>0,\nu}) = \mathcal{A}_\lambda\mathcal{A}_\lambda^{>0,\nu}$;
- the natural action of $\nu'(\mathbb{C}^*)$ on $C_\nu(\mathcal{A}_\lambda)$ is trivial.

The following result was established in [Los17] (see Lemma 5.8 therein).

Lemma 6.1.1. *Consider two elements $\nu, \nu' \in \mathrm{Hom}(\mathbb{C}^*, T)$, s.t. $\nu' \prec^\lambda \nu$. Then $C_\nu(\mathcal{A}_\lambda) = C_\nu(C_{\nu'}(\mathcal{A}_\lambda))$. Furthermore, there is an isomorphism of functors $\triangle_\nu = \triangle_{\nu'} \circ \underline{\triangle}$, where $\triangle_{\nu'} : C_{\nu'}(\mathcal{A}_\lambda)$ -mod $\to \mathcal{A}_\lambda$ -mod, $\underline{\triangle} : C_\nu(\mathcal{A}_\lambda)$ -mod $\to C_{\nu'}(\mathcal{A}_\lambda)$ -mod and $\triangle_\nu$ is the standardization functor given by Definition 1.5.8.*

Proposition 6.1.2. *Let $X = \overline{\mathcal{M}}^\theta(2,\ell)$ and consider the one-parameter subgroups $\nu = (t^{d_1}, t^{d_2}, \dots, t^{d_\ell})$ with $d_1 \gg d_2 > d_3 > \dots > d_\ell > 0$ and $\tilde{\nu} = (t^d, 1, \dots, 1)$ with $d > 0$. Then we have $\tilde{\nu} \prec^\lambda \nu$.*

Proof. We are going to find the sufficient condition on ℓ-tuples of weights $(d_1, d_2, \dots, d_\ell)$, so that for the corresponding one-parameter subgroup $\nu = (t^{d_1}, t^{d_2}, \dots, t^{d_\ell})$ the following containments hold

$$\overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\tilde{\nu}} \subseteq \overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\nu}, \tag{6.1.1}$$

For verifying the reverse containment it suffices to check that

$$\overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\nu} \subseteq \overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\tilde{\nu}} + (\overline{\mathcal{A}}_\lambda(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)})^{>0,\nu} \text{ and} \tag{6.1.2}$$

$$\overline{\mathcal{A}}_\lambda(2,\ell)^{\geq 0,\nu} \subseteq \overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\tilde{\nu}} + (\overline{\mathcal{A}}_\lambda(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)})^{\geq 0,\nu}. \tag{6.1.3}$$

Clearly, for such ν the equality

$$\overline{\mathcal{A}}_\lambda(2,\ell)(\overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\tilde{\nu}} + (\overline{\mathcal{A}}_\lambda(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)})^{>0,\nu}) = \overline{\mathcal{A}}_\lambda(2,\ell)\overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\nu} \tag{6.1.4}$$

holds. Recall that $C_\nu(\overline{\mathcal{A}}_\lambda(2,\ell)) = \overline{\mathcal{A}}_\lambda(2,\ell)^{\geq 0,\nu} / \left(\overline{\mathcal{A}}_\lambda(2,\ell)^{\geq 0,\nu} \cap \overline{\mathcal{A}}_\lambda(2,\ell)\overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\nu}\right)$ and the latter is equal to $\overline{\mathcal{A}}_\lambda(2,\ell)^{\geq 0,\nu} / \left(\overline{\mathcal{A}}_\lambda(2,\ell)^{\geq 0,\nu} \cap \overline{\mathcal{A}}_\lambda(2,\ell)(\overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\tilde{\nu}} + (\overline{\mathcal{A}}_\lambda(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)})^{>0,\nu})\right)$ due to equality (6.1.4), where the action of $\tilde{\nu}(\mathbb{C}^*)$ is trivial thanks to (6.1.3).

It remains to construct the tuples of numbers $(d_1, \dots, d_\ell)$, s.t. the containments (6.1.1)-(6.1.3) hold. The algebra of semiinvariants $\mathbb{C}[\overline{\mathcal{M}}(2,\ell)]^{\geq 0,\tilde{\nu}} = \mathrm{gr}(\overline{\mathcal{A}}_\lambda(2,\ell)^{\geq 0,\tilde{\nu}})$ is finitely generated (see Lemma 3.1.2 in [GL14]). Thus we can choose finitely many T-semiinvariant generators $f_1, \dots, f_s$ of the ideal $\mathbb{C}[\overline{\mathcal{M}}(2,\ell)]^{>0,\tilde{\nu}}$ with $f_i \in \mathbb{C}[\overline{\mathcal{M}}(2,\ell)]_{\chi_i}$ a T-semiinvariant of weight $\chi_i = (a^i_1, \dots, a^i_\ell)$. Let $\tilde{f}_1, \dots, \tilde{f}_s$ denote the lifts of the generators to $\overline{\mathcal{A}}_\lambda(2,\ell)^{\geq 0,\tilde{\nu}}$. These lifts generate $\overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\tilde{\nu}}$. Fix the collection of numbers $d_2 > d_3 > \dots > d_\ell > 0$, denote $\mathfrak{a}_i := \min_j\{a^j_i\}$ and pick $\nu' = (t^{d'_1}, t^{d_2}, \dots, t^{d_\ell})$ with

$$d'_1 > \max(d_2, -\sum_{1<i\leq \ell, \mathfrak{a}_i<0} \mathfrak{a}_i d_i). \tag{6.1.5}$$

We see that $\tilde{f}_i$ being in $\mathcal{A}_\lambda^{>0,\tilde{\nu}}$ imposes $a^i_1 > 0$ for all $i \in \{1, \dots, s\}$, hence, $\tilde{f}_i \in \mathcal{A}_\lambda^{>0,\nu}$ due to (6.1.5), so the containment (6.1.1) holds for ν' in place of ν.

Similarly, let $g_1, \dots, g_k$ be the T-semiinvariant generators of the algebra $\mathbb{C}[\overline{\mathcal{M}}(2,\ell)]^{\geq 0,\nu}$ with $g_j \in \mathbb{C}[\overline{\mathcal{M}}(2,\ell)]_{\theta_j}$ a T-semiinvariant of weight $\theta_j = (b^j_1, \dots, b^j_\ell)$. Introduce $\mathfrak{b}_i := \min_j\{b^j_i\}$ and pick $\nu'' = (t^{d''_1}, t^{d_2}, \dots, t^{d_\ell})$ with

$$d''_1 > \max(d_2, -\sum_{1<i\leq \ell, \mathfrak{b}_i<0} \mathfrak{b}_i d_i). \tag{6.1.6}$$

Notice that due to inequality (6.1.6) for all $\tilde{g}_j$ (lifts of g_j's, which generate $\mathcal{A}_\lambda^{\geq 0,\nu}$) $\tilde{g}_j \in \overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\nu}$ implies $\tilde{g}_j \in \overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\tilde{\nu}} + (\overline{\mathcal{A}}_\lambda(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)})^{>0,\nu}$, while $\tilde{g}_j \in \overline{\mathcal{A}}_\lambda(2,\ell)^{\geq 0,\nu}$ implies $\tilde{g}_j \in \overline{\mathcal{A}}_\lambda(2,\ell)^{>0,\tilde{\nu}} + (\overline{\mathcal{A}}_\lambda(2,\ell)^{\tilde{\nu}(\mathbb{C}^*)})^{\geq 0,\nu}$ and, therefore, containments (6.1.2) and (6.1.3) hold for ν'' in place of ν.

Finally, we put $d_1 > \max(d'_1, d''_1)$ so that the conditions (6.1.1)-(6.1.3) all hold true simultaneously for $\nu = (t^{d_1}, t^{d_2}, \dots, t^{d_\ell})$. □

Remark 6.1.3. Consider the pair of one-parameter subgroups $\nu = (t^{d_1}, t^{d_2}, \dots, t^{d_\ell})$ with $d_1 > d_2 > d_3 > \dots > d_{\ell-1} \gg d_\ell > 0$ and $\nu_0 = (t^{d_1}, t^{d_2}, \dots, t^{d_{\ell-1}}, 1)$. Similarly to the argument presented in the proof of Proposition 6.1.2, one shows that the containments (6.1.1)-(6.1.3) hold and hence $\nu_0 \prec^\lambda \nu$.

6.2 Restriction functor

Following [BE09] and [Los18], we define the restriction functor $\mathrm{Res} : \mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell)) \to \mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,\ell))$. Set $\overline{\mathcal{A}}_\lambda(2,\ell)^{\wedge_p} := \mathbb{C}[\overline{R}//G]^{\wedge_p} \otimes_{\mathbb{C}[\overline{R}]^G} \overline{\mathcal{A}}_\lambda(2,\ell)$ and $\overline{\mathcal{S}}_\lambda(2,\ell)^{\wedge_0} := \mathbb{C}[\mathbb{C}^{2\ell}//K]^{\wedge_0} \otimes_{\mathbb{C}[\mathbb{C}^{2\ell}]^K} \overline{\mathcal{S}}_\lambda(2,\ell)$, then anagolously to Lemma 6.4 in [Los18] there is a G-equivariant isomorphism $\Theta : \overline{\mathcal{A}}_\lambda(2,\ell)^{\wedge_p} \to \overline{\mathcal{S}}_\lambda(2,\ell)^{\wedge_0}$ of filtered algebras. It is the quantization of the Nakajima isomorphism of formal neighborhoods (see Section 5.4 of [BL15] for details). Let $\nu_0 = (t^{d_1}, t^{d_2}, \dots, t^{d_{\ell-1}}, 1)$ with $d_1 > d_2 > \dots > d_{\ell-1} > 0$ be a one-parameter subgroup. Consider the category $\mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,\ell))^{\wedge_0}$ consisting of all finitely generated $\overline{\mathcal{S}}_\lambda(2,\ell)^{\wedge_0}$-modules such that

1. $h_0 = d_e\nu_0$ (the differential of ν_0 at $e = (1,\dots,1)$) acts locally finitely with eigenvalues bounded from above;
2. the generalized h_0-eigenspaces are finitely generated over $\mathbb{C}[\mathcal{S}_p]^{\wedge_0}$.

We get an exact functor

$$\mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,\ell)) \to \mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,\ell))^{\wedge_0}, N \mapsto \mathbb{C}[\mathbb{C}^{2\ell}//K]^{\wedge_0} \otimes_{\mathbb{C}[\mathbb{C}^{2\ell}]^K} N.$$

Let h be the image of 1 under the quantum comoment map for $t \mapsto \nu(t)\nu_0(t)^{-1}$. For $N \in \overline{\mathcal{S}}_\lambda(2,\ell)^{\wedge_0}$-mod denote by N_{fin} the subspace of h-finite elements. The statement and proof of the following lemma is analogous to Lemma 6.5 in [Los18].

Lemma 6.2.1. *The functor $\bullet^{\wedge_0}$ is a category equivalence. A quasi-inverse functor is given by $N \mapsto N_{fin}$.*

Finally, define

$$\mathrm{Res}(N) := [\Theta_*(\mathbb{C}[\overline{R}//G]^{\wedge_p} \otimes_{\mathbb{C}[\overline{R}]^G} N)]_{fin}.$$

The following isomorphism of functors will be of crucial importance. It was established in Lemma 6.7 of [Los18].

$$\mathrm{Res} \circ \triangle_{\nu_0} \cong \triangle_{\nu_0} \circ \underline{\mathrm{Res}}, \tag{6.2.1}$$

where $\underline{\mathrm{Res}}$ is the functor $\mathcal{O}_\nu(C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell))) \to \mathcal{O}_\nu(C_{\nu_0}(\overline{\mathcal{S}}_\lambda(2,\ell)))$ defined analogously to Res.

Remark 6.2.2. Let $\lambda < 1-\ell$. As $\mathcal{SL}_p^{\nu_0}$ consists of $4\ell-3$ points and abelian localization holds for λ (Remark 4.3.5), we have $C_{\nu_0}(\overline{\mathcal{S}}_\lambda(2,\ell)) \cong \mathbb{C}^{4\ell-3}$. The variety of fixed points of $\overline{\mathcal{M}}^\theta(2,\ell)^{\nu_0}$ is $T^*\mathbb{CP}^1$ together with the disjoint union of $2(\ell-1)$ copies of $\mathbb{C}^2$ (see Remark 2.1.6). Arguing analogously to the proofs of Theorem 2.1.5 and Proposition 2.1.7 one checks that $C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell)) = \overline{\mathcal{A}}_\lambda(2,1) \oplus \mathcal{D}(\mathbb{C}^2)^{\oplus 2\ell-2} \simeq \mathcal{D}^\lambda(\mathbb{CP}^1) \oplus \mathcal{D}(\mathbb{C}^2)^{\oplus 2\ell-2}$.

Corollary 6.2.3. *Let $\lambda \in \mathbb{Z}_{<0}+1-\ell \cup \mathbb{Z}_{>0}+\ell-2$, then the images of standard and simple objects in $\mathcal{O}_\nu(C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell)))$ are given by*

$$\begin{aligned}
&\mathrm{Res}(\triangle_i) = \triangle_{\alpha_i} \oplus \triangle_{\beta_i}, i > \ell+1\\
&\mathrm{Res}(\triangle_i) = \triangle_{\alpha_{i+1}} \oplus \triangle_{\beta_{i+1}}, i < \ell,\\
&\mathrm{Res}(\triangle_j) = \triangle_{\alpha_{mid}}, j \in \{\ell, \ell+1\},\\
&\mathrm{Res}(S_i) = S_{\alpha_i} \oplus S_{\beta_i}, i > \ell+1,\\
&\mathrm{Res}(S_i) = S_{\alpha_{i+1}} \oplus S_{\beta_{i+1}}, i < \ell,\\
&\mathrm{Res}(S_j) = S_{\alpha_{mid}}, j \in \{\ell, \ell+1\}
\end{aligned}$$

In case $\lambda \in \mathbb{Z}_{<0}+\frac{1}{2}-\ell \cup \mathbb{Z}_{>0}+\ell-\frac{3}{2}$, the only difference is that $\mathrm{Res}(S_\ell)=0$.

Proof. We show that the analog of the assertion of the corollary holds for $\underline{\mathrm{Res}}$ in place of Res, then the result is a direct consequence of Lemma 6.1.1 and equality (6.2.1) (as $\nu_0 \prec^\lambda \nu$ due to Remark 6.1.3). The standard objects of $\mathcal{O}_\nu(C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell)))$ are $\underline{\triangle}(N_s)$, where N_s is the one-dimensional irreducible representation of $C_\nu(\overline{\mathcal{A}}_\lambda(2,\ell)) \simeq \mathbb{C}^{2\ell}$ with the action given by $(a_1,\dots,a_{2\ell})\cdot w := a_s w$ for $(a_1,\dots,a_{2\ell}) \in \mathbb{C}^{2\ell}$ and $0 \neq w \in N_s$. First, let us consider $i \notin \{\ell,\ell+1\}$, then

$$\begin{aligned}
&\underline{\mathrm{Res}}(\underline{\triangle}(N_i)) = \underline{\mathrm{Res}}(C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell))/\mathcal{I}^{>0,\nu} \otimes_{\mathbb{C}^{2\ell}} N_i) =\\
=&\underline{\mathrm{Res}}((\mathcal{D}^\lambda(\mathbb{CP}^1) \oplus \mathcal{D}(\mathbb{C}^2)^{\oplus 2\ell-2})/\mathcal{I}^{>0,\nu} \otimes_{\mathbb{C}^{2\ell}} N_i) = \underline{\mathrm{Res}}(\mathcal{D}(\mathbb{C}^2_s)/\tilde{\mathcal{I}}^{>0,\nu}) =\\
=&\underline{\mathrm{Res}}(\mathbb{C}[x_i,y_i]) \xrightarrow[\varphi]{\sim} M_{\alpha_k} \oplus M_{\beta_k},
\end{aligned}$$

where the map φ is the evaluation at points $(1,0)$ and $(-1,0) \in \mathbb{C}^2_s$, the two points on the s^{th} copy of $\mathbb{C}^2$ which are the $\nu_0(\mathbb{C}^*)$-fixed points with indices α_k and β_k on the slice (see Remark 3.2.2 for details). Here $\mathcal{D}(\mathbb{C}^2_s)$ stands for the algebra of differential operators on the sth copy of $\mathbb{C}^2$, while $\mathcal{I}^{>0,\nu} := C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell))C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell))^{>0,\nu}$, $\tilde{\mathcal{I}}^{>0,\nu} = \mathcal{I}^{>0,\nu} \cap \mathcal{D}_i(\mathbb{C}^2)$ and $M_{\alpha_k}, M_{\beta_k}$ are the one-dimensional irreducibles in $\mathcal{O}_\nu(C_{\nu_0}(\overline{\mathcal{S}}_\lambda(2,\ell)))$ with k as given in the statement of the corollary. In case $i \in \{\ell,\ell+1\}$, it is analogous to check that $\underline{\mathrm{Res}}(\underline{\triangle}(N_i)) = M_{\alpha_{mid}}$. This completes verification of the assertion on the images of standards.

Next we verify the assertion on the images of simples. Let $M \in \mathcal{O}_\nu(C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell)))$, we write $L_{\nu_0}(M)$ for the maximal quotient of $\triangle_{\nu_0}(M)$ that does not intersect the highest weight subspace. Analogously to Corollary 6.8 in [Los18], one checks that $\mathrm{Res}(L_{\nu_0}(M)) = L_{\nu_0}(\underline{\mathrm{Res}}(M))$. Therefore, if $\lambda \notin \mathbb{Z}_{<0}+\frac{1}{2}-\ell \cup \mathbb{Z}_{>0}+\ell-\frac{3}{2}$, then each $L_{\nu_0}(\underline{\triangle}(N_i))$ is irreducible, so $L_\nu(\underline{\triangle}(N_i)) = L_{\nu_0}(\underline{\triangle}(N_i))$ and we have $\operatorname{Res}(S_i) = \operatorname{Res}(L_\nu(\underline{\triangle}(N_i))) = \operatorname{Res}(L_{\nu_0}(\underline{\triangle}(N_i))) = L_{\nu_0}(M_{\alpha_k} \oplus M_{\beta_k}) = S_{\alpha_k} \oplus S_{\beta_k}$ if $i \notin \{\ell, \ell+1\}$ and $\operatorname{Res}(S_\ell) = \operatorname{Res}(S_{\ell+1}) = S_{\alpha_{mid}}$.

Notice, that in case $\lambda \in \mathbb{Z}_{<0} + \frac{1}{2} - \ell \cup \mathbb{Z}_{>0} + \ell - \frac{3}{2}$, we have $\underline{\triangle}(N_{\ell+1}) \subset \underline{\triangle}(N_\ell)$ (see Remark 6.2.2) and, hence, $S_\ell \neq L_{\nu_0}(\underline{\triangle}(N_\ell))$, instead, $S_\ell = L_{\nu_0}(\underline{\triangle}(N_\ell)/\underline{\triangle}(N_{\ell+1}))$, while $\underline{\operatorname{Res}}(\underline{\triangle}(N_\ell)/\underline{\triangle}(N_{\ell+1}))$ is already equal to zero. □

Corollary 6.2.4. *The restriction functor* Res *maps socles of standard objects to socles of their images.*

Proof. We start by noticing that Corollary 6.2.3 implies that Res maps simple objects to semisimple, hence, the containment $\operatorname{Res}(\operatorname{Soc}(\triangle_{\nu_i})) \subseteq \operatorname{Soc}(\operatorname{Res}(\triangle_{\nu_i}))$ follows. The reverse inclusion is a consequence of part (c), Proposition 4.3.6. Namely, it provides an explicit description of socles of standards in the target category, i.e.

$$\begin{aligned}
&\operatorname{Soc}(\triangle_{\alpha_k}) = S_{\alpha_{k+1}}, \operatorname{Soc}(\triangle_{\beta_k}) = S_{\beta_{k+1}} \text{ for } \ell < k < 2\ell - 2,\\
&\operatorname{Soc}(\triangle_{\alpha_{mid}}) = S_{\alpha_{\ell+1}} \oplus S_{\beta_{\ell+1}},\\
&\operatorname{Soc}(\triangle_{\alpha_k}) = S_{\beta_{2\ell-i}}, \operatorname{Soc}(\triangle_{\beta_k}) = S_{\alpha_{2\ell-i}} \text{ for } 1 \leq k < \ell.
\end{aligned}$$

Combining the above with the statement of Corollary 6.2.3, allows to conclude

$$\begin{aligned}
&0 \subsetneq \operatorname{Res}(\operatorname{Soc}(\triangle_k)) \subseteq \operatorname{Res}(S_{k+1}) = S_{\alpha_{k+1}} \oplus S_{\beta_{k+1}} \text{ for } \ell < k \leq 2\ell,\\
&0 \subsetneq \operatorname{Res}(\operatorname{Soc}(\triangle_\ell)) = \operatorname{Res}(\operatorname{Soc}(\triangle_{\ell+1})) \subseteq \operatorname{Res}(S_{\ell+2}) = S_{\alpha_{\ell+1}} \oplus S_{\beta_{\ell+1}},\\
&0 \subsetneq \operatorname{Res}(\operatorname{Soc}(\triangle_k)) \subseteq \operatorname{Res}(S_{k+1}) = S_{\beta_{2\ell-k}} \oplus S_{\alpha_{2\ell-k}} \text{ for } 1 \leq k < \ell,
\end{aligned}$$

so the nonstrict containments in every row must be equalities and the result follows. □

Corollary 6.2.5. *The socles of standards in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$ are as follows:*

1. *if* $\lambda \in \mathbb{Z}_{<0} + 1 - \ell \cup \mathbb{Z}_{>0} + \ell - 2$

$$\begin{aligned}
&\operatorname{Soc}(\triangle_k) = S_{k+1} \textit{ for } \ell + 1 < k < 2\ell,\\
&\operatorname{Soc}(\triangle_{\ell+1}) = \operatorname{Soc}(\triangle_\ell) = S_{\ell+2},\\
&\operatorname{Soc}(\triangle_k) = S_{2\ell-k+1}, \textit{ for } 1 \leq k < \ell.
\end{aligned}$$

2. *if* $\lambda \in \mathbb{Z}_{<0} + \frac{1}{2} - \ell \cup \mathbb{Z}_{>0} + \ell - \frac{3}{2}$

$$\begin{aligned}
&\operatorname{Soc}(\triangle_k) = \triangle_k \textit{ for } \ell < k \leq 2\ell,\\
&\operatorname{Soc}(\triangle_k) = S_{2\ell-k+1}, \textit{ for } 1 \leq k \leq \ell.
\end{aligned}$$

3. *otherwise, if* $\lambda \in (-\infty; 1-\ell) \cup (\ell-2; +\infty)$ *is neither an integer nor a half-integer*

$$\mathrm{Soc}(\triangle_k) = \triangle_k \text{ for } 1 \leq k \leq 2\ell.$$

Remark 6.2.6. Since Res maps simple objects to semisimple, the containment $\mathrm{Res}(\mathrm{Soc}(M)) \subseteq \mathrm{Soc}(\mathrm{Res}(M))$ is true for any $M \in \mathcal{O}_\nu(C_{\nu_0}(\overline{\mathcal{A}}_\lambda(2,\ell)))$.

Proposition 6.2.7. *Let* $\lambda \in \mathbb{Z}_{<0}+1-\ell \cup \mathbb{Z}_{>0}+\ell-2$, *then the support of the simple module* S_1 *has dimension* 2ℓ, *supports of simple modules* $S_2, \dots, S_{\ell+1}$ *have dimension* $4\ell-3$, *supports of simple modules* $S_{\ell+2}, \dots, S_{2\ell}$ *are of dimension* $4\ell-2$.

Proof. By Theorem 1.2 in [Los17] all irreducible components of $\mathrm{Supp}(S_i)$ have the same dimension (the arithmetic fundamental groups are finite due to the general result of [Nam17]). If $\mathrm{Res}(S_i) \neq 0$, there exists an irreducible component of $\mathrm{Supp}(S_i)$, containing the point p and, therefore, the symplectic leaf through it. Hence, codim $\mathrm{Supp}(S_i)$ in $\overline{\mathcal{M}}(2,\ell)$ is equal to codim $\mathrm{Supp}(\mathrm{Res}(S_i))$ in $\mathcal{SL}_p$. It remains to compute the dimensions of $\mathrm{Supp}(\mathrm{Res}(S_\alpha))$'s. It follows from Proposition 5.5 in [BLPW12] that the variety $\mathrm{Supp}(S_\alpha)$ is determined by the sign vector α corresponding to S_α. Namely, $\mathrm{Supp}(S_\alpha)$ is cut out in $\mathcal{SL}_p$ by the equations

$$x_s = 0 \text{ if } \alpha_i(s) = - \text{ and } y_s = 0 \text{ if } \alpha(s) = + \text{ for } s \in \{1, \dots, 2\ell-2\},$$
$$i_k = 0 \text{ if } \alpha(k) = - \text{ and } j_k = 0 \text{ if } \alpha(k) = + \text{ for } k \in \{2\ell-1, 2\ell\}.$$

The sign vectors for simple modules were provided in Proposition 4.3.6.

If $\alpha = \underbrace{-\dots-\overbrace{+\dots+}^{a}}_{\ell-1}\underbrace{-\dots---}_{\ell-1}--$ the coordinate ring $\mathbb{C}[\mathrm{Supp}(S_\alpha))]$ is generated by $u_{ij} := x_i y_j$ and $v_{js} := y_j y_s$ for $i \in \{\ell-a, \dots, \ell-1\}, j \in \{1, \dots, \ell-a-1\}, s \in \{\ell, \dots, 2\ell-2\}$ subject to relations:

$$u_{ij}u_{mn} = u_{mj}u_{in}$$
$$u_{ij}v_{kl} = u_{ik}v_{jl}$$
$$v_{ij}v_{kl} = v_{kj}v_{il}.$$

Therefore, dim $\mathrm{Supp}(S_\alpha) = \ell - a - 1 + \ell - 1 + a - 1 = 2\ell - 3$. The case $\alpha = \underbrace{-\dots---}_{\ell-1}\underbrace{\overbrace{+\dots+}^{a}-\dots-}_{\ell-1}--$ is completely analogous.

If $\alpha = \underbrace{\overbrace{+\dots+}^{a}-\dots-}_{\ell-1}\underbrace{-\dots---}_{\ell-1}+-$, the coordinate ring $\mathbb{C}[\mathrm{Supp}(S_\alpha)]$ is generated by polynomials in $u_{ij}, v_{kl} w_s = i_1 y_s j_2$ with i, j, s, u_{ij} and v_{js} as above. It is direct to check that dim $\mathrm{Supp}(S_\alpha) = 2\ell - 2$. The case $\alpha = \underbrace{-\dots---}_{\ell-1}\underbrace{\overbrace{+\dots+}^{a}-\dots-}_{\ell-1}-+$ is analogous.

Finally, if $\alpha = \underbrace{+\ldots+++}_{\ell-1}\underbrace{-\ldots---}_{\ell-1}--$ or $\underbrace{-\ldots---}_{\ell-1}\underbrace{+\ldots+++}_{\ell-1}--$, then the coordinate ring $\mathbb{C}[\mathrm{Supp}(S_\alpha)] = \mathbb{C}[x_1,\ldots,x_{\ell-1},y_\ell,\ldots,y_{2\ell-2},j_1,j_2]^{\mathbb{C}^*\times\mathbb{C}^*} = \mathbb{C}[y_1,\ldots,y_{\ell-1},x_\ell,\ldots,x_{2\ell-2},,j_1,j_2]^{\mathbb{C}^*\times\mathbb{C}^*} = \mathbb{C}$, so, $\mathrm{Supp}(S_\alpha)$ is a point. □

Example 6.2.8. If $\ell = 2$, then $\dim \mathrm{Supp}(S_1) = 4$, $\dim \mathrm{Supp}(S_2) = \dim \mathrm{Supp}(S_3) = 5$ and $\dim \mathrm{Supp}(S_4) = 6$.

6.3 Cross-walling functors and W-action

It was checked in Section 5 of [BLPW16] that the natural functor $\iota_\nu : D^b(\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))) \hookrightarrow D^b(\mathrm{Coh}(\overline{\mathcal{A}}^\theta_\lambda(2,\ell)))$ is a full embedding. As shown in Proposition 8.7 in [BLPW16], the functor ι_ν admits both left and right adjoints to be denoted by $\iota_\nu^!$ and ι_ν^* respectively.

Definition 6.3.1. Let ν, ν' be generic one-parameter subgroups. The *cross-walling functor* is given by

$$\mathfrak{CW}_{\nu\to\nu'} := \iota^!_{\nu'} \circ \iota_\nu.$$

The functor $\mathfrak{CW}_{\nu\to\nu'}$ has a right adjoint $\mathfrak{CW}^*_{\nu\to\nu'}$ given by $\iota_\nu \circ \iota^*_{\nu'}$.

We need to recall one more concept prior to formulating the property of cross-walling functors relevant for the purposes of the exposition. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two highest weight categories. Consider the full subcategories $\mathcal{C}_1^\triangle \subset \mathcal{C}_1$ and $\mathcal{C}_2^\nabla \subset \mathcal{C}_2$ of standardly and costandardly filtered objects. We say that $\mathcal{C}_2$ is *Ringel dual* to $\mathcal{C}_1$ if there exists an equivalence $\mathcal{C}_1^\triangle \xrightarrow{\sim} \mathcal{C}_2^\nabla$ of exact categories. This equivalence is known to extend to a derived equivalence $\mathcal{R} : D^b(\mathcal{C}_1) \xrightarrow{\sim} D^b(\mathcal{C}_2)$ to be called a *Ringel duality functor*.

The following result is obtained via a direct apllication of part 2 of Proposition 7.4 in [Los17].

Proposition 6.3.2. *The functor* $\mathfrak{CW}_{\nu\to-\nu}[2-3\ell]$ *is a Ringel duality functor that maps* $\triangle^\nu(p)$ *to* $\nabla^{-\nu}(p)$ *for all* $p \in \overline{\mathcal{M}}^\theta(2,\ell)^T$.

Let $W = N_G(T)/T \subset Sp_{2\ell}(\mathbb{C})$ be the Weyl group. The action of W on $\mathbb{C}[\overline{\mathcal{M}}^\theta(n,\ell)]$ lifts to an action on the quantization $\overline{\mathcal{A}}_\lambda(2,\ell)$. This gives rise to the functor $\Phi_w : \mathcal{O}_{\nu'}(\overline{\mathcal{A}}_\lambda(2,\ell)) \to \mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$, where $w\cdot\nu = \nu'$ (here we consider the action of W via conjugation, i.e. $w\cdot\nu = w\nu w^{-1}$). The functor Φ_w maps an object N to itself with the twisted action of $\overline{\mathcal{A}}_\lambda(2,\ell)$. More precisely,

$$a\cdot n := (wa)n,$$

with the ordinary action of $\overline{\mathcal{A}}_\lambda(2,\ell)$ on the r.h.s.

We conclude with an important result concerning the faithfulness of the functor Res.

Proposition 6.3.3. *The restriction of the functor* Res *to* $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))^\triangle$ *is faithful.*

Proof. The functor Res is exact (see Section 6.2 of [Los17]). Since it also preserves socles of the objects in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))^\triangle$ (see Corollary 6.2.4), it is sufficient that Res does not kill socles of standard objects to conclude that the functor is faithful (the socle of the image of a nontrivial homomorphism in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))^\triangle$ is nonzero). As established in Corollary 6.2.3 this is the case for $\lambda \notin \mathbb{Z}+\frac{1}{2}$, since $\mathrm{Res}(S_i) \neq 0$ for all i.

In case $\lambda \in \mathbb{Z}+\frac{1}{2}$, we have $\mathrm{Res}(S_\ell) = 0$ (here S_ℓ is the unique simple annihilated by Res as shown in Corollary 6.2.3), however, S_ℓ does not lie in the socle of any standard object in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$ (see Corollary 6.2.3). □

6.4 Main theorem

The results obtained above allow to describe the Hom spaces between standards in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))^\triangle$.

Theorem 6.4.1. *Let* $\lambda \in \mathbb{Z}_{<0}+1-\ell \cup \mathbb{Z}_{>0}+\ell-2$ *and* $\nu = (t^{d_1}, t^{d_2}, \ldots, t^{d_\ell})$ *with* $d_1 \gg d_2 \gg d_3 \gg \ldots \gg d_\ell > 0$. *The nontrivial Homs in* $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))^\triangle$ *are*

1. *Hom*($\triangle_i, \triangle_{i-1}$)*, where* $i \in \{2,\ldots,2\ell\}, i \neq \ell+1$*;*
2. *Hom*($\triangle_{\ell+2}, \triangle_\ell$)*, Hom*($\triangle_{\ell+1}, \triangle_{\ell-1}$)*;*
3. *Hom*($\triangle_{2\ell-i}, \triangle_{i+1}$) *with* $i \in \{0,\ldots,\ell-2\}$.

Let $\lambda \in \mathbb{Z}_{<0}+\frac{1}{2}-\ell \cup \mathbb{Z}_{>0}+\ell-\frac{3}{2}$. *The nontrivial Homs between standards in* $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))^\triangle$ *are Hom*($\triangle_{2\ell-i}, \triangle_{i+1}$) *with* $i \in \{0,\ldots,\ell-1\}$.

All the Hom spaces are one-dimensional.

Finally, if $\lambda \in (-\infty; 1-\ell) \cup (\ell-2; +\infty)$ *is none of the above, the category* $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$ *is semisimple.*

Proof. First consider $\lambda \in \mathbb{Z}_{<0}+1-\ell \cup \mathbb{Z}_{>0}+\ell-2$. For convenience of the exposition the proof will be broken down into several steps.

Step 1. Notice that $\mathrm{Soc}(\triangle_{2\ell-1}) = \mathrm{Soc}(\triangle_2) = \mathrm{Soc}(\triangle_1) = \triangle_{2\ell}$ (Corollary 6.2.5). Hence, $\mathrm{Hom}(\triangle_{2\ell}, \triangle_{2\ell-1})$, $\mathrm{Hom}(\triangle_{2\ell}, \triangle_2)$ and $\mathrm{Hom}(\triangle_{2\ell}, \triangle_1)$ do not vanish.

Step 2. Let $w_0 \in W$ be the longest element and consider the functor $\mathcal{F}_{w_0} := \Phi_{w_0} \circ \mathfrak{CW}_{\nu\to-\nu}$. Notice that $w_0 \cdot \nu = -\nu$ and the order on the T-fixed points corresponding to $-\nu$ is in reverse to the one associated with ν. Thus the functor $\mathcal{F}_{w_0}$ is an autoequivalence on $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))^\triangle$ with $\mathcal{F}_{w_0}(\triangle_i) = \triangle_{2\ell-i+1}$ (see Proposition 6.3.2). Hence, we see that $\mathrm{Hom}(\triangle_{2\ell-1},\triangle_1) = \mathrm{Hom}(\triangle_{2\ell},\triangle_2) = 0$ for $\ell \geq 3$ since $\mathrm{Soc}(\triangle_2) = S_{2\ell-1}$ does not contain $\triangle_{2\ell}$ (see Corollary 6.2.5). On the other hand if $\ell = 2$, then $\mathrm{Soc}(\triangle_2) = \triangle_4$, so $\mathrm{Hom}(\triangle_3,\triangle_1)$ does not vanish. Similarly, one shows that $\mathrm{Hom}(\triangle_2,\triangle_1) = \mathrm{Hom}(\triangle_{2\ell},\triangle_{2\ell-1})$ does not vanish either.

Step 3. We complete the proof for integral λ arguing by induction on the number of loops ℓ with $\ell = 2$ being the base. Assume the assertion holds for the variety $\overline{\mathcal{M}}^\theta(2,\ell)$ and take $\tilde{\nu} = (t^d, 1, \ldots, 1)$ with $d > 0$. Notice that $\overline{\mathcal{M}}^\theta(2,\ell) \subset \overline{\mathcal{M}}^\theta(2,\ell+1)^{\tilde{\nu}}$ as a component. Since $\tilde{\nu} \prec^\lambda \nu$ (see Proposition 6.1.2), Lemma 6.1.1 combined with the assumption that induction hypothesis holds in case of ℓ loops assure the existense of required homomorphisms between $\triangle_i$'s with indices $i \in \{2,\ldots,2\ell\}$ in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell+1))$. The remaining Homs between standard objects (not appearing in *Steps* 1, 2 above) vanish, since so do the Homs between their images in the category $\mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,\ell))$ and the functor Res is faithful (see Proposition 6.3.3).

Step 4. In case $\lambda \in \mathbb{Z}_{<0} + \frac{1}{2} - \ell \cup \mathbb{Z}_{>0} + \ell - \frac{3}{2}$ using that $\mathrm{Soc}(\triangle_{i+1})$ with $i \in \{0,\ldots,\ell-1\}$ is $S_{2\ell-i}$ (see Corollary 6.2.5), we establish the nonvanishing of Hom spaces in the statement of the theorem. Again the remaining Homs vanish since so do their images in the category $\mathcal{O}_\nu(\overline{\mathcal{S}}_\lambda(2,\ell))$ and the functor Res is faithful on standardly filtered objects (see Proposition 6.3.3).

Step 5. Finally, if $\lambda \in (-\infty; 1-\ell) \cup (\ell-2; +\infty)$ is neither an integer nor a half-integer, Corollary 6.2.5 asserts that all standards $\triangle_i$ in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$ are irreducible. Since for λ as above abelian localization holds (Theorem 5.0.3), the classes of standard and costandard objects in $K_0(\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell)))$ coincide (Corollary 6.4 in [BLPW16]), so we have that ∇_i's are simple as well. In particular, every simple lies in the head of a costandard object. The last condition is equivalent to $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$ being semisimple (see Lemma 4.2 in [Los18]). $\square$

Corollary 6.4.2. *Let* $\lambda \in \mathbb{Z}_{<0} + 1 - \ell \cup \mathbb{Z}_{>0} + \ell - 2$. *Then*

1. $\triangle_{2\ell} = S_{2\ell}$;
2. $\triangle_i$ *with* $\ell + 1 < i < 2\ell$ *has a socle filtration with subquotients* S_i *and* S_{i+1};
3. $\triangle_i$ *with* $i \in \{\ell, \ell+1\}$ *has a socle filtration with subquotients* S_i *and* $S_{\ell+2}$;
4. $\triangle_{\ell-1}$ *has a socle filtration with subquotients* $S_{\ell-1}, S_\ell, S_{\ell+1}$ *and* $S_{\ell+2}$;
5. *Finally,* $\triangle_i$ *with* $i < \ell - 1$ *has a socle filtration with subquotients* S_i, S_{i+1} *and* $S_{2\ell+1-i}$;

Let $\lambda \in \mathbb{Z}_{<0} + \frac{1}{2} - \ell \cup \mathbb{Z}_{>0} + \ell - \frac{3}{2}$. *Then*

1. $\triangle_i = S_i$ *for* $i > \ell$;
2. $\triangle_i$ *with* $i \leq \ell$ *has a socle filtration with subquotients* S_i *and* $S_{2\ell-i+1}$.

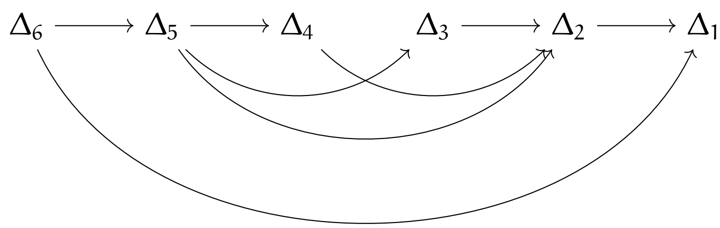


Figure 6.1: Homs between standard objects in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$ for $\ell = 3$ and $\lambda \in -2+\mathbb{Z}_{<0} \cup \mathbb{Z}_{>0}+1$

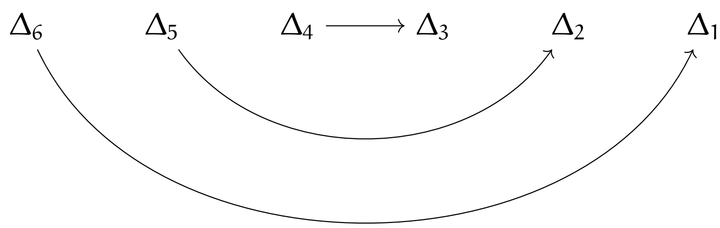


Figure 6.2: Homs between standard objects in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$ for $\ell = 3$ and $\lambda \in -\frac{5}{2}+\mathbb{Z}_{<0} \cup \mathbb{Z}_{>0}+\frac{3}{2}$

The multiplicity of each subquotient is equal to 1.

Proof. We check the assertion for $\ell+1 < i < 2\ell$, the remaining cases are established analogously. Let $0 = M_0 \subset M_1 \subset \ldots \subset M_j = \triangle_i$ be a socle filtration. Notice, that $M_1 = \mathrm{Soc}(\triangle_i)$, so $\mathrm{Res}(M_1) = \mathrm{Soc}(\triangle_{\alpha_i} \oplus \triangle_{\beta_i}) = S_{\alpha_{i+1}} \oplus S_{\beta_{i+1}}$. Next, $\mathrm{Res}(M_2/M_1) = \mathrm{Res}(\mathrm{Soc}(\triangle_i/M_1)) \subseteq \mathrm{Soc}((\triangle_{\alpha_i} \oplus \triangle_{\beta_i})/(S_{\alpha_{i+1}} \oplus S_{\beta_{i+1}})))$ (see Remark 6.2.6), but the latter is equal to $\mathrm{Res}(S_i)$ (see (c) of Proposition 4.3.6), hence, the nonstrict containment above must be an equality, so $j = 2$ and $M_2 = \triangle_i$, concluding verification of the claim. □

Δ_I	Δ_{II}	Δ_{III}	Δ_{IV}
S_I	S_{II}	S_{III}	S_{IV}
S_{II}	S_{IV}	S_{IV}	
S_{III}			
S_{IV}			

Table 6.1: Multiplicities of simples in standards in $\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell))$ for $\ell = 2$ and $\lambda \in \mathbb{Z}_{<0} - 1 \cup \mathbb{Z}_{>0}$

Chapter 7

Singular parameters

In this chapter we will combine the results of McGerty and Nevins from [MN16] and [MN14] to show that certain quantization parameters λ are *singular*, by which we understand that the derived localization does not hold. The following definitions are due.

Definition 7.0.1. Let M be a $D(\overline{R})$-module equipped with a rational action of G. This action gives rise to the map $\mathfrak{g} \to \mathrm{End}(M)$ with $x \mapsto x_M$. Recall that $x_{\overline{R}}$ stands for the image of x under the comoment map $\mathfrak{g} \to D(\overline{R})$. Then M is said to be a (G, λ)-*equivariant* $D(\overline{R})$-module provided $x_M m = x_{\overline{R}} m - \lambda(x)m$ for all $x \in \mathfrak{g}, m \in M$. The category of finitely generated (G, λ)-equivariant $D(\overline{R})$ modules will be denoted by $D(\overline{R}) - \mathrm{mod}^{G,\lambda}$.

7.1 Exactness of the functor of global sections

We have the functor $\pi_\lambda : D(\overline{R})\text{-mod}^{G,\lambda} \to \overline{\mathcal{A}}_\lambda(n, \ell)\text{-mod}$ of taking G - invariants and the functor $\pi^\theta_\lambda : D_{\overline{R}}\text{-mod}^{G,\lambda} \to \overline{\mathcal{A}}^\theta_\lambda(n, \ell)\text{-mod}$ (the latter category is the category of coherent $\overline{\mathcal{A}}^\theta_\lambda(n, \ell)$-modules) defined by first microlocalizing to the θ - semistable locus and then taking G - invariants.

Proposition 7.1.1. *The inclusion* $\ker \pi^{\det}_\lambda \subset \ker \pi_\lambda$, *where* $\pi_\lambda : D_{\overline{R}}\text{-mod}^{G,\lambda} \twoheadrightarrow \overline{\mathcal{A}}_\lambda(n, \ell)\text{-mod}$ *and* $\pi^{\det}_\lambda : D_{\overline{R}}\text{-mod}^{G,\lambda} \twoheadrightarrow \mathcal{A}^{\det}_\lambda(n, \ell)\text{-mod}$ *holds for* λ, *provided* $\lambda \notin \frac{\mathbb{Z}_{\leq 0}}{k} + (\ell-1)(n-k) - 1, k \in \{1, \ldots, n\}$. *We also have* $\ker \pi^{\det^{-1}}_\lambda \subset \ker \pi_\lambda$, *whenever* $\lambda \notin \frac{\mathbb{Z}_{\geq 0}}{k} + (\ell-1)(n-k), k \in \{1, \ldots, n\}$. *Moreover, for* λ *as above the functor of global sections* Γ_λ *is exact.*

Example 7.1.2. In case $n = 2$, we have $\ker \pi^{\det}_\lambda \subset \ker \pi_\lambda$ if $\lambda \notin \frac{\mathbb{Z}_{\leq 0}}{2} - 1 \cup \mathbb{Z}_{\leq 0} - \ell$ and $\ker \pi^{\det^{-1}}_\lambda \subset \ker \pi_\lambda$, if $\lambda \notin \frac{\mathbb{Z}_{\geq 0}}{2} \cup \mathbb{Z}_{\geq 0} + \ell - 1$.

Proof. First we recall the main results of [MN16]. Let X be a smooth, connected quasiprojective complex variety with an action of a connected reductive group G and $\lambda : G \to \mathbb{C}^*$

be a character. Assume, in addition, X is affine, the moment map $\mu : T^*X \to \mathfrak{g}^*$ is flat and the GIT quotient $\mu^{-1}(0)//_\chi G$ is smooth. The group G is equipped with a finite set of one-parameter subgroups of a fixed maximal torus $T \subset G$, depending on X and λ. These subgroups are known as the Kirwan-Ness one-parameter subgroups. Suppose that for each Kirwan-Ness subgroup β

$$\lambda(\beta) \in \mathtt{shift}(\beta) + I(\beta) \subseteq \mathtt{shift}(\beta) + \mathbb{Z}_{\geq 0},$$

where $\mathtt{shift}(\beta)$ is a numerical shift and $I(\beta) \subseteq \mathbb{Z}_{\geq 0}$. Then any λ-twisted, G-equivariant D-module with unstable singular support is in the kernel of quantum Hamiltonian reduction and the functor of global sections Γ_λ is exact.

Now we provide the proof of the second assertion (for $\theta = \det^{-1}$), the statement for $\theta = \det$ can be either shown analogously or derived from the isomorphism $\overline{\mathcal{A}}^\theta_\lambda(n, \ell) \cong \overline{\mathcal{A}}^{-\theta}_{-\lambda-1}(n, \ell)$ (see Lemma 2.0.1).

The computation is very similar to the one in Section 8 of [MN16], so we retain the notations. The multiplicity of each weight $e_i - e_j$ is ℓ and the weights e_i get substituted by $-e_i$ (alternatively, to avoid this substitution, one can use partial Fourier transform, 'swapping' V^* with V, see [MN16] for the details). The Kempf-Ness subgroups β_k correspond to the weights $-\sum\limits_{i=1}^{k} e_i, k \in \{1, \ldots, n\}$. The shift (in *loc. cit.*) becomes $(\ell-1)k(n-k) + \frac{k}{2}$ and $I(\beta) = \mathbb{Z}_{\geq 0}$. Therefore, we need

$$(-\lambda - \rho) \cdot \beta_k \notin \mathbb{Z}_{\geq 0} + (\ell - 1)k(n-k) + \frac{k}{2}$$

$$\frac{k}{2} + k\lambda \notin \mathbb{Z}_{\geq 0} + (\ell - 1)k(n-k) + \frac{k}{2}$$

$$\lambda \notin \frac{\mathbb{Z}_{\geq 0}}{k} + (\ell - 1)(n-k), k \in \{1, \ldots, n\},$$

where $\rho = \frac{1}{2}\sum\limits_{i=1}^{n} e_i$. For λ as above, the functor of global sections $\Gamma_\lambda : \mathcal{A}^\theta_\lambda(n, \ell) \to \mathcal{A}_\lambda(n, \ell)$ is exact (see [MN16]) and the inclusion $\ker \pi^\theta_\lambda \subset \ker \pi_\lambda$ holds. □

7.2 Complete form of the localization theorem

Theorem 7.2.1. *The algebra $\overline{\mathcal{A}}_\lambda(2, \ell)$ is not of finite homological dimension for $\lambda \in (-\ell; \ell - 1) \cap \mathbb{Z}$ or $\lambda = -\frac{1}{2}$, i.e. such λ are singular.*

Proof. The argument is completely anagolous to the one of a similar statement for Gieseker schemes in [Los18] (see Corollary 5.2). We give a brief outline. The statement is verified by contradiction. Assume $\overline{\mathcal{A}}_\lambda(2, \ell)$ is of finite homological dimension with λ as in the statement of the theorem. Then the main result (Theorem 1.1) of [MN14] implies that the derived localization functor $D^b(\overline{\mathcal{A}}_\lambda\text{-mod}) \to D^b(\overline{\mathcal{A}}^\theta_\lambda\text{-mod})$ is an equivalence, restricting to an equivalence $D^b(\mathcal{O}_\nu(\overline{\mathcal{A}}^\theta_\lambda(2,\ell))) \to D^b(\mathcal{O}_\nu(\overline{\mathcal{A}}_\lambda(2,\ell)))$. Since for our choice of λ the functor Γ_λ is exact (see Example 7.1.2), the abelian equivalence holds for λ. From this one can conclude that the long wall-crossing functor $\mathfrak{WC}_{-\theta\leftarrow\theta}$ induces an abelian equivalence $\mathcal{O}_\nu(\overline{\mathcal{A}}_{\lambda'}(2,\ell)) \to \mathcal{O}_\nu(\overline{\mathcal{A}}_{\lambda''}(2,\ell))$ (here $\lambda' = \lambda + s$ with $s \in \mathbb{Z}_{>0}$ a sufficiently large integer, so that the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_{\lambda'}(2,\ell))$ is a highest weight category and $\lambda'' - \lambda' \in \mathbb{Z}$). Since $\mathfrak{WC}_{-\theta\leftarrow\theta}$ is also a Ringel duality and for our choice of λ the category $\mathcal{O}_\nu(\overline{\mathcal{A}}_{\lambda'}(2,\ell))$ is not semisimple (see Theorem 6.4.1), we obtain a contradiction with Lemma 4.2 in [Los18], asserting that a highest weight category $\mathcal{C}$, where the classes of standard and costandard objects coincide is semisimple if and only if for any Ringel duality $\mathcal{R} : D^b(\mathcal{C}) \to D^b(\mathcal{C}^\vee)$, we have $H_0(\mathcal{R}(\mathcal{S})) \neq 0$ for any simple object $S \in \mathcal{C}$. □

Proposition 7.2.2. *Arguing completely analogously to the proof of Theorem 5.0.3, one shows that abelian localization holds for* $\theta < 0$ *and* $\lambda \in (-\ell; \ell-1), \lambda \notin \mathbb{Z}$ *and* $\lambda \neq -\frac{1}{2}$.

Proof. We notice that if $\lambda \notin \mathbb{Z}$ there are no finite dimensional irreducibles neither in the category $\mathcal{S}_\lambda$-mod nor the category of finitely generated modules over the corresponding quantization of the 2-dimensional slice, the type A_1 Kleinian singularity $\mathbb{C}^2/\mathbb{Z}/2\mathbb{Z}$ (see Remark 3.1.2). Since the aforementioned varieties expose the list of slices (Table 3.1) we conclude that there are no finite-dimensional irreducibles over the quantization $\mathcal{SL}_{x,\tilde{\lambda}}$ for any slice $\mathcal{SL}_x$.

On the other hand, if the equivalence does not hold, there exists a nontrivial bimodule M in the kernel or cokernel of one of the maps in (5.0.1) (see the proof of Theorem 5.0.3) and a point $x \in \overline{\mathcal{M}}(2,\ell)$, s.t. $\mathrm{Res}_{\dagger,x}(M) \neq 0$ is finite dimensional. Hence, we come up with a contradicion. □

Combining Theorems 5.0.3 and 7.2.1 with Proposition 7.2.2, we establish the abelian localization theorem.

Theorem 7.2.3. *The abelian localization holds for* $\lambda \notin (-\ell; \ell-1) \cap \mathbb{Z}$ *and* $\lambda \neq -\frac{1}{2}$.

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