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ANNUAL MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

PHILADELPHIA 1981

The 1981–1982 annual meeting of the Association for Symbolic Logic took place at the Franklin Plaza Hotel, Philadelphia, Pennsylvania, December 28–29, 1981, in conjunction with the annual meeting of the Eastern Division of the American Philosophical Association.

A symposium on Intuitionism was held, commemorating the centennial of L. E. J. Brouwer, jointly sponsored by the Eastern Division. The speakers were Scott Weinstein, W.W. Tait, and Dana Scott. Solomon Feferman served as Chairman. Kenneth McAloon gave a survey lecture, *Combinatorial versions of the incompleteness theorem: Survey and remarks*. J. Michael Dunn gave a survey lecture, *Topics in relevance logic*. Yuri Gurevich gave an invited address, *The monadic second-order logic in applications*.

Twenty contributed papers were presented in person and five by title. The paper of Lambek and Scott was presented by Scott; the paper of Friedman and Ščedrov was presented by Ščedrov.

The Council met the evening of December 28 and continued at noon December 29.

Abstracts of the contributed papers follow. The last five are of those presented by title.

SIMON KOCHEN
HUGUES LEBLANC
CHARLES D. PARSONS,
Chairman

ALASDAIR URQUHART, *Topological duality for projective algebras.*

A projective algebra is formulated by Everett and Ulam [4] as a Boolean algebra with three added operations. These algebras constitute an algebraic formulation of the operations of direct product and projections in a Boolean algebra of binary relations. They can be reformulated as an extension of cylindric algebras of dimension 2, as in Chin and Tarski [1]; the Chin-Tarski axioms require the addition of the postulates $(a.p^y)^x.p^y \leq a$, $(a.p^x)^y.p^x \leq a$. A topological duality is developed for these algebras, as an extension of Stone duality. The resulting duality allows easy solutions of the problems of Ulam [5, pp. 12–13]. Earlier solutions are in [2], [3].

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MICHAEL SHEARD, *Some independence results concerning ultrafilters.*

DEFINITION. An ultrafilter U over a cardinal κ is *indecomposable* if for every $f\colon \kappa \to \gamma$, $\gamma < \kappa$, there is a countable $C \subseteq \gamma$ such that $f^{-1}(C) \in U$.

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Indecomposability is a weakening of the concept of measurability; indeed, inaccessible cardinals carrying indecomposable ultrafilters exhibit some of the strong reflection properties of measurable cardinals. Silver asked whether the two notions were actually equivalent for inaccessible cardinals, i.e., whether an inaccessible cardinal carrying an indecomposable ultrafilter must be measurable. We prove that this is not the case:

THEOREM 1. Con(ZFC + $\exists$ *a measurable cardinal*) *implies* Con(ZFC + $\exists$ *an indecomposable ultrafilter over an inaccessible cardinal which is not measurable* (*in fact, not even weakly compact*)).

The basic forcing construction can be modified in a number of ways to produce various patterns in the Rudin-Keisler ordering on ultrafilters. Further forcing provides other corollaries; e.g.:

THEOREM 2. Con(ZFC + $\exists$ *a measurable cardinal*) *implies* Con(ZFC + *Martin's Axiom* + *there is an indecomposable ultrafilter over the continuum but no* $\aleph_1$*-saturated ideal*).

KENNETH L. MANDERS, *Transfer by definable relations between structures*.

An m-ary relation $R \subseteq \mathrm{Str}(\sigma_1) \times \cdots \times \mathrm{Str}(\sigma_m)$ between structures of similarity types $\sigma_1, \ldots, \sigma_m$ is (*finitely first-order*) *definable* if R, viewed as a class of m-sorted structures, is definable by a first-order sentence ψ_R with additional predicates (and possibly an extra sort). A *separation base for* R is a set Δ of m-tuples $(\phi_1, \ldots, \phi_m)$, ϕ_i a σ_i-sentence, such that:

(i) for any $(\phi_1, \ldots, \phi_m) \in \Delta$, $\psi_R \models \sim\bigwedge_{i=1}^{m} \phi_i$;

(ii) for any σ_i-sentences χ_i, $i = 1, \ldots, m$, such that $\psi_R \models \sim\bigwedge_i \chi_i$, there is a $(\phi_1, \ldots, \phi_m) \in \Delta$ such that $\chi_i \models \phi_i$.

By the completeness theorem and Craig's trick, any definable R has a decidable separation base, but not necessarily an informative one. In contrast, many classical preservation theorems specify syntactically informative separation bases.

THEOREM. *There is an effective procedure which generates, for any definable* R, *a decidable separation base for* R *by syntactic transformation of* ψ_R.

The transformation introduces no "extraneous syntactic material" except inequalities between variables. It first constructs a recursive closed game (Svenonius' Theorem); the separation base is constructed from the conjuncts in finite approximations to the game sentence by suitably coordinated positive Boolean combinations and distribution of prefix quantifiers over the ϕ_i of the correct sort. This latter part of the construction generalizes Lindström's theorem on regular relations.

JAMES W. GARSON, *The expressive power of modal logics*.

Let S be any propositional modal logic stronger than D ($= K$ plus $[\,]\,A \supset \Diamond A$), which is complete with respect to the set $R(S)$ of Kripke frames. A model $\langle D, R, \mathbf{a}\rangle$ *is standard for* S just in case its Kripke frame $\langle D, R\rangle$ is a member of $R(S)$. (Here $\mathbf{a}$ is the assignment function.) S *is canonical* iff every model for S is standard for S.

The first theorem shows that virtually all of the well-known modal logics have nonstandard models, and so are not canonical. S *is cancellable* iff removing all occurrences of $[\,]$ and $\Diamond$ in any theorem of S results in a tautology.

THEOREM 0. *If* S *is cancellable then* S *is not canonical.*

The next result shows a parallel between second order arithmetic and propositional modal logics with quantifiers over the propositional quantifiers. Let QS be the modal logic with propositional quantifiers that results from taking the propositional closure of the axioms of S. S is *in the Scott-Lemmon series* iff S can be axiomatized by adding axioms of the form $\langle k\rangle [l]\, A \supset [m]\langle n\rangle\, A$ to K. (Here $[i]$ and $\langle i\rangle$ stand for strings of i boxes and diamonds respectively.)

THEOREM 1. *If* S *is in the Scott-Lemmon series, then* QS *is canonical.*

The methods of Theorem 1 can be used to show canonicality of many modal logics (with propositional quantifiers) outside the Scott-Lemmon series.

Though modal logics without propositional quantifiers lack the expressive power to force their model's frames to take on the appropriate properties, they are expressively adequate in a weaker sense.

Frame $\langle D, R\rangle$ *satisfies* S iff all models $\langle D, R, \mathbf{a}\rangle$ satisfy S. S is *frame canonical* iff any frame which satisfies S is in $R(S)$.

THEOREM 2. *If* S *is in the Scott-Lemmon series, then* S *is frame canonical.*

DOLPH ULRICH, *The strict implicational fragments of some modal systems in the vicinity of* S4.9.

With wffs built in the usual way from letters p, q, r, s, t, ... and the binary connective C, *strict* wffs being those containing one or more C's, we consider various extensions of the calculus C4 whose axioms are *Cpp*, *CCpqCCpCqrCpr* and *CCpqCrCpq* and whose rules are substitution and detachment. $Z8_I$ is to be otained by adding *CCCCpqCrCCrrrCpqCpq* and *CCCCpqCCCCrstCrsCrsCpqCpq* to C4's axiom set, C4.4 by adding the latter and *CqCCCCCpqrCpqCCpqrCCpqr*, C4.9 by making both additions at once, and $K4_I$ by supplementing C4.9's axioms with *CpCCCpqpp*.

Adapting slightly a method the author has used elsewhere (*Strict implication in a sequence of extensions of* S4, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik***, vol. 27 (1981), pp. 201–212), semantic completeness results for these calculi relative to various Kripke-style, relational models may be obtained by exploiting a peculiar deduction theorem readily established for each extension L of C4: where M is a set of strict wffs, if $M, \alpha_1, \ldots, \alpha_j \vdash_L \beta$, then for some sequence $\gamma_1, \ldots, \gamma_k$ (typically, with many repetitions) of the α_i's, $M \vdash_L C\gamma_1 \cdots C\gamma_k\beta$. Comparing the completeness results thus obtained with standard, analogous results for the appropriate full modal calculi, $Z8_I$ emerges as an axiomatization of the strict implicational fragment of the modal system K3.2 (as well as of Z8), C4.4 is the strict fragment of S4.4, C4.9 is the strict fragment of S4.9, and $K4_I$ is the strict fragment of K4.

Finally, writing 'P' for '*Cpt*', let C4.9$^+$ result from C4.9 by addition of *CCCCCPqPPCrCsrCrCsr* and *CCCrCsrCCCPqPPCCCPqPP* while C5 comes from C4 by addition of *CCCPqPP*. Then C5 is well known (cf. the paper cited above) to be the strict fragment of S5. C4.9$^+$, on the other hand, lies strictly between C4.9 and C5; and by a Halldén-style argument, no proper extension of C4.9$^+$ is a proper subsystem of C5.

GARREL POTTINGER, *Uniform, cut-free formulations of T, S_4, and S_5.*

It is shown that the following rule schemata can be specialized so as to provide cut-free formulations of T, S_4, and S_5. *Consecutions* have the form $\Gamma \vdash \Theta$, where Γ and Θ are finite sequences of formulas. $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}_1$, ... are finite sequences of consecutions. Derivations are built up from such sequences. Commas punctuate sequences of formulas, and semicolons punctuate sequences of consecutions. *Axioms* have the form $\boldsymbol{\Sigma}_1; \Gamma_1, A, \Gamma_2 \vdash \Theta_1, A, \Theta_2; \boldsymbol{\Sigma}_2$. In S_5 and S_4, Γ^- is the result of deleting all formulas not of the form $\Box A$ from Γ, and in T, Γ^- is the result of deleting all formulas not of the form $\Box A$ from Γ and then deleting the initial $\Box$ from each formula occurrence in the residue. In S_5, $\Box A^\frown \Gamma_1 \vdash \Theta_1; \ldots; \Gamma_n \vdash \Theta_n = \Box A, \Gamma_1 \vdash \Theta_1; \ldots; \Box A, \Gamma_n \vdash \Theta_n$, and in T and S_4, $\Box A^\frown \boldsymbol{\Sigma} = \boldsymbol{\Sigma}$.

$$\& \vdash \quad \frac{\boldsymbol{\Sigma}_1; A, B, \Gamma_1, \Gamma_2 \vdash \Theta; \boldsymbol{\Sigma}_2}{\boldsymbol{\Sigma}_1; \Gamma_1, A \& B, \Gamma_2 \vdash \Theta; \boldsymbol{\Sigma}_2} \qquad \vdash \& \quad \frac{\boldsymbol{\Sigma}_1; \Gamma \vdash \Theta_1, \Theta_2, A; \boldsymbol{\Sigma}_2 \quad \boldsymbol{\Sigma}_1; \Gamma \vdash \Theta_1, \Theta_2, B; \boldsymbol{\Sigma}_2}{\boldsymbol{\Sigma}_1; \Gamma \vdash \Theta_1, A \& B, \Theta_2; \boldsymbol{\Sigma}_2}$$

$$\sim \vdash \quad \frac{\boldsymbol{\Sigma}_1; \Gamma_1, \Gamma_2 \vdash \Theta, A; \boldsymbol{\Sigma}_2}{\boldsymbol{\Sigma}_1; \Gamma_1, \sim A, \Gamma_2 \vdash \Theta, \boldsymbol{\Sigma}_2} \qquad \vdash \sim \quad \frac{\boldsymbol{\Sigma}_1; A, \Gamma \vdash \Theta_1, \Theta_2; \boldsymbol{\Sigma}_2}{\boldsymbol{\Sigma}_1; \Gamma \vdash \Theta_1, \sim A, \Theta_2; \boldsymbol{\Sigma}_2}$$

$$\Box \quad \frac{\Box A^\frown \boldsymbol{\Sigma}_1; \Box A, A, \Gamma_1, \Gamma_2 \vdash \Theta; \Box A^\frown \boldsymbol{\Sigma}_2}{\boldsymbol{\Sigma}_1; \Gamma_1, \Box A, \Gamma_2 \vdash \Theta; \boldsymbol{\Sigma}_2} \qquad \vdash \Box \quad \frac{\boldsymbol{\Sigma}_1; \Gamma \vdash \Theta_1, \Theta_2; \boldsymbol{\Sigma}_2; \Gamma^- \vdash A}{\boldsymbol{\Sigma}_1; \Gamma \vdash \Theta_1, \Box A, \Theta_2; \boldsymbol{\Sigma}_2}$$

A consecution is *true at* a world in a Kripke interpretation for the system S ($= T$, S_4, or S_5) if, and only if, the truth-functional conditional having the conjunction of the left formulas of the consecution as its antecedent and having the disjunction of the right formulas of the consecution as its consequent is true at every world in the interpretation. (Empty conjunctions are regarded as true, and empty disjunctions are regarded as false.) A sequence of consecutions is *true at* a world in a Kripke interpretation for S if, and only if, at least one of the consecutions in it is true at that world in that interpretation. A sequence of consecutions is S *valid* if, and only if, it is true at every world in every Kripke interpretation for S. The theorem is proved by showing that a sequence of consecutions is derivable in S by means of the rules given above if, and only if, it is S valid.

DAN DOUGHERTY, *The complexity of resolution and Gentzen systems.*

We consider the complexity of certain systems for propositional calculus, specifically, several formulations of Gentzen-style systems and of resolution.

DEFINITION (TSEITIN). A resolution derivation is *regular* if it contains no pair of clauses C_1 and C_2, with C_1 occurring in a subderivation of C_2, such that the same literal is resolved upon in both C_1 and C_2.

In [1], Tseitin proved that regular resolution is exponentially complex:

THEOREM (TSEITIN). *For all $c > 0$, and for all n, there exist unsatisfiable sets of clauses S_n, of length n, such that the shortest regular resolution refutation of S_n involves at least $2^{c\sqrt{n}}$ distinct clauses.*

Galil later improved this lower bound to 2^{cn}.

THEOREM. *Given any resolution refutation, of length k, there exists a regular refutation (of the same set of clauses), with at most k^3 clauses appearing.*

COROLLARY. *Unrestricted resolution has a 2^{cn} lower bound on proof length.*

Tseitin defined *resolution with extension*—the extension rule allows the introduction of certain auxiliary clauses to the original set. Tseitin showed that the sets of clauses which reguired exponentially long regular refutations had polynomial-length refutations in this system.

COROLLARY. *Resolution is not polynomially related to resolution with extension.*

In [2], Cook and Reckhow consider the relative complexity of various proof systems, among them Gentzen systems and resolution. If $\mathscr{A}$ and $\mathscr{B}$ are two proof systems, write $\mathscr{A} \leq_p \mathscr{B}$ if there exists a polynomial $p(x)$ such that for every proof in system $\mathscr{B}$, of length n, there is a corresponding proof in system $\mathscr{A}$ with length at most $p(n)$. We write $\mathscr{A} \leq \mathscr{B}$ if there is *no* increase in length in passing from $\mathscr{B}$ to $\mathscr{A}$.

A typical rule in a Gentzen system is as follows:

$$\frac{\Gamma_1, A, \vdash \Delta_1 \cdots \Gamma_k, A_k, \vdash \Delta_k}{\Gamma_1 \cdots \Gamma_k, A_1 \vee \cdots \vee A_k \vdash}.$$

DEFINITION. The system in which each inference must obey $\Gamma_i = \Gamma_j$ and $\Delta_i = \Delta_j$ is G_w. The system with no restrictions is G_s. The subsystem of G_s in which the A_i are atomic is G_l.

Cook and Reckhow report: Resolution $\leq_p G_w$ without cut, Resolution + extension $\equiv_p G_s$ with cut.

(Here, the *length* of a Gentzen proof is the number of distinct sequents appearing.)

THEOREM. *Resolution $\equiv G_l$ with cut, Resolution $\leq G_w$ with cut, Resolution + extension $\equiv G_s$ with cut.*

Thus the extension and cut rules do not correspond, as might be expected from Cook's and Reckhow's results above.

As corollaries, we obtain an exponential lower bound for G_w with cut, and the result that G_w with cut is not polynomially related to G_s with cut.

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J. LAMBEK and P. J. SCOTT, *New proofs of some intuitionistic principles.*

We present short proofs of various metarules of intuitionistic higher-order logic using category-theoretic techniques. In particular we show closure under the following rules: Existence, Disjunction, Uniformity, Independence of Premises, Markov's, and Indecomposability of PA: if $\vdash \forall x \in \mathrm{PA}(\varphi(x) \vee \psi(x))$ then $\vdash \forall x \in \mathrm{PA}\varphi(x)$ *or* $\vdash \forall x \in \mathrm{PA}\psi(x)$. Our method is a modification of Freyd's proof of the Existence and Disjunction Rules, which was further developed by us in [LS1], [LS2]. Rather than translating each rule into its algebraic equivalent, e.g. some statement about projectives, we use the internal logic of the Freyd Cover of the free topos [LS1]. Whereas Freyd's original proof is essentially equivalent to the Kleene-Friedman method [SS] as originally discussed by us [LS1], the proof here involves a higher-order version of the "Aczel Slash" and is more perspicuous.

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HUGUES LEBLANC, *Of truth-value functions and probability ones.*

Let L be a first-order language with a denumerable infinity of individual terms and '$\sim$', '&', and '$\forall$' as logical operators. Understand by a *truth-value function* for L any function P from the statements of L to $\{0, 1\}$ that meets the following three constraints:

A1. $P(\sim A) = 1 - P(A)$;

A2. $P(A \,\&\, B) = P(A) \times P(B)$;

A3. $P((\forall X)A) = 1$ iff $P(A(T/X)) = 1$ for each term T of L.

($A(T/X)$ in A3 is, of course, the result of putting T everywhere in A for X.) And understand by a *probability function* for L any function P from the statements of L to the reals that meets the following seven constraints:

B1. $0 \leq P(A)$;

B2. $P(\sim(A \,\&\, \sim A)) = 1$;

B3. $P(A) = P(A \,\&\, B) + P(A \,\&\, \sim B)$;

B4. $P(A) \leq P(A \,\&\, A)$;

B5. $P(A \,\&\, B) \leq P(B \,\&\, A)$;

B6. $P(A \,\&\, (B \,\&\, C)) \leq P((A \,\&\, B) \,\&\, C)$;

B7. $P(A \,\&\, (\forall X)B) = \mathrm{Limit}_{j\to\infty} P(A \,\&\, \prod_{i=1}^{j} B(t_i/X))$.

(t_i in B7 is the alphabetically ith term of L, and '$\prod_{i=1}^{j} B(t_i/X)$' is short for '$((\ldots(B(t_1/X) \,\&\, B(t_2/X)) \,\&\, \ldots) \,\&\, B(t_j/X))$'.)

I show that

(1) *A two-valued function P from the statements of L to the reals meets constraints* A1–A3 *iff it meets constraints* B1–B7.

Hence

(2) *Truth-value functions for L are two-valued probability functions for L, and vice-versa.*

Since a statement of L is provable in L iff it evaluates to 1 on every truth-value function for L, (2) substantiates Popper's claim on p. 356 of ***The logic of scientific discovery*** that "In its logical interpretation, the probability calculus is a genuine generalization of the logic of derivation."

Constraints B1–B7, incidentally, are counterparts for L of Kolmogorov's constraints on P in ***Foundations of probability,*** Chelsea, 1950, and are equivalent to constraints of Popper's in ***The logic of scientific discovery,*** New York, 1959.

JONATHAN P. SELDIN, *Remarks on β-strong equality.*

The most important property which distinguishes $\lambda\beta$-conversion from combinatory weak equality is

$$(\xi) \qquad X = Y \to \lambda x.X = \lambda x.Y,$$

which is satisfied by the former but not the latter. It might seem to follow from this that the relation $=_\beta$ between combinatory terms, which is defined by

$$(1) \qquad X =_\beta Y \rightleftarrows X_\lambda \operatorname{cnv}_\beta Y_\lambda,$$

where X_λ is the λ-term which corresponds to the combinatory term X under the normal mapping, is the same relation as $=_\xi$, which is defined by adding (ξ) to weak equality. But this is too simple, since the mapping from X to X_λ, and hence $=_\beta$, is independent of the definition of abstraction for combinatory terms but (ξ), and hence $=_\xi$, depends very much on that definition. It happens that the only definition of abstraction for which $=_\beta$ is the same as $=_\xi$ is, in some ways, unsatisfactory; for example, it is not always true that if x does not occur in Uv then $[U/v](\lambda x.X)$ is the same term as $\lambda x.[U/v]X$.

In this paper, $=_\xi$ is studied for a definition of abstraction which Lambek uses in [1] and which does not suffer from the above defect. It is shown that in this case $=_\xi$ can be characterized by adding to the postulates for weak equality a finite number of equations between constants, but that it does not correspond to cnv_β as well as does $=_\beta$. For this reason it appears that the main usefulness of $=_\xi$ is to illustrate how sensitive this theory is to the definition of abstraction.

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RAYMOND D. GUMB, *Programs for free logic*.

Free logics provide a natural framework for describing certain aspects of partial functions. As many results for standard first-order logic carry over into free logic, reasonable proof and model theories can be developed. Further, the interpretation of the quantifiers in free logic can be given a constructive flavour, and Scott and others have developed a variety of free intuitionistic theories.

In this talk, we describe another kind of constructive application of free logic. We develop a Hoare "logic" of total correctness for the class of while-programs using a free version of Peano arithmetic. In the intended interpretation of the latter, the natural numbers belong to the inner domain, and a unique error element belongs only to the outer domain. Execution of a program can fail to terminate because the error element is referenced (through an uninitialized integer variable, an array bounds violation, or a function applied to an argument outside its domain) as well as because of a diverging while-loop.

In the Hoare part of our axiomatization, the dummy axiom and the composition and consequence rules are as in the literature. The assignment axiom and the if-then-else and while-rules require modification. For example, the assignment axiom (schema) for simple integer variables becomes:

$$\{P(e/i) \ \& \ \exists x(x = e)\} \ i := e \, \{P\}.$$

In the case of the if-then-else and while-rules, we require that the outermost integer expressions occurring within the boolean condition of the program statement exist. In the case of the while-rule, we impose additional existence conditions on what is roughly a "good function" (in the sense of Manna) applied to the integer variables occurring within the body of the while-loop. The Hoare system is sound with respect to an operational semantics.

Further work remains to be done on our free Peano arithmetic and the underlying free second-order logic.

IRVING H. ANELLIS, *Constructive analysis is non-Hausdorff*.

Brouwer [1] argued that several important theorems of analysis are false. In particular he focused on the Heine-Borel and Extreme Value Theorems.

(Heine-Borel Theorem). *Any cover of a closed interval* $[a, b]$ *by a system of open intervals* (*or open sets*) *has a finite subcover*.

We are reminded by Kolmogorov and Fomin [3], however, that the generalization of this covering property of closed intervals leads to "a key concept of real analysis," the definition of compact space and the compactum of Hausdorff space.

Brouwer also rejects the Extreme Value Theorem, or at least that part of it which holds that a continuous function f defined for all points on a closed interval i has a maximum.

Ceĭtin [2] shows that Rolle's Theorem, that a continuous function differentiable on all points of an interval i and vanishing at the endpoints will also vanish for some value on i between the endpoints, fails to hold for constructive analysis. Ceĭtin also proves that many of the Mean-Value Theorems of calculus, including Rolle's Theorem, Cauchy's Theorems, the Lagrange Theorem, and others, fail for constructive analysis. These proofs are far stronger than the proofs by counterexample presented originally by Brouwer.

The central theorem for constructive analysis may be the Heine-Borel Theorem, since, as shown by Kolmogorov and Fomin, it permits development of topological models, especially Hausdorff models, for the continuum. K.L. Singh suggested (oral communication) that a differentiable function continuous on an interval of the Reals has a unique limit in Hausdorff space, but that its limit may not be unique in nonmetric or non-Hausdorff space. I suggest that constructive mathematics is too weak to permit the reconstruction of real analysis, and conjecture that (a) the theorems of classical analysis hold only in Hausdorff space, and (b) that constructive analysis requires non-Hausdorff space for its Mean-Value Theorems to hold, and that it is in fact non-Hausdorff.

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KENDALL D'ANDRADE, *Which logic should you use?*

Since there are many different logics, each of which is complete, but with many pairs incompatible, the problem seems to reduce to the question "Which logic most adequately mirrors reality?" Then there would be at most one logic dealing with real objects; all other logics would mirror only mathematical constructs, shadowy objects best avoided in respectable metaphysics.

Tragesser shows the futility of this hankering after a single logic. "The right logic" is an incomplete phrase which should be interpreted as "The appropriate logic for ___" where the dash is filled in with the task at hand. Along the way he vindicates the reality of any mathematical object embedded in significant mathematics by offering a new, phenomenological, explanation of mathematical existence. We thus return to psychological criteria for choosing an appropriate logic, but with a psychology as respectable as the mathematics we are doing.

ZORAN MARKOVIĆ, *Forcing in Kripke models versus the classical satisfiability relation.*

In Gödel's and other translations we have a method for determining which classically valid formulas are intuitionistically valid. Here we investigate formulas which whenever satisfied in a classical structure at a node of some Kripke model are forced by that node.

Let P be the set of all formulas of Heyting predicate calculus (HPC) which are formed from atomic formulas with the connectives $\wedge$, $\vee$, $\exists$. It is easy to see that for any $\varphi \in P$, any Kripke model $M = \langle (T, 0, \leq); \mathfrak{A}_t : t \in T \rangle$, any $t \in T$ and any $a_1, \ldots, a_n \in A_t$ we have

$$(*) \qquad \mathfrak{A}_t \models \varphi[a_1, \ldots, a_n] \text{ iff } t \Vdash \varphi_1[a_1, \ldots, a_n].$$

We describe now the set of all formulas for which one half of this equivalence holds.

Theorem 1. *Let $\varphi(x_1, \ldots, x_n)$ be a formula of HPC. The following are equivalent:*

(a) *For any Kripke model $M = \langle (T, 0, \leq); \mathfrak{A}_t : t \in T \rangle$, any $t \in T$, and any $a_1, \ldots, a_n \in A_t$ we have:*

$$\text{if } \mathfrak{A}_t \models \varphi[a_1, \ldots, a_n] \text{ then } t \Vdash \varphi[a_1, \ldots, a_n].$$

(b) *There is a formula $\psi(x_1, \ldots, x_n) \in P$ such that $\vdash_C \varphi \leftrightarrow \psi$ and $\vdash_{\mathrm{HPC}} \psi \rightarrow \varphi$.*

We can prove now, as a corollary, the converse to our first statement:

Theorem 2. *If a formula φ satisfies $(*)$, then it is intuitionistically equivalent to a formula from P.*

HARVEY FRIEDMAN and ANDREJ ŠČEDROV, *Large sets in intuitionistic set theory.*

We consider properties of sets in intuitionistic set theory corresponding to large cardinal properties in (classical) ZF. Adding such "large set axioms" to ZF in intuitionistic logic does not violate well-known metamathematical properties of intuitionistic systems. Moreover, we consider statements in constructive analysis equivalent to the consistency of such "large set axioms".

DEFINITION. (a) A set z is said to be *inaccessible* if it is transitive, $\omega \in z$, z is closed w.r.t. pairs, unions, power-sets, and collection (w.r.t. $\langle x, y\rangle \in t$, for t not necessarily in z).

(b) *A* set z is said to be *Mahlo* if it is inaccessible and for each $u \in z$ and each set t, there exists inaccessible $v \in z$ with $u \in v$ and

$$\forall x \in v[\exists y \in z(\langle x, y\rangle \in t) \to \exists y \in v(\langle x, y\rangle \in t)].$$

One defines an n-Mahlo set accordingly.

Let $T_i\,(i = 1, 2)$ be theories obtained from intuitionistic ZF by adding:

(1) $\forall x \exists z[x \in z \wedge z$ inaccessible].

(2) $\exists z[z$ Mahlo].

The theory T_3 is obtained from intuitionistic ZF by expanding the language with a unary predicate symbol M, and a unary function symbol j, and requiring that M is an inner model containing ω, and that $j\colon V \to M$ is an elementary embedding such that $\exists x\,[x \in j(x)]$.

THEOREM 1. *For every sentence A provable in $T_i + CT$ ($+MP$, $+MP + RDC$, resp.) there is a* 1945-*realizing numeral n so that n **r** A is provable in T_i ($+MP$, $+MP + RDC$, resp.)*

THEOREM 2. *T_i has disjunction and number existence properties (by q-realizability).*

THEOREM 3. *T_i is equiconsistent with its classical counterpart (by negative interpretation).*

THEOREM 4. *Classical counterpart of T_i is conservative over T_i w.r.t. Π^0_2-sentences.*

THEOREM 5. *T_i + RDC in finite types is conservative over T_i w.r.t. arithmetic sentences.*

THEOREM 6. *"For every n, every Σ^0_1-sentence provable in $T_2(n)$ is true" is equivalent to:*

"For all integers $m, r > 0$ there is an integer q so *large that the following holds. Let $F_1, \dots F_q\colon (R^q)^{n+1} \to [-1, 1]$ be continuous functions satisfying $|F_k(x) - F_k(y)| \le k \cdot |x - y|$. Then there are $\{x_k\}$ from R^q, $k \le m$, and integers $0 < a_1 < a_2 < \cdots < a_r$ such that for all integers $\alpha < \beta < \gamma \le r$, $s < t \le m - n + 1$, we have*

$$\left| x_{s+1}(1) - \max_{\substack{g \in \mathrm{SL}(a_\alpha,\, [-a_\alpha,\, a_\alpha]) \\ k \in [a_\beta,\, a_\gamma]}} F(x_s, gx_t, \dots, gx_{t+n-1}) \right| < \frac{1}{a_\alpha},$$

where SL(p, Z) *acts on first p coordinates in R^q, $q \ge p$.*

REMARK. Intuitionistic ZF under consideration here has Collection.

WILLIAM C. POWELL, *Calculus with standard infinitesimals.*

We assume every real number x is standard in the sense that

$$\forall y(\forall n \in \omega\,|x - y| \le 1/(n + 1) \to x = y).$$

Nonetheless, in the absence of the law of bivalence ($p \vee \neg p$), it is consistent that there are invertible h that are infinitesimal in the sense that

$$\forall n \in \omega\ \neg\neg|h| \le 1/(n + 1).$$

We develop this possibility to the point of giving a logically correct proof of the Fundamental Theorem of the Calculus. Derivatives and integrals are defined so that they are approximated within an infinitesimal distance. Since ideally lengths at least as large as positive rationals can be experienced, we call approximations to within $1/(n + 1)$ empirical approximations in contrast to infinitesimal approximations. Thus, our approximations are more precise in that they are infinitesimal rather than empirical. Furthermore, the limits we consider are not approximation processes, but rather precise algebraic operations.

Our theory of the calculus is interpreted in set theory by constructing the reals from the rationals in essentially the same way the integers are constructed from the finite ordinals. Rather than defining the integral closure $I(\omega)$ of the finite ordinals to be certain equivalence classes of pairs of finite ordinals, a finite ordinal is replaced by the set of ordinals less than *or equal to* it. Thus, for our theory of *integration*, we define the *integral* closure $I(Q)$ of the rationals to be certain equivalence classes of pairs of initial segments of the nonnegative rationals.

Since our infinitesimals h fail to be negatively standard in the sense that

$$\forall y(\forall n \in \omega\ \neg\neg|h - y| \le 1/(n + 1) \to h = y),$$

the infinitesimals become nonstandard when our real variable theory is interpreted negatively. Nonetheless, we maintain that the reals of our theory are classically standard. For when our real variable theory is first interpreted into set theory by $I(Q)$, and then in set theory we pass to the negative interpretation, $I(Q)$ becomes isomorphic to the order completion of the rationals, which is (negatively) standard. Thus, when classical *set* theory is considered, infinitely close reals do get identified.

CARL J. POSY, *Kant's debt to Brouwer and Brouwer's debt to Kant.*

Because of his views about such issues as the incompletability of infinite totalities and because of his frequent constructivist leaning remarks, Kant is acknowledged by many (including Brouwer himself) as a forerunner of twentieth century intuitionism. That is Brouwer's debt to Kant. This paper investigates the relation between Kant and the modern intuitionists in a series of three steps:

(i) It is shown that techniques developed for the study of intuitionism can be used to interpret several Kantian arguments concerning empirical science, and to vindicate those arguments in the face of some traditional criticisms. That is Kant's debt to Brouwer.

(ii) Some recent work on the philosophical foundations of intuitionism is used to interpret the general themes of Kant's "Copernican Revolution." It is argued that this reading fits Kant's texts in many respects, and indeed is superior on several counts to other interpretations of Kant's main philosophical themes. Thus the seemingly anachronistic moves of (i) are not totallv artificial.

(iii) It is argued, however, that the interpretation suggested in (ii), when applied to Kant's remarks about pure mathematics and when combined with some Kantian doctrines about "faculties" and about mathematical constructions, supports the surprising claim that Kant's approach to pure *mathematics* (as opposed to empirical science) is realistic (or at least classical) rather than intuitionistic.

JOHN CORCORAN, *Deduction and reduction: Two proof-theoretic processes in Prior Analytics.* I.

Aristotle's system of direct and indirect *deductions* is discussed by Smiley and other recent authors. Aristotle's system of direct and indirect *reductions* is discussed in older works by Mill, DeMorgan, Łukasiewicz, and many others. Although the distinction between them is clear in Aristotle's work, the two seem not to have been contrasted by commentators. *A deduction* of a conclusion from a premise set is a *sentence*-sequence constructed by chaining simple inferences to show that the conclusion is implied by the premise set. A *reduction* is an *argument*-sequence wherein each argument after the first is constructed from *the* previous one by "weakening" the premise set, by "strengthening" the conclusion, or by "contraposition," i.e., replacing one premise with the contradictory of the conclusion and taking as the new conclusion the contradictory of the replaced premise. "Weakening" here means replacing a sentence by its Aristotelian converse and "strengthening" is the reverse. The initial argument is thereby "reduced to" the final argument. Some valid arguments reduce to invalid: *Aab Abc*/*Iac* reduces to *Aab Icb*/*Iac* and to *Aab Abc*/*Aca*. However, as Aristotle noticed, invalid arguments reduce only to invalid and, therefore, any reduction ending with an obviously valid argument must have started with a valid argument, thus providing a mark of validity. Exaggeration, or misapplication, of this point seems to have led to confusion of reduction with deduction and to mistaking the role of reduction to be that of deduction. This paper shows, to the contrary, (1) that Aristotle held to a sharp separation between the two processes (discussing now one, now the other, and also implicitly contrasting them by mentioning both together) and (2) that reduction is not a method of inference but rather a process for studying relationships among syllogisms (cf. 29b26), an *almost* totally separate enterprise. Although reading *certain* reductions *backward* suggests a deduction system, the system thus suggested is a Gentzen-sequent natural deduction system (with arguments as lines) rather than a Jaskowski-supposition natural deduction system (with sentences as lines) as attributed to Aristotle by recent authors. Taking arguments to be conditional sentences leads to transformation of a Gentzen-sequent system into an axiomatic system of the sort proposed by Łukasiewicz.

MICHAEL SCANLAN, *Veblen and the definability of congruence.*

Veblen [3] states that although it was known that definitions can be added to the axioms of Hilbert [1] so that only six undefined nonlogical constants remain, the number of undefined constants can be reduced further by defining the two congruence relations on the basis of the between relation of points. Veblen [4] presents an axiom set for Euclidean geometry in which only "point" and "between" appear as undefined nonlogical constants. But Lindenbaum showed using Padoa's Method that congruence of line segments is not definable on this basis (Tarski and Lindenbaum [2]).

Veblen held two beliefs about definability which led to his mistake. Veblen [4] used a general procedure which would have been appropriate to prove that Hilbert's congruence axioms are interpretable in the Veblen axiom set. That is, he tried to show that Hilbert's congruence axioms are consequences of his axioms and added definitions. He failed to realize that the interpretability of a set of sentences in an axiom set using only a proper subset of the original nonlogical constants does not mean that the omitted constants are definable in the larger language.

Veblen does not prove that the Hilbert congruence axioms are interpretable in his own. His actual procedure is only sufficient to show that the Hilbert axioms are quasi-interpretable in his own. If S is a finite set of axioms, an existentialization of S is a conjunction of the sentences of S in which one or more constants are uniformly replaced by variables and the resulting open formula is closed with existential quantifiers. S is *quasi-interpretable* in an axiom set T if an existentialization of S is interpretable in T. Veblen appears to have believed that quasi-interpretability is sufficient for the definability of the constants on which the existentialization is based.

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NUEL D. BELNAP, JR., *Display logic.*

"Display logic" is a refinement of Gentzen's sequenzen-kalküle which permits the simultaneous study of a variety of classical and nonclassical logics.

$(I, *, .)$ is a family of "structure connectives," respectively, 0, 1, and 2-place. I is like Gentzen's emptiness, $*$ is negative, and . is like Gentzen's comma (but two-place). Structures are built from formulas by structure connectives. There can be many families of structure connectives. With X, Y structures, statements have the form $X \vdash Y$.

The basic postulates permit the "display" of any positive part of the antecedent or negative part of the consequent as the antecedent itself (and similarly for the consequent); for example, $(X. Y) \vdash Z$ is equivalent to $X \vdash (Y^*.Z)$, permitting the display of X.

Each family has roughly the *same* rules governing the usual connectives ($t, f, \sim, \Box, \Diamond, \wedge, \vee, \rightarrow$); e.g., from $X \vdash A$ and $B \vdash Y$ to infer $A \rightarrow B \vdash (X^*.Y)$—with the principal constituent always displayed (the $\rightarrow$ and the structure connectives must all be from the same family). The families *differ* as to structural rules, e.g., from $(W.(X.Y)) \vdash Z$ to infer $((X.W).Y) \vdash Z$, can be postulated or not. Under easily verifiable conditions on these rules, a Subformula theorem and a generalized Elimination theorem (from the provability of $X \vdash M$ and $M \vdash Y$ to infer the provability of $X \vdash Y$) follow without relying on structural rules and without tricks.

The possibility of multiple families permits a sequenzen formulation of the relevance logic ***R***, including negation, with its intensional connectives in one family and its extensional in another. There are also families for the intuitionist connectives, those of the various modal logics such as ***S2***, ***S3***, ***S4***, and ***S5***, the Brouwersiche system ***B***, and other relevance logics such as ***E*** and ***T***. The formulations of ***S2*** and ***S3*** are simpler than others; that of ***S4*** has no restricted rules; that of ***S5*** introduces no ad hoc features; that of ***B*** is perhaps the first.

THOMAS JOHN, *On countable stable ordinals.*

For any $X \subseteq \omega$, let $\sigma_0(X)$ be the least ordinal such that $L_{\sigma_0(X)}[X]$ is a Σ_1 elementary substructure of $L[X]$. Friedman has proved that if ω_1 is an inaccessible cardinal of L, then for any countable stable ordinal σ, there is an $X \subseteq \omega$ such that $\sigma = \sigma_0(X)$. We derive this property of countable stable ordinals just from the assumption that there is a nonconstructible real number. The proof is done by forcing with 'stablely' pointed perfect conditions as in Sacks' pointed forcing with perfect sets.

J.M.B. MOSS, *Some neglected results about the systems ML and NF.*

(ML_0 = the inconsistent system of Quine V.163; ML = the revised system of Quine XVII.149, first presented in Wang XV.25. Theorems †801 to †844 are in Rosser VII.1 = Rosser A, in which the inconsistency of ML_0 was proved. NF = the system of Quine II.86 and XIX.134, developed in Rosser XVIII. 326 = Rosser B). (Bibliographic references are in the style of the Reviews of this JOURNAL.)

1. $ML \vdash$ †838, $ML \vdash$ †841.

The first follows from Orey XX.95 = Orey A, Theorem 2.3; the second is *ibid.* Theorem 3.6. Consequently by Rosser A †844, if ML is consistent, then $ML \nvdash$ †842. The class $\leq$ of †842 is, however, determined by a stratified condition.

2. Let A be the argument: "NF is unsatisfactory because:

(1) $NF \vdash \neg$ †841 (Rosser B XII.3.15).

(2) $NF \vdash \neg$ 'Axiom of Choice' (Specker XIX.127).

(3) $NF \vdash \mathrm{Card}(USC(V)) < \mathrm{Card}(V)$ (Rosser B XI. 1.8, Specker 4.7).

(4) In NF, the cardinals cannot be well-ordered by the $\leq$ relation defined in Rosser B at p. 375.

Consequently ML, which is a conservative extension of NF, is equally unsatisfactory."

Some version of A occurs in the writings of set theorists, and has entered the oral tradition. Nevertheless:

Argument A is unsound.

The universe of ML consists of classes, whereas that of NF consists only of sets, which, however, exist in the same sense as the classes (Quine's thesis of the univocity of "exists"). This difference between ML and NF is important, in that the unsatisfactory features of NF arise solely because some notions correctly difinable within ML, with the use of (essential) quantification over classes, are not so definable within NF. ML is hence not to be rejected for any of the reasons above.

3. If r and s range over elements of ML that are (intuitively) well-ordering relations, then $ML \vdash (\forall r, s)(r \text{ smor } s \leftrightarrow r \text{ SMOR } s)$ (Orey A). However, $ML \vdash \neg (\forall x, y)(x \text{ sm } y \leftrightarrow x \text{ SM y})$. Indeed, NF has more 'cardinals' than ML, and also (Orey A, Theorem 3.7) more 'ordinals'. However the NF pseudo-ordinals, unlike the NF pseudo-cardinals, all come at the end.

4. Quine has argued (Preface to 1981 printing of V.163) that the main inadequacy of ML arises from the result of Rosser XVII.238 that, if ML is consistent, $ML \nvdash (\exists y)(Nn \in y)$, where Nn is the class of natural numbers; consequently, $ML \nvdash Nn = nn$, where 'nn' is the NF definition of the set of natural numbers. This difficulty is best resolved by strengthening the background logic to include the infinitary quantifier 'U' (= 'for infinitely many'). To the argument that logic becomes noncompact, the reply must be that logic *is not* compact; that is, there are logical consequences of premises that do not follow from any finite number of them.

Orey XXXIV.649 proves: $NF \nvdash$ 'Axiom of Counting' (see Rosser B p. 485). Nevertheless, for the derivability relation $\vdash'$ of the noncompact logic mentioned above, $NF \vdash'$ 'Axiom of Counting'.

5. It has been suggested that the Burali-Forti paradox is avoided in ML only by restricting *200 of ML_0, whereas the Cantor paradox is avoided by the restriction on stratification in *202. However, a variant of the latter can be avoided only by the restriction on class quantification in ML (hence without going through the argument of Rosser A). For, let $C =_{\mathrm{df}} \hat{x}\{(\forall y)(S(x, y) \to x \notin y)\}$ where S is is 1-1 between $USC(V)$ and $SC(V)$, and so specified by a stratified condition, though one that requires class quantification. It follows that $C \notin V$.

6. $ML \vdash$ 'There are denumerable proper classes' (by the argument in Specker op. cit.) Perhaps the only explicit mention of this intriguing result is at XL.510(21) 219, where its bearing upon the semantic paradoxes is noted.

7. NF, though finitely axiomatizable (Hailperin IX.1), is an impredicative system; Hailperin's axiom P6 violates the vicious circle principle, and his rules of inference would require a restriction similar to that suggested for rule 2 at p. 125 of XIX.135. ML, as formulated in first-order logic, is not finitely axiomatizable (Levy). Formulated in the above noncompact logic, ML is an

impredicative extension of a second-order object-property theory that is adequate for predicative mathematics; this theory might be found acceptable from a logicist point of view. (See the following abstract.)

Problems. 1. To determine whether the *prohibition* of class quantifiers in *ML*'s version of *200 can be weakened to a restriction that permits sets definable in Δ_1-form from class quantifiers.

2. To prove the consistency of *NF* (or *ML*) formulated with the above noncompact logic.

3. To determine whether or not *ML* is finitely axiomatizable in the above noncompact logic.

J.M.B. MOSS, *The Burali-Forti paradox*.

(For some abbreviations and background results, see the previous abstract.)

In 86.12, Cesare Burali-Forti (1861–1931) did not really establish the paradox that bears his name, though his argument lies at the core of contemporary proofs. On balance, however, priority in respect of publication of the Burali-Forti paradox can justifiably be credited to him.

This paradox has remained a key puzzle for any foundation of mathematics formulated in the language of set-theory, even though it is resolved by 'limitation-of-size' theories that characterize segments of the cumulative hierarchy; these theories fail to account either for the truth or for the applicability of mathematics. However three alternative approaches proposed by Quine, Martin-Löf and Bostock succumbed to some form of the paradox, and their later, presumably consistent, versions lack either the strength or the conceptual simplicity needed for a satisfactory foundation. (References to Quine and to Rosser A are in the previous abstract; the mimeographed version of P. Martin-Löf, *An intuitionistic theory of types*, was proved inconsistent by J.Y. Girard in his Thèse du Doctorat, Paris 1972, III A pp. 1–7, and the version published in ***Logic Colloquium '73*** (Rose and Shepherdson, Editors), is restricted to the predicative part; D. Bostock, *A study of type neutrality. Part* II (***Journal of Philosophical Logic,*** vol. 9 (1980), pp. 363–414, especially at pp. 373–385) notes the inconsistency of the "general principles" that lie behind some earlier versions of his approach.)

A Fregean approach to the foundations of mathematics requires at least domains of objects (V), of properties of objects (V'), and of typeless properties of properties (including quantifiers), through which the objects are provided in bootstrap fashion. Properties of objects divide into object-like or I-II properties (cf. the I-II *Dinge* of von Neumann 299.2), and ultimate properties. Certain typeless equivalence properties of I-II properties, including cardinals, ordinals and order-types, excluding the Burali-Forti ordinal etc., determine, and can be identified with, objects. A type structure could be superadded, which might result in more cardinals and ordinals being introduced (cf. 194.3), so that the extension of certain I-II properties, e.g. V itself, would change.

The basal logic is first-order logic with identity, together with the additional quantifier 'U' of the previous abstract. It is proposed that the I-II properties are those definable in a stratified way using quantification over objects and I-II properties, together with Δ_1 quantification over all properties of objects. In the enriched logic, the property of being a natural number can be proved to be a I-II property, using the permutability of the U quantifier with a second-order $\forall$ quantifier to obtain also a Σ_1 characterization; the proposed logic thus resembles, in important ways, Quine's *ML* (revised version) with the additional axiom of the elementhood of the class *Nn*, which is defined in XVII.149 at p. 242.

The $\leq$ relation between ordinals, though homogeneous between objects, fails to be a I-II property of objects, since it cannot be defined without essential use of a Π_1 property quantifier. This result corresponds to the presumed unprovability in *ML* of †842 of Rosser A. Corresponding to †838 of Rosser A, which was essentially proved for *ML* in Orey A, is the result: if S is a I-II well-ordering relation, with ordinal α, then α determines an object. The Burali-Forti paradox is thus resolved by the absence of a Σ_1 characterization of the $\leq$ relation between ordinals.

Predicative mathematics can be obtained in the standard way, though apparently harmless impredicativities, which do not violate the vicious circle principle, arise in such theorems as $(\exists F)(1(F)\ \&\ F(1))$.

From the results of Orey A, a Π_1 definition of the property of being an ordinal can be obtained. However the divergence between the equivalence relations (for cardinals) sm and SM (see 3 of the previous abstract) apparently precludes a Π_1 definition of the property of being a cardinal. Indeed, the theory of cardinals is less tidy than that of ordinals, since an ultimate property can bear SM to a I-II property and thus have the same cardinal, which must determine an object. For example, the Chang cardinal Q, for which $(Qx)\ (x = x)$, that is $Q(V)$, determines an object.

ALBERT A. MULLIN, *Another look at Hilbert's eighth class.*

This note complements earlier results on Hilbert's eighth class of problems [1] by proposing a new approach to A. de Polignac's variation on the Goldbach Conjecture.

LEMMA 1. *The asymptotic density of positive integers m for which the diophantine defining relation* $\phi(n)\sigma(n) = (n - m - 1)(n + m - 1)$ *characterizes the product n of two primes that differ by that* $m \geq 1$ *is* 1.

Note 1. There exist infinitely many odd m and infinitely many even m for which there is *not* such characterization by the diophantine defining relation.

COROLLARY 1. *For almost every positive even integer m, the relation* $\phi(n)\sigma(n) = (n - m - 1) \times (n + m - 1)$ *characterizes the product n of two primes that differ by that* $m \geq 1$.

LEMMA 2. *For almost every positive even integer m there exists a positive odd composite integer n such that* $\phi(n)\sigma(n) = (n - m - 1)(n + m - 1)$.

Conjecture 1. Lemma 2 can be strengthened in two directions so as to hold: (1) "for *each* positive even integer m" and (2) "there exist *infinitely* many positive odd composite n". Hence by Lemmata 1 and 2:

COROLLARY 2. *Almost every positive even integer is a difference of two primes.*

Conjecture 2. *For each* $m \geq 1$, the diophantine defining relation has only finitely many *exceptional* solutions n (i.e., where a solution n is *not* the product of two primes that differ by $m \geq 1$).

Note 2. From Conjectures 1 and 2 it would follow that each positive even integer is a difference of two primes in *infinitely* many ways.

Problem 1. Do there exist infinitely many n for which $(2^n + 1)$ is the product of two distinct primes?

Problem 2. Do there exist infinitely many odd n for which $(2^n - 1)$ is the product of two distinct primes?

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PAULO A. S. VELOSO, *On the nonfinitary character of Γ-logics.*

The familiar notion of ω-model has a natural generalization, the α-models of da Costa [1] or the Γ-models of Henkin [2]. The addition of the natural generalization of Carnap's ω-rule to the usual first-order deductive calculus gives a complete one for α-logics with denumerable α [1]. The purpose of this note is to pinpoint the reason why finitely long proofs will not suffice for Γ-logics.

Let Γ be a set of variable-free terms of the first-order language L. A Γ-model is one in which every element is denoted by a symbol from Γ. Now, assume Γ infinite and L to have at least one nonlogical symbol. Then, there exists a set Δ of first-order sentences of L (with $|\Delta| \leq |\Gamma|$) such that (1) for every finite $n > 0$, Δ has one (and only one, up to isomorphism) Γ-model with n elements; (2) Δ has no infinite Γ-model (of course, Δ has infinite models).

Hence, the compactness theorem fails for Γ-logics. Therefore, there exists no sound and complete deductive system for Γ-logics with proofs of finite length (even if the restriction of effectiveness of axioms and rules of inference is lifted).

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