Part I

The art of logic

Chapter 1

Grammar

After some preliminary grammatical considerations in this chapter, we collect the material on truth-functional logic in chapter 2; the material on quantifiers and identity in chapter 3 (proofs), chapter 5 (symbolization), and chapter 6 (semantics); some applications in chapter 7 (logical theory, arithmetic, set theory), and a discussion of definitions in chapter 8. Chapter 13 is an unfulfilled promise of a discussion of the chief theorems about modern logic.

1A Logical Grammar

Rigorous logic ought to begin with rigorous grammar; otherwise bungle is almost certain. Here by a "logical" grammar we mean one of the sort that guides the thinking of most logicians; such a grammar is a more or less language-independent family of grammatical categories and rules clearly deriving from preoccupation with formal systems but with at least prospective applications to natural languages. A logical grammar is accordingly one which is particularly simple, rigorous, and tidy, one which suppresses irregular or nonuniform or hard to handle details (hence differentiating itself from a "linguistic" grammar whether of the MIT type or another), one which idealizes its subject matter, and one which by ignoring as much as possible leads us (perhaps wrongly) to the feeling of "Aha; that's what's *really* going on!" Among such logical grammars we think a certain one lies at the back of most logicians' heads. Our task is to make it explicit, not in order to criticize it—indeed though it would be dangerous to suppose it eternally ordained, we think very well of it—but because only when it is brought forth in all clarity can we sensibly discuss how it ought to be applied. The version here presented derives from Curry and Feys (1958) and Curry (1963).

Of course here and hereafter application of any *logical* grammar will be far more straightforward and indisputable in the case of formal languages than in the case of English; for in the case of the former, possession of a tidy grammar is one of the design criteria. But English is like Topsy, and we should expect a fit only with a "properly understood" or "preprocessed" English—the preprocessing to remove, of course the hard cases. So read on with charity.

1A.1 Sentence and term

"Logical grammar," as we understand it, begins with three fundamental grammatical categories: the "sentence," the "term," and the "functor." The first two are taken as primitive notions, hence undefined; but we can say enough to make it clear enough how we plan to apply these categories to English and to the usual formal languages.

By a *sentence* is meant a *declarative* sentence in pretty much the sense of traditional grammar, thus excluding—for expository convenience—interrogatives and imperatives. A sentence is of a sort to express the content of an assertion or of a conjecture, and so on. Semantically, a sentence is of a kind to be true or false, although typically its truth or falsity will be relative to various items cooked up by linguists and logicians. Examples: The truth or falsity of "It is a dog" often depends on what the speaker is pointing at, and that of the symbolic sentence "Fx" depends on how "F" is interpreted and on the value assigned to "x."

Traditional grammar gives us less help in articulating the concept of a *term*, although the paradigm cases of both traditional nouns and logical terms are proper names such as "Wilhelm Ackermann."

Some more examples of terms from English:

Terms	*What some linguists call them*
Wilhelm Ackermann	proper noun
the present king of France	noun phrase
your father's mustache	noun phrase
triangularity	(abstract) noun
Tom's tallness	(abstract) noun phrase
that snow is white	that clause; factive nominal; nominalized sentence
what John said	factive nominal; nominalized sentence
his going	gerund; nominalized sentence
he	pronoun

We add some further examples from formal languages.

Terms	*What some logicians call them*
$3+4$	closed term
x	variable
$3+x$	open term
{x: x is odd}	set abstract
ιxFx	definite description

What the logical grammarians contrast with these are so-called "common nouns" such as "horse," as well as plural noun phrases such as "Mary and Tom." And although we take the category of terms to be grammatical, it is helpful to heighten the contrast by a semantic remark: The terms on our list purport—at least in context or when fully interpreted ("assigned a value")—to denote some single entity, while "horse" and "Mary and Tom" do not. Perhaps the common noun is the most important English grammatical category not represented anywhere in logical grammar. Contemporary logicians (including us) uniformly torture sentences containing common nouns, such as "A horse is a mammal," into either "The-set-of-horses is included in the-set-of-mammals" or "For-anything-you-name, if it is-a-horse then it is-a-mammal," where the role of the common noun "horse" is played by either the term (our sense) "the-set-of-horses" or the predicate (see below) "_ is-a-horse."

Exercise 1 *(Logical-grammar terms)*

Taking "term" in the sense given to it by logical grammar, give several *fresh* examples of terms, trying to make them as diverse as possible. Include both English examples and formal examples—perhaps even an example from advanced mathematics.

▷ . . . . ◁

1A.2 Functors

So much for sentence and term.[1] By a *functor* is meant a way of transforming a given ordered list of grammatical entities (i.e., a list the members of which are terms, sentences, or functors) into a grammatical entity (i.e., into either a term, a sentence, or a functor). That is to say, a functor is a function—a grammatical function—taking as inputs (arguments) lists of items from one or more grammatical categories and yielding uniquely as output (value) an item of some grammatical category. For each functor, as for any function, there is defined its *domain*, that is, the set of its input lists and its *range*, which is the set of its outputs.

For our limited purposes, however, we can give a definition of a functor which, while not as accurate as the foregoing, is both easier to understand and adequate for our needs:

1A-1 DEFINITION. *(Functor)*

A *functor* is a pattern of words with (ordered) blanks, such that when the blanks are filled with (input) any of terms, sentences, or functors, the result (output) is itself either a term, sentence or functor.[2]

We wish henceforth to ignore the cases in which functors are used as either inputs or outputs. For this reason, we define an "elementary functor."

1A-2 DEFINITION. *(Elementary functor)*

An *elementary functor* is a functor such that either all of its inputs are terms or all of its inputs are sentences; and whose output is either a term or a sentence.

[1] There is in fact a little more to say about terms, but you will understand it more easily if we delay saying it until after introducing the idea of a functor.

[2] This is the first official (numbered and displayed) definition in this book. Don't bother continuing your study of logic unless you commit yourself to memorizing each and every such definition. Naturally learning how to *use* these definitions is essential as well; but you cannot learn how to use them unless you first memorize them.

Hence, "input-output analysis" leads us to expect four kinds of elementary functors, depending exclusively on whether the inputs are terms or sentences, and whether the outputs are terms or sentences. The following is intended as a definition of "operator," "predicate," "connective," and "subnector."

1A-3 DEFINITION. *(Four kinds of elementary functor)*

Inputs	*Output*	*Name*	*Examples*
Terms	Term	Operator	_ + _ ; _'s father
Terms	Sentence	Predicate	_ < _ ; _ is nice
Sentences	Sentence	Connective	_ and _ ; John wonders if _
Sentences	Term	Subnector	"_" ; that _

For example, the table tells us that *a predicate is a pattern of words with blanks such that when the blanks are filled with terms, the result is a sentence.*

Exercise 2 *(Kinds of elementary functors)*

1. Write out a definition of each of "operator," "connective," and "subnector" that is parallel to the foregoing definition of "predicate."
2. Use an example of each of the four, filling its blanks with short and simple words.
3. Give a couple of examples such that the *output* of some functor is used as the *input* of a different functor.

   We will discuss these in class.

▷ . . . . ◁

Here are some more examples.

Connectives

_ and _; both _ and _; (_&_) (_ ∧ _).

_ or _; either _ or _; (_∨_).

if _ then _; _ if _; _ only if_; only if _, _; (_→_).

_if and only if _; (_↔_).

it is not the case that _; ∼_.

The above are often studied in logic; the following, however, are also connectives by our definition.

> John believes (knows, wishes) that _; it is possible (necessary, certain) that _; that _ implies (entails) that _; if snow is red then either _ or if Tom is tall then _.

We note in passing that in English an expression can often have an analysis supplementary to the one given above; for example, English allows us to fill the blanks of "_ and _" with the terms "Mary" and "Tom," obtaining "Mary and Tom," which is a plural noun phrase and hence not a creature of logical grammar. We therefore only intend that our English examples have at least the analysis we give them, without excluding others.

Predicates. _ hit _; John hit _; (_ =_); _ was the first person to notice that _ 's mother had stolen the pen of _ from _.

Operators. $\_^2$ (i.e., the square operator); the sister of _.

Subnectors. The probability that _; the desire that _.

Note that we can sort functors by how many *blanks* they contain; a predicate with two blanks is called a *two-place predicate*, and so forth. Negation is a one-place connective.

Reflection on sentences, terms, and functors leads us to reinstate an old distinction among expressions.

1A-4 DEFINITION. *(Categorematic vs. syncategorematic expressions)*

- A *categorematic expression* is an expression in logical grammar such as a sentence or a term that can be the input or output of a functor, but is not itself used only in making up functors.

- A *syncategorematic expression* is an expression in logical grammar such as "and" or "+" or "(" that is used only to help in making up a functor. In other words, a syncategorematic expression serves only as part of a pattern of words with blanks.

It is often said that categorematic expressions have their meaning "in isolation" (or have independent meaning) whereas syncategorematic expressions have meaning only "in context." This will be true enough for the symbolic language that you will be learning in these notes, but it is dangerous to take it as a "deep" thought about natural language. The reason is this: The very idea of "meaning in isolation" seems to slip out of reach when applied to something as essentially social as natural language. In any event, (1) we will be applying the distinction only within the confines of logical grammar, and (2) we won't be needing it for a while.

1A.3 English quantifiers and English quantifier terms

English quantifier terms. In English many terms are constructed by applying an English quantifier such as "each" or "at least one" to a common noun phrase; for example, "each person wearing a brown hat" or "at least one giant firecracker." We know that in the logical grammar of English these are *terms* because they go in exactly the same blanks as any more standard term such as a proper name like "Jackie" or a variable such as "x." To give but one example, if you can put "Jackie" into the blank of "_ went out," then you can also insert "each person wearing a brown hat" into the same blank:

> *Each person wearing a brown hat* went out.

English quantifiers and quantifier terms are important to English because they are a principal way in which English expresses "quantificational propositions." For this reason, it is all the more important to be aware that typical formal languages (such as the one we shall be mastering) do *not* express "quantificational propositions" in this way. They instead use the variable-binding connectives "$\forall$x" and "$\exists$x." For example, instead of "Each person wearing a brown hat went out," one would have something like

$$\forall x(x \text{ is a person wearing a brown hat} \to x \text{ went out}).$$

We will later study this matter in detail; see §5A and §10A. For now, simply file away three facts: (1) Both English and typical formal languages have some terms that—like proper names—purport to denote a single entity. (2) English includes quantifier terms, but typical formal languages do not. (3) English uses quantifier terms to help express "quantificational propositions," whereas typical formal languages accomplish that end by means of variable-binding connectives such as "$\forall$x" and "$\exists$x." (Note relevant to a coming exercise: In a subsequent section, §10A.3, we will learn that "the" is an English quantifier and that "the golden mountain" counts as an English quantifier term.)

Refined grammatical analysis. The above grammatical analysis given in terms of input-output is gross; it can, however, be usefully refined without alteration of the spirit of the enterprise by dividing the set of terms into categories, and stating, for each blank in a predicate or operator, and for each result of a subnector or operator, to what category the term belongs. This is particularly useful when introducing new technical terms.

We give a few examples, where we take the following as categories of terms: *number*-terms (they name numbers), *physical-object*-terms (they name physical objects), *sentence*-terms (they name sentences), *proposition*-terms (they name propositions), *person*-terms (they name persons).[3]

- "_ $<$ _" is a *number*-predicate; i.e., inputs must be number terms. "_ $+$ _" is a *number*-operator; both inputs and outputs must be number-terms.
- "_ has the same mass as _" is a *physical-object* predicate; the input terms must be physical-object-terms. "The mass of _ (in grams)" is a *physical-object-to-number*-operator; the input must be a physical-object-term, while the output is a number-term.
- Most logicians use "_ is true" as a *sentence*-predicate; the input must be a sentence-term, i.e., a term naming a sentence. Some logicians use "_ is true" as a *proposition*-predicate, where the input must be a name of a proposition. Most logicians use "_ implies _" as a sentence-predicate. (What does this mean?)
- "The probability that _" is a *number*-subnector; the output is a number-term.

[3]A *sentence* is a piece of language. A *proposition* is an abstract entity that is, in some sense, the "meaning" of a sentence. The term "proposition" is philosophical jargon whose meaning varies from philosopher to philosopher. The only thing you need to know just now, however, is that some philosophers take "that-clauses" to name propositions, so that, for example, the sentence "Dick runs" has, as its meaning in some sense, the proposition that Dick runs.

- "_ believes _" is a predicate that requires person-terms in its first blank, and proposition-terms in its second blank. ("_ believes that _" is a non-elementary functor. Why?)

Exercise 3 *(Logical grammar)*

1. Identify each of the following as a (plain) term (*t*), English quantifier term (*Eqt*), sentence (*s*), operator (*o*), predicate (*p*), connective (*c*), subnector (*sub*), or none of these (*none*).

   (a) Sam
   (b) each quick brown fox
   (c) the quick brown fox
   (d) _ 's teacher
   (e) _ jumped over the lazy brown dog
   (f) _ $\rightarrow$ _
   (g) Where is Mary?
   (h) snow is puce $\leftrightarrow$ _
   (i) if _ then both _ and _
   (j) "_"
   (k) _ is true
   (l) John's hope that _
   (m) it is true _
   (n) All _ are mortal
   (o) _ play baseball

2. The following is a quote from these notes. List all the terms (whether plain terms or English quantifier terms) and functors (stating which kind) you can find, but if there are many, *stop* after a few.

   > From at least one theoretical point of view, identity is just another two-place predicate, but the practice of the art of logic finds identity indispensable.

3. Considering the "refined grammatical analysis" above, which of the following make grammatical sense? Assume that "Jack" is a physical-object term, and that "Mary" is both a physical-object term and a person term. *Explain* your answers as best you can.

   (a) The mass of Jack (in grams) < 4.
   (b) $3+4$ has the same mass as $4+3$.
   (c) Jack has the same mass as the mass of Mary (in grams).

(d) The mass of Mary (in grams) $<$ the probability that Mary believes that $3+4=4+3$.

4. Optional (skippable). Alter each of the following to make it grammatically correct while retaining the meaning it would ordinarily be taken to have.

(a) Snow is white is a long sentence

(b) Snow is white implies that snow is colored

(c) If "snow is white" then "snow is colored"

(d) If what Tarski said, then snow is white.

▷ . . . . ◁

1B Some symbolism

Each branch of logic is distinguished by

- its grammar,
- its proof theory,
- its semantics,
- its applications.

As indicated in Figure 1.1, symbolic logic has *grammar* as its foundation; each of the other parts depends on first specifying the grammar of the symbolic language at issue. We throw out the bare bones of our grammar, and indicate what counts as practical mastery thereof.

We introduce some standard symbolism for logical ideas that come from several parts of logical grammar in the sense of §1A. Our idea is to introduce the grammar of the symbolism, while postponing learning about semantics and proof theory and applications. Be reassured that when we turn to those later tasks, we will take them up piecemeal. Only in this section on grammar do we lay out everything all at once.

Quantifiers are an important part of logic. Although we do not study the logic of symbolic quantifiers and their variables until a later chapter, we want to be able to start writing them now and then. Until we introduce quantifiers as a topic of study in chapter 3, however, when we use quantifiers to say something, we will always

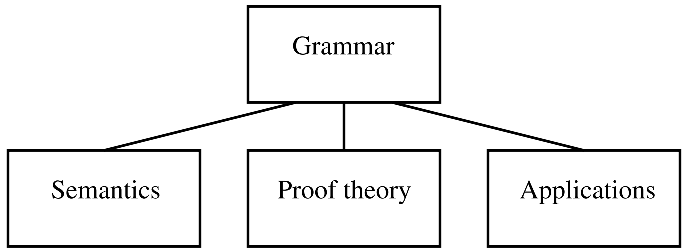

Figure 1.1: Four fundamental parts of any symbolic logic

say the same thing in English (or "middle English," as we soon say). The aim is to accustom you to seeing and reading quantifiers and variables of quantification without prematurely forcing you to master their use. Here is the notation that we shall use.

1B-1 SYMBOLISM. *(Quantifiers)*

We use "∀" for "universal quantification" and "∃" for "existential quantification." These are always used with variables such as "x," "X_1," "A_1," and so on, in such a way that for example "∀x" is to be read "for every x," and "∃x" is to be read "there is an x such that." In logical grammar, "∀x_" and "∃x_" are *connectives*: In each case the blank must be filled by a sentence, and then the result is a sentence. Thus, "∀x(if x is a cat then x is a mammal)" is read (in a language that we sometimes call "middle English") as "for every x, if x is a cat then x is a mammal"; and "∃x(x is a cat and x has claws)" is read "there is an x such that x is a cat and x has claws." The sentence that goes into the blank of "∀x_" will almost certainly contain an occurrence of "x." The variables that are used with symbolic quantifiers (or in middle English) are "*variables of quantification*."

Later, especially in Convention **1B-19** and Convention **1B-20** we state two conventions governing our use of variables in these notes.

1B-2 SYMBOLISM. *(Variables, constants, parameters)*

In our jargon, a *variable* is always a variable of quantification, to be used with explicit symbolic or middle-English quantifiers. The contrast is with *constants*. We shall be dealing with constants associated with various types in logical grammar: Individual constants, operator constants, predicate constants, and connective constants will be foremost. A constant always has in each use a definite meaning. Constants in logic have two roles. In one role the constant has a meaning that is fixed by previous (human) use. But sometimes we want to have available a constant that we just *assume* has a fixed denotation, without saying exactly what it is. Sometimes such constants are called *parameters*, to distinguish them from both variables of quantification and constants that have a fixed predetermined meaning. Parameters are often used to express generality without the help of explicit universal quantification. Other times—and often by us—they are used for toy examples.

Here is a summary.

Variables (of quantification). To be used with explicit symbolic quantifiers or with quantificational idioms in middle English.

Constants. Can be associated with any type in logical grammar. They always have a meaning that is assumed to be fixed in each use. They have two roles.

1. A constant can be used with a meaning fixed by previous usage. This includes the case where we introduce a new constant accompanied by an explanation of its meaning. Occasionally we call these **"real" constants**.
2. **Parameters**, in contrast, are constants that we just assume ("pretend" is not quite right) have a fixed meaning, without saying exactly what it is, either to help express generality (without explicit universal quantification) or as auxiliaries of one kind or another or just as examples.

Parameters work so well as examples in learning logic because *logic itself makes no distinction between "real" constants and parameters.* Their grammar, proof theory, and semantics are identical. The only difference lies in how we think about them. That is why we almost always speak of just "constants" without distinguishing the "real" ones from the parameters.

1B-3 REMARK. *(Middle English)*

In these notes we very often use single letters with *English* quantifier expressions, saying, for example, For every A, if A is a sentence then A is either true or false.

The difficulty is that the displayed sentence is neither correct English (English itself has no variables) nor correct symbolism ("for all" is not part of our symbolic language).[4] Therefore we call the (eminently useful) language containing the quoted sentence *middle English*. So we could and should call "A" in the quoted sentence a "middle English variable of quantification" in contrast to a "symbolic variable of quantification." Instead, however, we almost always just ignore the topic.

1B-4 SYMBOLISM. *(Individual constants)*

An *individual constant* is an "atomic" singular term (no complex structure of interest to logic) that is not used as a variable of quantification. To say that an individual constant is an *individual* constant is to say that it can go into the blanks of predicates and operators. To say that an individual constant is an individual *constant* is to say that it cannot be used quantificationally with $\forall$ or $\exists$. An individual constant has in each use a definite denotation. Symbols such as "0" and "1" and "Fred Fitch" may be taken as individual constants.

Individual constants, like all constants, have two roles. One role is like that of the examples just cited, whose denotation is fixed by previous use (by we humans). We shall be introducing a variety of them for our special logical purposes; for example, we shall be introducing "T" and "F" as individual constants respectively denoting the two truth values Truth and Falsity. But sometimes we want to have available an individual constant that we just *assume* has a fixed denotation, without saying exactly what it is; these individual constants can therefore also be called "parameters" in order to distinguish them from both variables of quantification and "real" individual constants that have a fixed predetermined meaning. When we want to make up some new individual constants to use as parameters, we use letters "a," "b," and "c," and so on; we will need to assume an unending (infinite) supply of individual parameters, created perhaps by numerical subscripting.

1B-5 SYMBOLISM. *(Operator constants and operators)*

An expression is an *operator constant* if it is used in just one way: in connection with blanks and (perhaps) parentheses to make up an operator. Therefore an

[4]Furthermore, there is no English part of speech that suits *all* occurrences of "A" in the displayed sentence. An English common noun must come after "every," and an English singular term must come before "is."

operator constant is syncategorematic, Definition **1A-4**. In particular, no operator constant can be put into the blanks of any elementary functor. The symbol "+" is an operator constant and "(_ + _)" is its associated operator. We shall be introducing some important operators in our study of logic; for example, "Val(_)" is an operator whose blank is restricted to (not sentences but) sentence-terms (that is, terms that denote sentences), and will be introduced with the reading "the truth value of _."

In practice, when "f(_)" is a "real" operator, it will usually be a "Kind 1 to Kind 2" sort of operator, with its inputs restricted to terms that denote entities of Kind 1, and with its outputs always denoting entities of Kind 2. For example, "Val('_')" is a sentence-to-truth-value operator.

When we want to make up some new operators to use as parameters, we use "f" and "g" and the like for operator constants, saying on each occasion how many blanks are presupposed, and where they go. Usually we write the blanks after the operator constant, with or without parentheses: "f_" or "f(_)" or "g(_, _)" or the like, whereas in the case of "real" operator constants, we let the blanks fall where they may. Operator parameters are almost always of the most general kind: They form anything-to-anything operators.

1B-6 SYMBOLISM. *(Terms and closed terms)*

A *term* is constructed from individual constants (either "real" constants or parameters) and variables of quantification by means of operators, for example, "g(f(a) + 0)." A term can be *atomic* (that is, an individual constant or a variable) or complex. A *closed term* or *variable-free term* is a term that does not contain a variable of quantification.

1B-7 CONVENTION. *(t for terms)*

We use "t" as a variable ranging over terms. Thus, any use of "t" carries the presupposition that t is a term.

1B-8 SYMBOLISM. *(Predicate constants and predicates)*

An expression is a *predicate constant* if it is used in just one way: in connection with blanks and (perhaps) parentheses to make up a predicate. Therefore a predicate constant is syncategorematic, Definition **1A-4**. In particular, no predicate constant can be put into the blanks of any elementary functor. For example, "=" is a predicate constant since "(_ = _)" is a predicate; and you will see numerous further examples. For predicate parameters, we use "F" and "G" and the like, saying on each occasion how many blanks are presupposed, and where they go. Usually we write the blanks after the predicate parameter, with or without parentheses: "F_" or "F(_)" or "G(_, _)" or the like, whereas in the case of "real" predicate constants, we let the blanks fall where they may.[5]

A *predication* is defined as the result of applying a predicate to an appropriate sequence of terms, so that a predication is a special kind of sentence. For example, "$(3+5)$ is greater than 7" and "Fab" are predications.

1B-9 SYMBOLISM. *(Identity predicate)*

Identity is perhaps the most important predicate in logic. The *identity predicate* "_=_" expresses the simple-minded relation that holds only between each thing and itself. An identity sentence $t_1 = t_2$ is to be read in English as "t_1 is identical to t_2." Since a given pair of terms t_1 and t_2 might or might not denote the same thing, it always makes sense to ask whether or not $t_1 = t_2$. For example, $2+2 = 4$ (two names for the one thing), but $2+3 \neq 4$ (two names, two things). From time to time we use "$t_1 \neq t_2$" as the negation of $t_1 = t_2$. We shall state some logical rules for identity just as soon as we can, in §3D, where rules for it are introduced.

1B-10 SYMBOLISM. *(Sets: two predicates and an operator)*

A *set* is a collection that has members (unless of course it doesn't). The set of even numbers contains infinitely many members. The set of integers between three and five has just one member. The set of numbers between three and three is empty (it has no members). Three pieces of notation (and a convention) come in right away.

- When we wish to say that something is a set, we just say so: "_ is a set" is a predicate. (The predicate constant is "is a set.")
- Set membership is a predicate that as logicians we should not like to do without. We use "∈" as the predicate constant for set membership; the predicate is "_ ∈ _," to be read (from left to right) as "_ is a member of the set _." The left blank is unrestricted, but the right blank should be restricted to terms that denote sets.

[5]Curry calls "=," for example, an "infix," which means that the predicate constant goes in the middle of its blanks. Others are called "prefixes" or "suffixes."

- We use the notations "{_}," "{_, _}," "{_, _, _}," etc. as operators whose *inputs* are terms (e.g. individual constants), and whose *output* is a term that denotes a *set* that contains exactly the things denoted by its inputs. For example, "{a, b}" is read "the set containing just a and b." Thus, the following are invariably true:

  $a \in \{a\}$
  $b \in \{a, b\}$

  Whether "$c \in \{a, b\}$" is true depends, however, on what "c" denotes; it's true iff either c = a or c = b. We'll come back to this after we have in hand enough logical machinery to do so with profit.

1B-11 CONVENTION. *(*X, Y, Z *for sets)*

Convention: We use "X," "Y," and "Z" as variables or parameters ranging over sets.

This means three things. (1) In any context in which we use e.g. "X," the information that X is a set is automatically presupposed. (2) Whenever we say "For all X ... ," we intend "For all X, if X is a set, then ... ." (3) Whenever we say "For some X ... ," we intend "For some X, X is a set and ... ."

1B-12 SYMBOLISM. *(Truth-functional connectives)*

An expression is a *connective constant* if it is used in just one way: in connection with blanks and (perhaps) parentheses to make up a connective. Therefore a connective constant is syncategorematic, Definition **1A-4**. In particular, no connective constant (except $\perp$, which takes zero arguments) can be put into the blanks of any elementary functor. We introduce six connective constants: $\rightarrow$, $\vee$, $\leftrightarrow$, $\sim$, &, and $\perp$, which help us to symbolize some connectives of English. They are all, in a sense to be explained, "truth functional." The six symbolic connectives and their English readings are as follows:

Symbolic connective	English connective
$(\_ \rightarrow \ldots)$	if _ then …
$(\_ \vee \ldots)$	_ or …
$(\_ \leftrightarrow \ldots)$	_ if and only if …
$\sim \ldots$	it's not true that …
$(\_ \& \ldots)$	_ and …
$\perp$	standard logical falsehood; absurdity

Four can be seen to be two-place, while one is one-place, and one is 0-place. The sentence $\perp$ will count as Very False; we do not add it for theory, but because it is practically useful to have it available, as we will see.

The connective constants and the connectives each have names, as follows:

Constant	Name of constant	Name of connective
$\rightarrow$	Arrow	Conditional
$\vee$	Wedge	Disjunction
$\leftrightarrow$	Double arrow	Biconditional
$\sim$	Curl	Negation
&	Ampersand	Conjunction
$\perp$	Bottom	*falsum*, absurdity

1B-13 SYMBOLISM. *(Individual variables)*

We have introduced *constants* of the types of individuals, operators, predicates, and connectives. We shall be using *"real" constants* of all of these types, and—for all but the connectives—we shall also be dealing with some *parameters* (without any definite meaning fixed by previous usage).

Note, however, that operator and predicate and connective constants are syncategorematic. As a companion decision, we will not be introducing variables of quantification except for the type of individuals. *There will be no operator or predicate or connective variables of quantification (to be bound by a quantifier).* It is this feature of the logic that we are studying that chiefly earns it the title "first order logic." To introduce quantified variables of the type of operators or predicates or connectives counts as proceeding to a "higher order logic," something that we do not do in these notes.[6] All of our variables are therefore of the type of individuals.

[6]Example. Higher order logic permits writing "∀F(Fa↔Fb)" in order to say that individuals a

1B-14 SYMBOLISM. *(Sentence and closed sentence)*

We know that *terms* are made from individual constants and variables by means of operators. We know that *predications* are made from terms by means of predicates. Finally, *sentences* are made from predications by means of (symbolic) quantifiers and the six listed truth-functional connectives. A sentence is *closed* if every variable of quantification that it may contain occurs in the scope of an explicit quantifier that uses that variable.

1B-15 CONVENTION. *(*A, B, C *for sentences;* G *for sets of sentences)*

- We use "A" or "B" and "C" as parameters or variables ranging over sentences.
- We use "G" and "H" as parameters or variables ranging over sets of sentences.

1B-16 CONFUSION. *(Use and mention)*

Experience teaches that when we wish to *speak about* the very language that we wish to *use*, confusion is unavoidable. Therefore do not hold either yourself or us to blame if the following sounds like double talk. (You are not responsible for understanding it. But *try*.)

When we *use* the symbolic language that we are introducing (including illustrative uses), we use e.g. "A" as *being* a sentence and we use e.g. "_→_" as a connective, with sentences to be put into its blanks. When, however, we *talk about* the symbolic language, we use e.g. "A" as a term rather than a sentence—a term that denotes a sentence. In this case, the pattern "_→_" is an operator, a "sentence-to-sentence operator," that takes names of sentences in its blanks and outputs the name of a sentence. Context tends to resolve the ambiguity between these two cases.

For certain purposes, it is good to think of the grammar of our symbolic language "inductively" as follows. (See also §2A.1.)

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and b have exactly the same properties (this is discussed in §6G). This sentence is not part of first order logic. (There is more to say about this, but not at this stage.)

1B-17 SYMBOLISM. *(Inductive explanation of "term")*

Base clause for terms. Each individual constant and variable is a term.

Inductive clauses for terms. For each operator, if the blanks of the operator are filled with terms, the result is a term.

1B-18 SYMBOLISM. *(Inductive explanation of "sentence")*

Base clause for sentences. Each predication is a sentence.

Inductive clause for sentences. The result of putting a sentence into the blanks of either one of the symbolic quantifiers or into the blanks of one of the six truth-functional connectives is a sentence.

The similarity is obvious: In inductive explanations, the "base" clause gets you started, and the "inductive clauses" tell you how to carry on.

There is, however, more to grammar than the base and inductive clauses. Typically an inductive grammatical explanation will add a so-called "closure clause" saying that nothing is of the given grammatical type (e.g., term or sentence) unless it is so in virtue of one of the base or inductive clauses. Furthermore, "no ambiguity" is an essential part of logical grammar: If you are given any expression, it is uniquely determined (1) whether it is a term or sentence (never both), (2) whether it is "atomic" (in the sense of coming in by a base clause) or rather results from applying some functor, and (3) if it comes by a functor, it is uniquely determined by which functor it comes, and to which arguments the functor has been applied. These important features of logical grammar we will just take for granted.

Before summarizing, we call attention to two almost indispensable conventions governing our use of variables in these notes.

1B-19 CONVENTION. *(Variables of limited range)*

In many places in these notes we announce that by convention we are using a certain letter or group of letters as variables that *range over* some specified set of entities. So far we have laid down Convention **1B-11** for the use of "X" (etc.) to range over sets, Convention **1B-15** for the use of "A" (etc.) to range over sentences and the use of "G" to range over sets of sentences, and Convention **1B-7** for the use of "t" to range over terms; and there are further examples in the offing. In every such case we intend the following.

1. Every use of the restricted letter involves a presupposition that what the letter denotes is *of the appropriate sort*.
2. When a restricted letter is used with a universal quantifier, the meaning is that all entities *of the appropriate sort* satisfy the condition at issue.
3. When a restricted letter is used with an existential quantifier, the meaning is that some entity *of the appropriate sort* satisfies the condition at issue.

1B-20 CONVENTION. *(Dropping outermost quantifiers)*

We often drop outermost universal quantifiers. (This convention applies only to *outermost* and only to *universal* quantifiers.) For example, the first line of Examples **2B-16** on p. 40 is

A, B $\vDash_{TF}$ A & B.

By application of the present convention, we should restore outermost universal quantifiers, giving

For all A and for all B: A, B $\vDash_{TF}$ A & B.

Now by application of Convention **1B-19** just above, and noting that we have declared that "A" and "B" range over sentences, we come to the "full" meaning, namely,

For all A and for all B, if A is a sentence and B is a sentence, then A, B $\vDash_{TF}$ A & B.

As you can see, the conventions governing the use of variables of restricted range together with the convention permitting us to drop outermost universal quantifiers make for perspicuity by eliminating an overdose of henscratches.

1B-21 SYMBOLISM. *(Summary)*

These are the key ideas of the symbolism that we have introduced in this section.

Quantifiers. $\forall$x ("for all x" in middle English) and $\exists$x ("for some x" in middle English). See Symbolism **1B-1**.

Variables, constants, and parameters. See Symbolism **1B-2**.

Variables of limited range. See Conventions **1B-7**, **1B-11**, **1B-15**, and **1B-19**.

Dropping outermost quantifiers. See Convention **1B-20**.

Individual constants (including parameters). Individual parameters are a, b, c, and so on. See Symbolism **1B-4**.

Operator constants (including parameters) and operators. Operator parameters are f, g, and so on. See Symbolism **1B-5**.

Terms and closed terms. See Symbolism **1B-6**.

Predicate constants (including parameters) and predicates. Predicate parameters are F, G, H, and so on. See Symbolism **1B-8**.

Identity predicate. See Symbolism **1B-9**.

Sets. Two predicates ("_ is a set" and "_ ∈ _") and a family of anything-to-set operators ("{_}," "{_, _}," and so on). See Symbolism **1B-10**.

Truth-functional connective constants and connectives. See Symbolism **1B-12**.

Individual variables. See Symbolism **1B-13**

Sentences and closed sentences. See Symbolism **1B-14**

Variables of restricted range.[7] So far introduced:

- t for terms.
- X, Y, Z for sets.
- A, B, C for sentences.
- G, H for set of sentences.

Other letters used in special ways.

- x, y, z for individual variables of quantification (general purpose).
- a, b, c for individual constants (general purpose).
- f, g, h for operator constants.

[7] In a few cases, restricted letters are used in other ways—without, we think, confusion.

- F, G, H for predicate constants.

Exercise 4 *(English into symbols)*

Symbol-English dictionary: Let j = Jack, Y = the set of cats, f(v) = the father of v, and let $T(v_1, v_2) \leftrightarrow (v_1$ is taller than $v_2)$. Put the following into notation.

1. Jack is a member of the set of cats and the set of cats is a set.
2. Jack is taller than his father.
3. Jack's father is a member of the set of cats if and only if Jack's father is not a member of the set of cats.
4. If the father of Jack is identical to Jack then Jack is identical to the father of Jack.
5. The set containing just Jack and the father of Jack is a set.
6. If x belongs to the set containing just Jack and his father, then either x is identical to Jack or x is identical to the father of Jack.
7. For every x, x is identical to x.
8. There is an x such that x is identical to Jack.

▷ . . . . ◁

Exercise 5 *(Symbols into English)*

Using the same "dictionary" as for Exercise 4, put the following into English.

1. $j = f(j) \rightarrow \sim$(Y is a set)
2. $j \in Y \vee Y \in Y$
3. $\forall x(x \in Y \rightarrow x \in Y)$
4. $\sim T(f(j), f(j))$
5. $\{j, f(j)\}$

▷ . . . . ◁

Chapter 2

The art of the logic of truth-functional connectives

In this chapter we study the logic of the six truth-functional connectives that we have listed in Symbolism **1B-12**. Sometimes we call this "TF-logic," and we also use "TF" in other contexts as an indicator that we are restricting attention to a small (but important) branch of logic.

2A Grammar of truth-functional logic

Recall that each branch of logic has its grammar, its semantics, its proof theory, and its applications. Almost all of our work will be on semantics and proof theory; and we will find that it is proof theory that occupies our chief attention in this chapter. Here, however, we spend a little time—as little as possible—on grammar.

2A.1 Bare bones of the grammar of truth-functional logic

To study the logic of the six truth-functional connectives, we need very little in the way of grammatical background. That is to say, the grammar can be as rich as you like, as long as you keep separate track of the special six. One thing is certain: (1) There is a set of sentences, (2) there are the six listed connectives, and (3) there is a set of TF-atoms. (Read "TF-atom" as "truth-functionally atomic sentence.") There are two ways to organize our thought using (1)–(3).

- Sometimes it is good to take the notion of (3) "TF-atom" and (2) "listed connective" as primitive, and to define (1) "sentence" in terms of these. This is often the procedure in creating a formal language to be studied.
- The second way takes the notion of (1) "sentence" as primitive (i.e., as a given) along side of (2) "listed connective," and defines (3) "TF-atom" in terms of these. This is the best way to think when taking the apparatus as a grammar of an existing language, for example English, or our language including operators, predicates, and quantifiers.

The following summarizes the relations between (1), (2), and (3) that must hold on *either* approach.

2A-1 EXPLANATION. *(Basic grammar)*

Sentences. There is a collection or set of *sentences*.[1] (Later we let Sent = the set of all sentences.) We do not define "sentence," nor shall we be entirely specific about what counts as a sentence, although we always pretend that the matter is definite enough. Imagine that at least some sentences are bits of English, and others are more symbolic. And feel free to vary exactly what counts as a sentence from context to context.

Base clause. $\perp$ and each TF-atom is a sentence.[2]

Inductive clauses. A *grammar* is imposed on these sentences by means of the six listed connectives. Since, aside from $\perp$, connectives make new sentences out of old ones, there is an *inductive clause* for each connective, as follows. Suppose that A and B are sentences taken as inputs; then the following outputs are also sentences:

$$(A \rightarrow B) \quad (A \leftrightarrow B) \quad (A \& B) \quad (A \vee B) \quad \sim A$$

Closure clause. That's all the sentences there are; nothing counts as a sentence unless it does so in virtue of one of the above clauses.

[1] More logicians say "formula" (or sometimes "well-formed formula" or "wff") than say "sentence," reserving "sentence" for those expressions that are not only sentential, but also have no free variables of quantification. Our minority usage is designed to emphasize the crucial similarities between English and an artificial language that are revealed by the considerations of logical grammar such as are sketched in §1A .

[2] If "sentence" is taken as primitive and "TF-atom" is defined, this paragraph turns out to be a provable "fact."

No ambiguity. Furthermore, we write down explicitly something that you never thought to doubt: If you are given any sentence, it is uniquely determined (1) whether or not it is a TF-atom, and if it is not a TF-atom, (2) which of the six kinds you have and (3) what the inputs were. These clauses say that the grammar of our language is unambiguous. Each sentence has a "unique decomposition" in terms of the six connectives, right down to TF-atoms; and about these we have no information whatsoever.

Not all sentences come by inductive clauses.

2A-2 DEFINITION. *(*TF-*atom)*

Any sentence that does not come by an inductive clause is a "TF-*atom*." That is, a TF-atom is any sentence that is not subject to "decomposition" by means of one of the six inductive clauses for the connectives. To repeat: A is a TF-*atom* just in case A is a sentence, but does not have any of the six forms listed as part of the inductive clause.[3]

In still other words (words that you can profitably learn by heart), A is a TF-atom iff A is a sentence, but does not have one of the six privileged forms $A_1 \rightarrow A_2$, $A_1 \vee A_2$, $A_1 \& A_2$, $A_1 \leftrightarrow A_2$, $\sim A_1$, or $\perp$.

It is good philosophy to keep in mind how negative is the idea of a TF-"atom." You should also keep in mind to what an enormous extent "atomicity" is relative to a particular grammar. It is bad philosophy to forget either of these things.

Sometimes, as in application to English, a TF-atom may be long and complicated—as long as its *outermost* complication does not arise from one of our six listed connectives. At other times, as in constructing a toy language, it is good to envision the TF-atoms as very short, perhaps single letters such as "p" or "q." In the latter case, TF-atoms are sometimes called "sentence letters" or "proposition letters." In any case, every TF-atom is itself a sentence, every sentence is either a TF-atom or comes by one of the inductive clauses, and there is no overlap between TF-atoms and sentences that arise out of one of the inductive clauses.

For example, if p is a sentence in virtue of being an TF-atom, then p&p is a sentence in virtue of one of the inductive clauses.

[3]If "TF-atom" is taken as primitive and "sentence" is defined, this paragraph turns out to be a provable "fact."

For example, suppose p is a TF-atom, and the only one. Then p is a sentence. So by the inductive clause ∼p is also a sentence. So (p & ∼p) must also be a sentence (by a second use of the inductive clause).

Also, "→(∼" is not a sentence, by the closure clause; since it is not a TF-atom (the only one was given as p), nor is it admissible by any inductive clause.

Sometimes it is convenient to introduce a name for a particular set. At this point all we have to go on is the grammar just described in Explanation **2A-1**, and the chief sets there of which at present we want names are the set of sentences and the set of TF-atoms. We use "Sent" and "TF-atom" respectively.

2A-3 DEFINITION. *(*Sent *and* TF-atom*)*

- Sent $=_{df}$ the set that contains all and only the sentences in the sense of §2A.1. Therefore, in our usage, A ∈ Sent iff A is a sentence.[4]
- TF-atom $=_{df}$ the set that contains all and only the TF-atoms in the sense of §2A.1. Therefore, in our usage, A ∈ TF-atom iff A is a TF-atom.

Recall that a TF-atom is simply any sentence that does not arise by using one of our six listed connectives.

2A.2 Practical mastery

You will have mastered the art of the grammar of truth-functional logic if:

1. You can create *bona fide* sentences. (This is part of "speaker competence.")
2. You can tell a sentence when you see one, and have some feel for why it is a sentence. (This is part of "hearer competence.")
3. You can tell when a candidate fails to be a sentence, and say something as to why it fails.

[4] "_ $=_{df}$ _" is grammatically a predicate; it has exactly the same meaning as the standard identity predicate "_ = _." We use it instead of the standard identity to signal the reader that the identity in question is intended as a *definition*. In exact analogy, "_ $\leftrightarrow_{df}$ _" is grammatically a connective with exactly the same meaning as the standard truth-functional equivalence "_↔_," except that it signals an intended definition.

4. You can figure out the "structure" of a given sentence in terms of TF-atoms and the connectives. There is a fair amount of useful grammatical jargon to this purpose, e.g., a sentence A→B is a "conditional" with A as its "antecedent" and B as its "consequent," but we suppose that you have learned it elsewhere.

5. You can manage substitution: Given a TF-atom p and a sentence (. . . p . . .) that contains some occurrence(s) of p, if you are also given a sentence A, then you can write down the result (. . . A . . .) of substituting A for every occurrence of p in (. . . p . . .). Furthermore, you can tell when a candidate is *not* the result of substituting A for every occurrence of a TF-atom p in (. . . p . . .), and say something as to why.

6. You can manage replacement: Given a sentence A and a sentence (. . . A . . .) containing a specified occurrence of A, if you are also given a sentence B, then you can write down the result (. . . B . . .) of replacing that occurrence of A by B. Furthermore, you can tell when a candidate is *not* the result of replacing some specified occurrence of A in (. . . A . . .) by B, and say something as to why.

Exercise 6 *(Grammar of truth functional logic)*

These trivial exercises are provided just to slow you down.

1. Make up a small stock of TF-atoms. (Lower case letters like p and q are common choices. You may, however, also let one or more of your TF-atoms be English.) Then make up some sentences, using each of the connectives at least once.

   Example. My TF-atoms are p and q and "London is large." One example sentence: (p → (p & London is large)),

   Keep your chosen TF-*atoms for the rest of this exercise.*

2. Now pretend you are a hearer, and exhibit part of hearer competence: Say something vague as to why one of your sentences is in fact a sentence.

   Example. p and q are sentences by the base clause. So p&q is a sentence by the inductive clause for conjunction. So ∼(p&q) is a sentence by the inductive clause for negation.

3. Make up a couple of more or less interesting examples of non-sentences, using symbols; and explain what is wrong with each. (Don't try too hard for precision.)

   Example. "&→" isn't a TF-atom, nor does it come from other sentences by any of the inductive clauses. So it isn't a sentence.

4. Skippable (because you don't really have enough to go on). Show you understand substitution by giving and explaining an example.

5. Skippable (ditto). Show you understand replacement by giving and explaining an example.

▷ . . . . ◁

2B Rough semantics of truth-functional logic

We are going to say enough about some of the key semantic concepts of truth-functional logic in order to get on in a practical way. We assume that you can do truth tables, so that here we are largely just introducing our own jargon. These concepts are absolutely essential to our further progress. Here is a check-list; do not proceed without being certain that you can explain each.

2B-1 CHECK-LIST FOR MASTERY. *(Basic semantic ideas)*

- Truth values T and F.
- Truth-value-of operator, Val(A).
- Truth-functional connective.
- Truth (two concepts of).
- Interpretation (generic account).
- TF (truth-value) interpretation and usage of $\mathbf{i}$.
- Relativized truth-value-of operator, $Val_{\mathbf{i}}$(A).

2B.1 Basic semantic ideas

Truth values. There are two truth values, Truth or T, and Falsity or F. Henceforth "T" and "F" are parts of our technical vocabulary.[5] We say very little about T and F, except that we can be sure that T$\neq$F. Observe that {T, F}, which has two members, counts as a name of the set of truth values.

Truth-value-of operator. We think of each sentence as *having* a truth value according as it is true or false. This leads us to introduce a truth-value-of operator, namely, Val(_): When A is a sentence, Val(A) is the truth value of A. Naturally Val(A)=T iff A is true, and Val(A)=F iff A is not true. Our usage makes "Val(_)" a "sentence-to-truth-value" operator: Put the name of a sentence in the blank, and you have a name of a truth value. You can even say this symbolically: A$\in$Sent $\rightarrow$ Val(A)$\in$ {T, F}.

Truth-functional connective. The simplest part of the art of logic involves truth-functional connectives only. What are they? We first give a metaphorical account and then an "official" definition; it is part of the art of logic to begin to appreciate the difference so as to be able to value both without confusing them.

2B-2 METAPHORICAL DEFINITION. *(Truth-functional connective)*

A connective is truth functional if for *every* pattern of "inputs" (we use this term for the sentences filling the blanks), the truth value of the "output" (we use this term for the result of filling in the blanks with sentences) is *entirely determined by* the truth value of the inputs.

For example, the connective _&_ is truth-functional because Val(A&B) "depends entirely on" (note the metaphor) Val(A) and Val(B).

In non-metaphorical language, we can give a definition based on our "input-output" analysis of the nature of connectives:

2B-3 DEFINITION. *(Truth-functional connective)*

A connective is truth functional $\leftrightarrow_{df}$ it is *not* true that there are two patterns of inputs that are *alike* in truth values, while the respective outputs are *different* in truth value.

[5]We use each of "T" and "F" in more than one way; let us know if this turns out to confuse you (we don't think that it will).

For example, to say that $\sim\_$ is truth-functional is to say that there is no pair of sentences A and B such that Val(A) = Val(B), while $Val(\sim$A$)\neq Val(\sim$B).

This second form, Definition **2B-3**, is most useful when establishing *non*-truth-functionality. That is, by the second form of the definition, we can see that to show that a connective is *not* truth functional, it is required that we produce two patterns of inputs that are *alike* in truth values while the respective outputs are *different* in truth values.

2B-4 EXAMPLE. *(Truth functionality)*

Consider the connective,

In 1066, William the Conqueror knew that _.

To show that this connective is not truth functional, we must produce two inputs with like truth values that nevertheless give different results when put in the blank.

Try 1. First input = "$2+2=5$," second input = "England is west across the Channel from Normandy." This will not work, because the input sentences have different truth values: Val("$2+2=5$")$\neq Val$("England is just west across the Channel from Normandy").math envir.

Try 2. First input = "$2+2=5$," second input = "England is east of Normandy." This is O.K. on the side of the inputs: They do have like truth values (both are false): Val("$2+2=5$")$=Val$("England is to the east of Normandy")$=$F. But it still won't work to show that our connective is non-truth-functional, since the two outputs are, alas, also the same in truth value: Both "In 1066, William the Conqueror knew that $2+2=5$" and "In 1066, William the Conqueror knew that England is to the east of Normandy" are false: Val("In 1066, William the Conqueror knew that $2+2=5$")$=Val$(In 1066, William the Conqueror knew that England is to the east of Normandy") = F. Indeed, since no one can know (as opposed to merely believe) falsehoods, not even in 1066, *no* pair of false inputs will work![6]

[6]That is, if two inputs of this connective are alike in being false, then regardless of how they may differ, the outputs will also be alike in being false. Should we then call our connective "semi-truth-functional" since at least sometimes knowing the truth value of the input is enough for knowing the truth value of the output? No, we shouldn't—it's too much bother.

Try 3. First input = "2+2=4," second input = "Channel waters are two parts hydrogen and one part oxygen (H_2O)." These inputs are indeed alike in truth value: Val("2+2=4") = Val("Channel waters are two parts hydrogen and one part oxygen (H_2O)") = T. Furthermore, the respective results, "In 1066, William the Conqueror knew that 2+2=4" and "In 1066, William the Conqueror knew that Channel waters are two parts hydrogen and one part oxygen" are *different* in truth value: Val("In 1066, William the Conqueror knew that 2+2=4") $\neq$ Val("In 1066, William the Conqueror knew that Channel waters are two parts hydrogen and one part oxygen"). "In 1066, William the Conqueror knew that 2+2=4" is (doubtless) true, whereas "In 1066, William the Conqueror knew that Channel waters are two parts hydrogen and one part oxygen" is (doubtless) false. *So on Try 3, we have succeeded in showing that our connective is not truth functional*, according to our non-metaphorical Definition **2B-3** of "truth-functional connective" based on "input-output analysis."

Exercise 7 *(Semantics of truth functional logic)*

1. Show that the connective "most folks today believe that _" is not truth functional. Be sure that your demonstration relies explicitly on Definition **2B-3** of "truth functional connective" based on "input-output" analysis.

2. Now show that "most folks today believe that it's false that _" is also not truth functional.

▷ . . . . ◁

In contrast to many English connectives, each of the six connectives introduced in §2A.1 is truth-functional *by our decision or convention*: In each case, the truth value of the output depends entirely on the truth values of the inputs *because we say so*. How this dependence works is conveniently expressed in the following table.

2B-5 CONVENTION. *(Truth tables)*

We agree to use the following six connectives as truth functional in accord with the following table.

A	B	A&B	A∨B	A→B	A↔B	∼A	⊥
T	T	T	T	T	T	F	F
T	F	F	T	F	F	F	F
F	T	F	T	T	F	T	F
F	F	F	F	T	T	T	F

Exercise 8 *(A truth-value computation)*

1. Let A and B and C be sentences. Suppose Val(A) = T and Val(B) = T and Val(C) = F. Compute Val((A&∼B)∨(∼A→C))

2. That's enough of that.

▷ . . . . ◁

So some connectives of English are not truth functional, and all of our symbolic connectives are truth functional. Two questions remain. (1) Are such English connectives as "if_then_," "_and_," and "_or_" truth functional? (2) Are there any symbolic connectives that are not truth functional? These questions are left for your deliberation. (Hint: (1) No; but philosophers differ in their opinions on this matter. (2) Yes. Everyone agrees.)

Truth (two concepts of) There are two concepts of truth of use in semantics. The first is the ordinary or garden-variety or "absolute" or "non-relativized version," according to which it is meaningful to say directly of a sentence that it is, or isn't, true. Let the sentence be "England is just west across the channel from Normandy." That sentence is just plain outright true, without the need to "relativize" its truth to anything.[7] In our jargon (see Explanation **2A-1**), we may say that Val("England is just west across the channel from Normandy") = T.

Of use in semantics is also a *relativized* notion of truth. What we relativize truth to, in semantics, is: an "interpretation."

[7]Sort of. The statement in the text represents an idealization of the inescapably sloppy facts of English. One of the lessons of logic is that without idealization there is no theory of anything.

Interpretation We do not give a sharp definition of the general notion of interpretation.

2B-6 ROUGH IDEA. *(Interpretation)*

The idea of an interpretation is that it assigns some kind of "meaning" (including a truth value as a degenerate sort of meaning) to each *non*-logical part of each sentence. So *an interpretation always gives enough information to determine the truth value of each sentence*; and this is its essential nature.

In our usage, we will always let an interpretation be a function **i** of some sort, reading "**i**(__)" as "the interpretation of __."

Although we give only a rough idea of the general notion of interpretation, we *do* give a sharp definition of a "truth-value interpretation" or "truth-functional interpretation" or, as we will say, "TF interpretation." Note how the *semantic* idea of a "TF interpretation" depends on the *grammatical* idea of a "TF-atom"; that's critical.

2B-7 DEFINITION. *(*TF *interpretations)*

- **i** is a TF *interpretation* (or *truth-value interpretation)* $\leftrightarrow_{df}$ **i** is a function from the set of TF-atoms into the set of truth values such that for every TF-atom A, **i**(A) $\in$ {T, F}. In symbols: $\forall A[A \in$ TF-atom $\rightarrow$ $\mathbf{i}(A) \in$ {T, F}].
- Thus, $\forall$**i**[**i** is a TF interpretation $\rightarrow \forall A[A \in$ TF-atom $\rightarrow$ $\mathbf{i}(A) \in$ {T, F}]].

2B-8 CONVENTION. *(***i** *for* TF *interpretations)*

We use **i** as a variable or parameter ranging over TF interpretations.

So "**i**(__)" is a "TF-atom-to-truth-value" operator.[8] A TF interpretation is therefore an assignment of a truth value to each sentence not having one of the six privileged forms: A&B, A$\vee$B, A$\rightarrow$B, A$\leftrightarrow$B, $\sim$A, $\perp$. In our grammatical jargon, a

[8]But **i** is an *individual* parameter or variable, not an *operator* parameter or variable, since by Symbolism **1B-5** the latter is syncategorematic and so can be used only to make up an operator. Were **i** *only* an operator parameter or variable, we could not extract it for consideration apart from its operator, using it categorematically as we do when we say to which sets **i** belongs, or as we do when we use it with a quantifier.

sentence not having one of these forms is a "TF-atom." So, in other words, *a* TF *interpretation assigns a truth value to each* TF-*atom*. You can then see that a TF interpretation satisfies the generic requirement on an interpretation: A TF interpretation gives us enough information to compute the truth value of *every* sentence. Just use the usual truth-tables.

The "geometrical correlate" of a (single) TF interpretation $\mathbf{i}$ is a (single) row of a truth-table. For that is exactly what such a row does—it assigns a truth value to each "atomic" sentence (in the so-called "reference columns" on the left), as in the following example.

2B-9 EXAMPLE. *(A truth table)*

TF-interp:	p	q	(p&q)∨∼p
$\mathbf{i}_1$:	T	T	T
$\mathbf{i}_2$:	T	F	F
$\mathbf{i}_3$:	F	T	T
$\mathbf{i}_4$:	F	F	T

Keep this picture firmly in mind. As you know, the fundamental truth-concept for truth-tables relativizes truth to row: Given a sentence such as (p&q)∨∼p, you mark it as T or F *in a particular row* such as that marked $\mathbf{i}_2$. Relativized "truth-value-of" works the same way: The truth value of A is relativized to (depends on) a TF interpretation.

2B-10 NOTATION. *($Val_{\mathbf{i}}$(A))*

- "$Val_{\mathbf{i}}$(A)" is to be read "the truth value of A on $\mathbf{i}$."
- "$Val_{\mathbf{i}}$(A)=T" may be read "A is true on $\mathbf{i}$," which is short for "the truth value of A on $\mathbf{i}$ is T."
- "$Val_{\mathbf{i}}$(A)=F" may be read "A is false (or not true) on $\mathbf{i}$," which is short for "the truth value of A on $\mathbf{i}$ is F."

This interpretation-dependent idea is fundamental to truth-functional semantics. Theoretically speaking, it is a sharply defined idea, and here we pedantically record the essentials of its definition.

2B-11 DEFINITION. *($Val_{\mathbf{i}}$(A))*

- $Val_{\mathbf{i}}$(A) = F iff $Val_{\mathbf{i}}$(A)$\neq$T (bivalence).
- If A is a TF-atom, then $Val_{\mathbf{i}}$(A) = T iff $\mathbf{i}$(A) = T.
- $Val_{\mathbf{i}}$(A&B) = T iff $Val_{\mathbf{i}}$(A) = T and $Val_{\mathbf{i}}$(B) = T.
- $Val_{\mathbf{i}}$(A$\vee$B) = T iff $Val_{\mathbf{i}}$(A) = T or $Val_{\mathbf{i}}$(B) = T.
- $Val_{\mathbf{i}}$(A$\rightarrow$B) = T iff $Val_{\mathbf{i}}$(A)$\neq$T or $Val_{\mathbf{i}}$(B) = T.
- $Val_{\mathbf{i}}$(A$\leftrightarrow$B) = T iff $Val_{\mathbf{i}}$(A) = $Val_{\mathbf{i}}$(B).
- $Val_{\mathbf{i}}$($\sim$A) = T iff $Val_{\mathbf{i}}$(A)$\neq$T.
- $Val_{\mathbf{i}}$($\perp$)$\neq$T.

You really do not need to study or use this definition in the immediate future, although later we feel free to refer back to it when wanted. If you keep in mind that a TF interpretation $\mathbf{i}$ is like a row of a truth-table, you cannot go wrong.

Observe that now it makes no sense to ask for the truth value of, say, "p&q" outright; we can only ask for its truth value relative to some TF interpretation—relative to some row of the truth-table. We care about whether $Val_{\mathbf{i}}$(p&q)=T or not, where the role of $\mathbf{i}$ is essential. The following example works out how you can begin with a truth-table picture and make out of it a series of statements involving $\mathbf{i}$(p) and $Val_{\mathbf{i}}(A)$.

2B-12 EXAMPLE. *(Use of $\mathbf{i}$(p) and $Val_{\mathbf{i}}(A)$)*

As we said, each TF interpretation corresponds to a row in a truth table. Let us indicate this by borrowing the labeling of the rows of the truth table displayed in Example **2B-9**; and at the same time, we will add further sentences to the right of the vertical line.

TF-interp.:	p	q	p	q	p&q	$\sim$p	(p&q)$\vee\sim$p
$\mathbf{i}_1$:	T	T	T	T	T	F	T
$\mathbf{i}_2$:	T	F	T	F	F	F	F
$\mathbf{i}_3$:	F	T	F	T	F	T	T
$\mathbf{i}_4$:	F	F	F	F	F	T	T

The following statements convey in symbols the same information that is represented in the first and third lines:

- $\mathbf{i}_1$(p) = T, $\mathbf{i}_1$(q) = T, $Val_{\mathbf{i}_1}$(p) = T, $Val_{\mathbf{i}_1}$(q) = T, $Val_{\mathbf{i}_1}$(p&q) = T, $Val_{\mathbf{i}_1}$(∼p) = F, $Val_{\mathbf{i}_1}$((p&q)∨∼p) = T.
- $\mathbf{i}_3$(p) = F, $\mathbf{i}_3$(q) = T, $Val_{\mathbf{i}_3}$(p) = F, $Val_{\mathbf{i}_3}$(q) = T, $Val_{\mathbf{i}_3}$(p&q) = F, $Val_{\mathbf{i}_3}$(∼p) = T, $Val_{\mathbf{i}_3}$((p&q)∨∼p) = T.

In the picture, the TF-atoms all occur on the *left* of the double vertical. The various interpretations $\mathbf{i}_n$ tell you which truth value has been assigned to each such TF-atom. The picture has all the relevant sentences (whether simple or complex) written to the *right* of the double vertical; and for each sentence A, $Val_{\mathbf{i}_n}(A)$ tells you the truth value that A has based on the truth values assigned by $\mathbf{i}_n$ to each of the TF-atoms that A contains.

Exercise 9 *(Truth tables)*

1. Do for the second and fourth rows ($\mathbf{i}_2$ and $\mathbf{i}_4$) of the truth table displayed in Example **2B-12** what is done, just below that truth table, for the first and third rows.
2. Let p and q be all the TF-atoms. (In symbols from Definition **2A-3** and Symbolism **1B-10**, TF-atom = {p, q}.) Now consider the following sentences:

   ∼(p ∨q), ((p &q) →p) ∨∼q, ⊥→∼(p ↔p)

   Make a "truth table" that shows the truth values of these sentences on each TF interpretation (that is, assignment of truth values to p and q). (If you do not know how to do this, consult almost any elementary logic text.)
3. Your truth table should have four rows. Thinking of each row as a TF interpretation, give each a name, say $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, and $\mathbf{i}_4$. Now write down half a dozen statements each having the form

   $Val\_(\_)=\_$,

   where, to make sense, the subscript on Val should be the name of one of your rows (i.e., one of $\mathbf{i}_1$, etc.); the second blank should be filled with the name of one of three sentences mentioned above, and the third blank should be filled with the name of a truth value.

▷ . . . . ◁

2B.2 Defined semantic concepts: tautology, etc.

In terms of the (two-place) operator $Val\_(\_)$ that carries the idea of truth relativized to TF interpretations, we can define several crucial semantic concepts.

2B-13 CHECK-LIST FOR MASTERY. *(Defined semantic concepts)*

- Tautological implication; $A_1, \ldots, A_n \vDash_{TF} B$.
- Tautological equivalence; $A \approx_{TF} B$.
- Tautology; $\vDash_{TF} A$.
- Truth-functional falsifiability; $\nvDash_{TF} A$
- Truth-functional inconsistency; $A_1, \ldots, A_n \vDash_{TF}$.
- Truth-functional consistency; $A_1, \ldots, A_n \nvDash_{TF}$.

We will give each in pretty-much-English, in middle English, and then symbolically. These will be easier to read if we use our conventions governing the use of certain letters: A, B, and C are sentences, and G is a set of sentences, by Convention **1B-15**; **i** is a TF interpretation by Convention **2B-8**; and by Convention **1B-20**, any variables not already bound to explicit quantifiers are to be interpreted as bound on the very-most outside of the entire display. Here, then, are the definitions, accompanied by important examples.

Tautological implication. Although it is not the simplest, far and away the most important concept for logic is tautological implication. Be certain to learn the definition of this concept.

2B-14 DEFINITION. *(Tautological implication and* $G \vDash_{TF} A$*)*

We supply alternative ways of saying exactly the same thing; any one can count as a definition of "tautological implication."

1. Where G is a set of sentences and A is a sentence, G *tautologically implies* A $\leftrightarrow_{df}$ on every TF interpretation in which every member of G is true, so is A.

2. Abbreviative definition: $G \vDash_{TF} A \leftrightarrow_{df}$ G tautologically implies A.
3. In other words, $G \vDash_{TF} A \leftrightarrow_{df}$ for every TF interpretation **i** [if (for every sentence B, if $B \in G$ then $Val_{\mathbf{i}}(B) = T$) then $Val_{\mathbf{i}}(A) = T$].
4. That is to say, $G \vDash_{TF} A \leftrightarrow_{df} \forall \mathbf{i}[\forall B[B \in G \rightarrow Val_{\mathbf{i}}(B) = T] \rightarrow Val_{\mathbf{i}}(A) = T]$.
5. Therefore, $G \vDash_{TF} A$ iff there is no TF interpretation in which all of the members of G are true and A is not.
6. That is, $G \vDash_{TF} A$ iff it is false that there is a TF interpretation **i** such that (both for all B [if $B \in G$ then $Val_{\mathbf{i}}(B) = T$] and $Val_{\mathbf{i}}(A) \neq T$).

Pay particular attention to the structure of the right side of the definition, noting the scope of each quantifier.

Because for instance $\{A, A \rightarrow B\}$ is a set (Symbolism **1B-10**), the grammar of tautological implication and the double turnstile allows us to write the truth that

$$\{A, A \rightarrow B\} \vDash_{TF} B$$

This says that no matter the interpretation **i**, if every member of the set $\{A, A \rightarrow B\}$ is true on **i**, then so is B. The curly braces make this a little hard to read. It is therefore usual to drop the curly braces on the left of the double turnstile, writing simply

$$A, A \rightarrow B \vDash_{TF} B.$$

We promote this into a full-fledged convention.

2B-15 CONVENTION. *(Dropping curly braces)*

Instead of

$$\{A_1, \ldots, A_n\} \vDash_{TF} B,$$

one may drop the curly braces, writing instead simply

$$A_1, \ldots, A_n \vDash_{TF} B.$$

In the same spirit we write the English "$A_1, \ldots, A_n$ tautologically implies B," omitting the curly braces.

This convention is used heavily from now on.

The following is a list of useful tautological implications. *They have no special logical status* aside from historically proven usefulness. More or less standard names are indicated, and should be memorized.[9] In the names with "int" and "elim," int = introduction and elim = elimination, for historical reasons explained later.

2B-16 EXAMPLES. *(Tautological implication and* $G \vDash_{TF} A$*)*

A, B	$\vDash_{TF} A \& B$	Conjunction, Conj., &int
$A \& B$	$\vDash_{TF} A$	Simplification, Simp., &elim
$A \& B$	$\vDash_{TF} B$	Simplification, Simp., &elim
A	$\vDash_{TF} A \vee B$	Addition, Add., $\vee$int
B	$\vDash_{TF} A \vee B$	Addition, Add., $\vee$int
$A \vee B, \sim A$	$\vDash_{TF} B$	Disjunctive Syllogism, DS
$A \vee B, \sim B$	$\vDash_{TF} A$	Disjunctive Syllogism, DS
$\sim A \vee B, A$	$\vDash_{TF} B$	Disjunctive Syllogism, DS
$A \vee \sim B, B$	$\vDash_{TF} A$	Disjunctive Syllogism, DS
$A \vee B, A \rightarrow C, B \rightarrow C$	$\vDash_{TF} C$	Constructive Dilemma, Dil.
$A \rightarrow B, A$	$\vDash_{TF} B$	Modus Ponens, MP, $\rightarrow$elim
$A \rightarrow B, \sim B$	$\vDash_{TF} \sim A$	Modus Tollens, MT
$A \rightarrow \sim B, B$	$\vDash_{TF} \sim A$	Modus Tollens, MT
$\sim A \rightarrow B, \sim B$	$\vDash_{TF} A$	Modus Tollens, MT
$\sim A \rightarrow \sim B, B$	$\vDash_{TF} A$	Modus Tollens, MT
$A \rightarrow B, B \rightarrow C$	$\vDash_{TF} A \rightarrow C$	Hypothetical Syllogism, HS
$A \leftrightarrow B, B \leftrightarrow C$	$\vDash_{TF} A \leftrightarrow C$	Biconditional syllogism, BCS
$A \leftrightarrow B, A$	$\vDash_{TF} B$	Modus ponens for the biconditional, MPBC, $\leftrightarrow$elim
$A \leftrightarrow B, B$	$\vDash_{TF} A$	Modus ponens for the biconditional, MPBC, $\leftrightarrow$elim
$A \leftrightarrow B, \sim A$	$\vDash_{TF} \sim B$	Modus tollens for the biconditional, MTBC
$A \leftrightarrow B, \sim B$	$\vDash_{TF} \sim A$	Modus tollens for the biconditional, MTBC
$A, \sim A$	$\vDash_{TF} B$	*ex absurdum quodlibet*, XAQ, $\sim$elim
$A, \sim A$	$\vDash_{TF} \bot$	$\bot$ int
$\bot$	$\vDash_{TF} A$	$\bot$ elim

[9] The reservation "more or less standard" signals that in some cases, for convenience, we have extended the meaning of a standard name by allowing it to apply to cases that are like the standard cases except for a "built in" double negation. For example, "disjunctive syllogism" standardly applies only to the first two of the four cases we list.

Observe, then, that you have two jobs to carry out in connection with tautological implication: (1) You must learn what it means (you must memorize Definition **2B-14**), and (2) you must learn the listed Examples **2B-16**.

Tautological equivalence. The next important concept is tautological equivalence.

2B-17 DEFINITION. *(Tautological equivalence and $\approx_{\text{TF}}$)*

Again we offer the definition in several equivalent forms.

1. Two sentences A and B are *tautologically equivalent* $\leftrightarrow_{df}$ A and B have exactly the same truth values on *every* TF interpretation.
2. Abbreviative definition:

   $$\text{A} \approx_{\text{TF}} \text{B} \leftrightarrow_{df} \text{A and B are tautologically equivalent.}$$

3. That is, $\text{A} \approx_{\text{TF}} \text{B} \leftrightarrow_{df}$ for every TF interpretation $\mathbf{i}$, $Val_{\mathbf{i}}(\text{A}) = Val_{\mathbf{i}}(\text{B})$.
4. In other words, $A \approx_{\text{TF}} \text{B} \leftrightarrow_{df} \forall \mathbf{i}[Val_{\mathbf{i}}(A) = Val_{\mathbf{i}}(\text{B})]$.

The following is a list of useful tautological equivalences; standard names are indicated, and *should be memorized.*

2B-18 EXAMPLES. *(Tautological equivalence)*

A	$\approx_{TF}$	$A \& A$	Duplication, Dup.
A	$\approx_{TF}$	$A \vee A$	Duplication, Dup.
$A \& B$	$\approx_{TF}$	$B \& A$	Commutation, Comm.
$A \vee B$	$\approx_{TF}$	$B \vee A$	Commutation, Comm.
$A \leftrightarrow B$	$\approx_{TF}$	$B \leftrightarrow A$	Commutation, Comm.
$A \& (B \& C)$	$\approx_{TF}$	$(A \& B) \& C$	Association, Assoc.
$A \vee (B \vee C)$	$\approx_{TF}$	$(A \vee B) \vee C$	Association, Assoc.
$A \& (B \vee C)$	$\approx_{TF}$	$(A \& B) \vee (A \& C)$	Distribution, Dist.
$A \vee (B \& C)$	$\approx_{TF}$	$(A \vee B) \& (A \vee C)$	Distribution, Dist.
$A \to (B \to C)$	$\approx_{TF}$	$(A \& B) \to C$	Exportation, Exp.
$A \leftrightarrow B$	$\approx_{TF}$	$(A \to B) \& (B \to A)$	Biconditional Exchange, BE
$A \to B$	$\approx_{TF}$	$\sim B \to \sim A$	Contraposition, Contrap.
$A \to B$	$\approx_{TF}$	$\sim A \vee B$	Conditional Exchange, CE
$\sim A \to B$	$\approx_{TF}$	$A \vee B$	Conditional Exchange, CE
A	$\approx_{TF}$	$A \& (A \vee B)$	Absorption
A	$\approx_{TF}$	$A \vee (A \& B)$	Absorption

The entries for Absorption are useless in proofs, but have the following philosophical point: It is difficult or impossible to base a theory of what a sentence is "about" by considering any feature common to all its tautological equivalents.

Here is an additional list of tautological equivalents that are also especially important. They all involve negated compounds.

2B-19 EXAMPLES. *(Removing outermost negations)*

$\sim(A \& B)$	$\approx_{TF}$	$\sim A \vee \sim B$	De Morgan's, DeM, $\sim \&$
$\sim(A \vee B)$	$\approx_{TF}$	$\sim A \& \sim B$	De Morgan's, DeM, $\sim \vee$
$\sim(A \to B)$	$\approx_{TF}$	$A \& \sim B$	Negated Conditional, NCond, $\sim \to$
$\sim(A \leftrightarrow B)$	$\approx_{TF}$	$\sim A \leftrightarrow B$	Negated Biconditional, NBC, $\sim \leftrightarrow$
$\sim(A \leftrightarrow B)$	$\approx_{TF}$	$A \leftrightarrow \sim B$	Negated Biconditional, NBC, $\sim \leftrightarrow$
$\sim\sim A$	$\approx_{TF}$	A	Double Negation, DN, $\sim\sim$

As the title indicates, it is good to think of these equivalences as driving negations from the outside to the inside, insofar as that is possible.

Tautology and truth-functional falsifiability. Much simpler but also of much less use is the notion of a tautology.

2B-20 DEFINITION. *(Tautology and* $\vDash_{\text{TF}}$ A*)*

As before, we say the same thing in different ways.

1. A is a *tautology* $\leftrightarrow_{df}$ A is a sentence that is true on *every* TF interpretation.
2. Abbreviative definition: $\vDash_{\text{TF}}$ A $\leftrightarrow_{df}$ A is a tautology.
3. That is, $\vDash_{\text{TF}}$ A $\leftrightarrow_{df}$ for every **i**, $Val_{\mathbf{i}}(\text{A})=\text{T}$.
4. That is, $\vDash_{\text{TF}}$ A $\leftrightarrow_{df}$ $\forall\mathbf{i}[Val_{\mathbf{i}}(\text{A})=\text{T}]$.

With regard to each of the above displays, it is none too early for you to begin to observe the following: Each occurrence of "A" is implicitly bound to an outermost universal quantifier. That is, each of the displays should be interpreted as beginning with "For any sentence A." In contrast, the quantifier "for every **i**" or "$\forall$**i**" is not and cannot be omitted since it is not outermost; it *must* be made explicit. That is to say, it must be made explicit that to be a tautology is to be true on *every* TF interpretation; the "every" cannot be dropped.

Definition **2B-20** of "tautology" is important for our work—although not as important as tautological implication and tautological equivalence—and should be *both* understood and *memorized*. The following is a list of tautologies, of which only the first is of much use; standard names are indicated, *and should be memorized*.

2B-21 EXAMPLES. *(Tautology)*

$\vDash_{\text{TF}}$ A$\vee\sim$A	Excluded middle
$\vDash_{\text{TF}}$ $\sim$(A$\,\&\sim$A)	Noncontradiction
$\vDash_{\text{TF}}$ A$\rightarrow$A	Identity
$\vDash_{\text{TF}}$ A$\leftrightarrow$A	Identity
$\vDash_{\text{TF}}$ $\sim\perp$	Negated $\perp$, Truth, $\sim\perp$ int

Any sentence that is not a tautology is truth-functionally falsifiable.

2B-22 DEFINITION. *(*TF*-falsifiability)*

A is TF-falsifiable (or truth-functionally falsifiable) $\leftrightarrow_{df}$ A is not a tautology. In symbols:

A is TF-falsifiable $\leftrightarrow_{df}$ $\nvDash_{\text{TF}}$ A.

Here '$\nvDash_{\text{TF}}$ A' is intended as merely the negation of $\vDash_{\text{TF}}$ A.

Truth-functional inconsistency and consistency. The following also plays a useful role in logic.

2B-23 DEFINITION. *(Truth-functional inconsistency and* $G \vDash_{TF}$*)*

All of the following say the same thing.

1. For any sentence A, A is *truth-functionally inconsistent* $\leftrightarrow_{df}$ A is false on *every* TF interpretation.
2. Abbreviative definition: $A \vDash_{TF}$ $\leftrightarrow_{df}$ A is truth-functionally inconsistent.
3. That is, A is truth-functionally inconsistent $\leftrightarrow_{df}$ for every TF interpretation $\mathbf{i}$, $Val_{\mathbf{i}}(A) \neq T$.
4. In other words, A is truth-functionally inconsistent $\leftrightarrow_{df}$ $\forall \mathbf{i}[Val_{\mathbf{i}}(A) \neq T]$.
5. Furthermore, for any *set* G of sentences, G is *truth-functionally inconsistent* $\leftrightarrow_{df}$ on every TF interpretation, at least one member of the set G is false. That is, there is no TF interpretation on which each member of the set G is true.
6. Abbreviative definition: $G \vDash_{TF}$ $\leftrightarrow_{df}$ G is truth-functionally inconsistent.
7. In other words, $G \vDash_{TF}$ $\leftrightarrow_{df}$ for every TF interpretation $\mathbf{i}$ [there is a sentence A such that [$A \in G$ and $Val_{\mathbf{i}}(A) \neq T$]].
8. That is, $G \vDash_{TF}$ $\leftrightarrow_{df}$ $\forall \mathbf{i} \exists A[A \in G \,\&\, Val_{\mathbf{i}}(A) \neq T]$.

2B-24 EXAMPLES. *(Truth-functional inconsistency and* $G \vDash_{TF}$*)*

$A \,\&\, {\sim}A \vDash_{TF}$	Explicit contradiction
$A, {\sim}A \vDash_{TF}$	Explicit contradiction
$\perp \vDash_{TF}$	Absurdity

Go back and look at Definition **2B-14**(5) of tautological implication. Notice that if in fact there is no interpretation in which all the members of G are true, then of course there is no interpretation in which all the members of G are true and B is not. Therefore, if the set G is truth-functionally inconsistent, then it will tautologically imply each and every arbitrary B. By definition. That is:

For every G and A, if $G \vDash_{TF}$ then $G \vDash_{TF} A$.

When G is truth-functionally inconsistent, one has G $\vDash_{\text{TF}}$ A quite regardless of the nature of A. There is a little philosophy here. Most logicians associate "tautological implication" with "good argument" or "valid argument"—i.e., with the arguments we *ought* to use. They do this because tautological implication satisfies the logician's promise: Never to lead you from truth to error (safety in all things). Other logicians complain that some tautological implications are not good arguments because of an absence of *relevance* between premisses and conclusion. These complaints are more than elementary and we cannot pursue them.[10] But you should understand that what leaves room for the complaints is the nontrivial step to be taken from the *descriptive* concept of "tautological implication" to the *normative* concept of a good or valid argument.

There is one more concept to be defined: truth-functional consistency.

2B-25 DEFINITION. *(Truth-functional consistency and $\nvDash_{\text{TF}}$)*

The following are intended as equivalent ways of defining TF-consistency of a single sentence A and of a set of sentences G.

1. For any sentence A, A is TF-*consistent* or *truth-functionally consistent* $\leftrightarrow_{df}$ A is not TF-inconsistent.

   In symbols:

   A is TF-consistent $\leftrightarrow_{df}$ A $\nvDash_{\text{TF}}$.

2. Therefore, A $\nvDash_{\text{TF}}$ iff there is a TF interpretation $\mathbf{i}$ such that $Val_{\mathbf{i}}(\text{A}) = \text{T}$.

3. In other words, A $\nvDash_{\text{TF}}$ iff $\exists \mathbf{i}[Val_{\mathbf{i}}(\text{A}) = \text{T}]$.

4. Furthermore, if G is a *set* of sentences, then the set G is *truth-functionally consistent* $\leftrightarrow_{df}$ G $\nvDash_{\text{TF}}$.

5. Accordingly, G is TF-consistent iff there is at least one TF interpretation on which every sentence in the set G is true. That is, there is some TF interpretation on which every member of G is true.

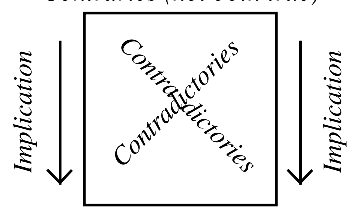
6. Therefore, G is TF-consistent iff there is a TF interpretation $\mathbf{i}$ such that for every sentence A, if A $\in$ G then $Val_{\mathbf{i}}(\text{A}) = \text{T}$.

[10] If you are interested, see *Entailment: the logic of relevance and necessity*, with volume 1 by A. R. Anderson and N. Belnap (1976) and volume 2 by A. R. Anderson, N. Belnap, and J. M. Dunn (1992), Princeton University Press.

Tautology: True on *every* TF-interpretation

TF-inconsistent: False on *every* TF-interpretation

Contraries (not both true)



Subcontraries (not both false)

TF-consistent: True on *some* TF-interpretation

TF-falsifiable: False on *some* TF-interpretation

Figure 2.1: Semantic square of opposition for TF concepts

7. That is, $G \nvDash_{TF}$ iff $\exists \mathbf{i} \forall A[A \in G \rightarrow Val_{\mathbf{i}}(A) = T]$.

Observe that from (5) on, there are *two* quantifiers in the *definiens* (right side). You need eventually to get them straight: To say that a set G is consistent is to say that there is a single TF interpretation on which they are all true together. It does not say merely that for each member of G there is a (different) TF interpretation on which it is individually true.[11] For example, $\{p, \sim p\}$ is *not* consistent even though there is (at least) one TF interpretation that makes p true and a different TF interpretation making $\sim$p true. Let us add that we may use "$G \nvDash_{TF} A$" for "G does not tautologically imply A" and "$A \not\approx_{TF} B$" for "A and B are not tautologically equivalent."

It helps to understand the foursome of tautology, inconsistency, consistency, and falsifiability by means of what the medieval logicians called a "square of opposition" as illustrated in Figure 2.1.

We will be *using* the double turnstile and the double squiggle almost immediately, and you need to be able to *read* these notations. The next exercise asks you to write them a little, but apart from that, you will be asked *only* to read these notations. You will not be asked to work with $\vDash_{TF}$ and $\approx_{TF}$ until §7A.

[11]With embarrassment we report that even after working entirely through these notes, some students confuse these two ideas. Try not to be one of them.

2B-26 REVIEW. *("Positive"* TF *semantic concepts)*

You can use the following list to check your understanding. You should know the definition of each of the listed examples.

1. Tautological implication: expressed with "$A_1, \ldots A_n \vDash_{TF} B$." See p. 38.
2. Tautological equivalence: expressed with "$A \approx_{TF} B$." See p. 41.
3. Tautology: expressed with "$\vDash_{TF} A$." See p. 43.
4. Truth-functionally inconsistent sentence: expressed with "$A \vDash_{TF}$." See p. 44.
5. Truth-functionally inconsistent set: expressed with "$A_1, \ldots A_n \vDash_{TF}$." See p. 44.

Exercise 10 *(Tautological implication, etc.)*

These exercises involve *important* memory-work. It will be easy if you have already mastered an elementary logic text such as Klenk (2002), as mentioned in the Preface. If you have *not* already mastered such a text, now is the time to do so: Go back to such a text and do all the exercises! Otherwise, this book will be a waste of your time.

1. Pick one each of the listed tautologies (Examples **2B-21**), tautological implications (Examples **2B-16**), truth-functional inconsistencies (Examples **2B-24**), and tautological equivalences (Examples **2B-18**). For each one you pick, show by a truth table that it is what is claimed.
2. For each of the "positive" semantic TF properties or relations listed in Review **2B-26**, (1) give one example that is not on any of the lists, and also (2) give one counterexample (i.e., something that does *not* have the property, or in the case of a relation, a pair of items that do *not* stand in the relation). Use sentences built from p and q.
3. Write down all the tautologies and all the tautological implications and all the truth-functional inconsistencies and all the tautological equivalences from the four lists Examples **2B-21**, Examples **2B-16**, Examples **2B-24**, and Examples **2B-18**. Then supply their names without peeking. Check this yourself. Do not hand in your work; however, if you are not prepared to certify to your instructor that you have completed this memory task, please abandon trying to learn the material of these notes.

4. Write down all the names. Then supply the principles without peeking. Check this yourself. Do not hand in your work; however, if you are not prepared to certify to your instructor that you have completed this memory task, please abandon trying to learn the material of these notes.

5. Optional (skippable). We have used the double turnstile in three ways: with something only on its right (expressing tautologyhood), with something only on its left (expressing truth-functional inconsistency), and with something on both sides (expressing tautological implication). Furthermore, what is on the left could be either a sentence or a set of sentences. Can you think of any principles that relate any of these usages one to the other? Example: For every sentence A, $\vDash_{TF} A$ iff $\sim A \vDash_{TF}$.

▷ . . . . ◁

2C Proof theory for truth-functional logic

We outline a proof system for truth-functional logic that combines ideas from the "method of subordinate proofs" of Fitch (1952) with ideas from Suppes (1957), adding a thought or two borrowed elsewhere. The aim has been to provide a *useful* system, one which is applicable to the analysis of interesting arguments. The style of presentation will presume that you have had some prior experience with some system of formal logic; that is, although we won't leave anything out, we shall be brief.

2C-1 CONVENTION. *(*Fi *as system name)*

Occasionally, when we need a name for the system, we call it "Fi" (for "Fitch")—keeping the same name even when we gradually add to the system.

This section gives the rules in prose and pictures or examples, but certainly the blackboard will be needed for elaboration.

2C-2 CHECK-LIST FOR MASTERY. *(First items to learn)*

- Rule of hypothesis (hyp)
- Subproof and scope

- Rule of reiteration (reit)
- Availability
- Wholesale rule TI
- Wholesale rule TE
- Rule for negated compounds
- Wholesale rule Taut
- Conditional proof (CP)
- Fitch proof (proof in Fi)
- Hypothetical vs. categorical proof
- ADF procedure
- Reductio (RAA⊥)

2C.1 The structure of inference: hyp and reit

The first two rules define the very structure of inference itself, as described by Fitch. Sometimes these two are called "structural rules."

2C-3 RULE. *(Rule of hypothesis* (hyp)*)*

At any stage, a new set of hypotheses may be made. Notation: Start a *new* vertical line, and list the one or more hypotheses to its right, giving "hyp" as reason. Enter a short horizontal line under the last of the one or more hypotheses. As alternative notation for the reason, a hypothesis could be marked "P" for "premiss" or "assp" for "assumption"; the logical idea is the same. Pictures:

Single hypothesis:

```
 . | .
 . | .
 . | .
 n |  |_A   hyp
   |  |
   |  |
   |
```

Three hypotheses:

.	.		
.	.		
.	.		
n		A_1	hyp
n+1		A_2	hyp
n+2		A_3	hyp

The horizontal line helps the eye in seeing what is a hypothesis. The vertical line gives the *scope* of the newly entered hypotheses. It signifies that the hypotheses are merely *supposed* true, and not stated outright as true. What is proved in this scope is proved *under* those hypotheses, not "outright."

2C-4 DEFINITION. *(Subproof and scope)*

A vertical line, with all its hypotheses and other steps, is called a *subordinate proof*, or *subproof*. The vertical line represents the *scope* of the hypotheses. Here is an example of the use of hyp.

1	$(A\&B)\to C$		hyp
2	B		hyp
.	.		.
.	.		.
.	.		.
10		$(\sim C\vee\sim B)$	hyp
11		.	.
12		.	.
13		.	.
14		.	.
15	.		.
16	.		.
17	.		last line of proof

The rule of hyp was used three times: at lines 1, 2, and 10. The *scope* of the hypotheses at steps 1 and 2 extends through line 17 (the end of the proof). The *scope* of line 10 extends through line 14. Then it stops. It is critical that lines beyond 14 are not within the scope of the hypothesis at line 10.

The figure 10–14 counts as a *subproof*. The entire proof 1–17 also counts as a *subproof*. As you can see, one subproof can contain another subproof.

The rule of hypothesis is crucial to logic; without it, informative proofs would be impossible. We must be able to *suppose* whatever we like and whenever we like. It is also essential that we not pretend our supposals are assertions. We must mark them carefully as hypotheses, and we must with equal care indicate their scopes.

2C-5 RULE. *(Reiteration* (reit)*)*

You can reiterate or repeat a formula *into* a subordinate proof (but never out of one). Notation: Write the formula, giving "reit" plus the line number of the repeated step.

Picture:

```
.  | .
.  | .
j  | B
.  | .
.  | .
k  |  |_A   hyp
.  |  | .
.  |  | .
n  |  | B   j, reit
   |  |
```

We don't use this rule much, because instead we obtain the effect of it by the definition of availability, as follows.

2C-6 DEFINITION. *(Availability)*

A step is *available* at a line in a subproof if (1) it is above that line and (2) it is either already in that subproof, or could have been obtained in that subproof by (possibly repeated) reiteration. In other words, an *available* step occurs both above, and also either on the same vertical line, or on a vertical line to the left of, the given subproof line.

Definition **2C-6** is used in the statement of the rules TI and TE below.

Here is an example illustrating various possibilities of availability and nonavailability at The Sstep Under Consideration.

```
| .
| Available
| .
| | Available                         hyp
| | ----------
| | .
| | | Not                             hyp
| | | ----
| | | .
| | | Not
| | | .
| | Available
| | .
| | | Available                       hyp
| | | ----------
| | | .
| | | Available
| | | .
| | | | Not                           hyp
| | | | ----
| | | | .
| | | | Not
| | | | .
| | | The Step Under Consideration
| | | .
| | | Not
| | | .
| | Not
| .
| Not
```

Exercise 11 *(Availability)*

In the following example, which steps are available at the step marked The Step Under Consideration? Do this exercise without peeking at the previous example, then check it yourself (do not hand in your work).

```
|  .
|  1
|  .
|  | 2                      hyp
|  |  .
|  |  | 3                   hyp
|  |  |  .
|  |  |  4
|  |  |  .
|  |  5
|  |  .
|  |  | 6                   hyp
|  |  |  .
|  |  |  7
|  |  |  .
|  |  |  | 8                hyp
|  |  |  |  .
|  |  |  |  9
|  |  |  .
|  |  |  The Step Under Consideration
|  |  |  .
|  |  |  10
|  |  |  .
|  |  11
|  .
|  12
```

▷ . . . . ◁

2C.2 Wholesale rules: TI, TE, and Taut

The following "wholesale rules" TI, TE, and Taut use the concept of availability in order to enable you to make great logical leaps (if you wish to do so).

2C-7 RULE. *(Tautological implication* (TI)*)*

Let $A_1, \ldots, A_n \vDash_{TF} B$ in the sense of Definition **2B-14** on p. 38. Let steps $A_1, \ldots, A_n$ be "available" at a given line in a given subproof. Then B can be written at that line in that subproof, giving as reason, TI, together with the line numbers of the premisses $A_1, \ldots, A_n$. Add (or substitute) the name of the particular tautological implication if it is one of those listed in Examples **2B-16** on p. 40.

Instead of giving you a general picture, here is an example of a proof involving some uses of TI.

1	| (A→B) & E	hyp
2	| | (∼C ∨ ∼E) & A	hyp
3	| | ∼C	1, 2 TI
4	| | B	1, 2 TI
5	| | B & ∼C	3, 4 TI
6	| | E	1, TI

You may not be able to see right off that each of the above "TI" annotations is correct. You can, however, see that each of the cited premisses is "available," and for the rest, just do a little truth table.[12] Here is another example.

1	| A ∨ B	hyp
2	| ∼A	hyp
3	| | C	hyp
4	| | B	1, 2 TI (DS)
5	| | B & C	3, 4 TI (Conj)
6	| | B ∨ ∼∼B	4, TI (Add, or ∨int)

In all of your exercises, try to write a standard name if there is one. This practice will help you learn. Drop the "TI" if your writing hand becomes weary. For example, line 4 could be annotated as just "1, 2 DS."

2C-8 RULE. *(Tautological equivalence* (TE)*)*

Let A and B be tautologically equivalent in the sense of Definition **2B-17** on p. 41. Let (. . . A . . .) be a sentence containing A, and let it be "available" at a given line in a given subproof. Then (. . . B . . .) can be written at that line in that subproof, giving as reason, TE, together with the line number of the premiss (. . . A . . .). Add (or substitute) the name of the particular tautological equivalence if it was listed in Examples **2B-18** on p. 42.

[12] It is critical that you can check by truth tables whether or not the cited premisses really do tautologically imply the desired conclusion. This checkability is essential to our taking TI as a legitimate rule.

Note: Given a sentence (. . . A . . .) said to contain A as a part (or indeed as the whole), by (. . . B . . .) we mean the result of replacing A in (. . . A . . .) by B, leaving everything else as it was. (We mentioned this idea of replacement in §2A.2(6) on p. 28.) Here is an example of the use of TE.

```
1 | A&(B→C)       hyp
  | ·
  | ·
n | A&(∼C→∼B)     1, TE
```

As we have emphasized by the list Examples **2B-19** headed "Removing outermost negations," it is logically important that there is a TE for ∼A, whenever A is a *complex* formula. This TE permits removing that outermost negation—usually by driving it inwards. It is convenient to refer to this group of special cases of TE as "the Rule for Negated Compounds."

2C-9 DEFINITION. *(Rule for negated compounds)*

An application of the rule TE is considered also as an application of the *rule for negated compounds* if the particular tautological equivalence applied is contained in Examples **2B-19** on p. 42.

There is a final rule based on truth-table ideas.

2C-10 RULE. *(Tautology* (Taut)*)*

If $\vDash_{TF}$ A in the sense of Definition **2B-20** on p. 43, then A can be entered at any line of any subproof. As reason, just write Taut. Add (or substitute) the name of the particular tautology from Examples **2B-21** on p. 43 if it has one and you know it.

Remark: This rule is hardly ever useful. The following, however, is an example of the use of Taut.

```
· | ·
· | ·
· | ·
j | | B        hyp
· | | ·
· | | ·
n | | A∨∼A     Taut
```

Note that on line n, no previous lines are mentioned. A tautology needs no premisses, and it is a confusion to suggest otherwise.

So far, in addition to the structural rules hyp and reit, we have the rules TI, TE (including the rule for outermost negations as a special case), and Taut. You can prove a lot with these rules, and they become powerful tools when used in conjunction with other logical equipment; but proofs using *only* these rules tend to be a waste of time. Therefore, we insert here only a brief exercise asking you to annotate with the names that you have learned.

Exercise 12 *(*TI, TE, Taut*)*

Annotate the following proofs. Include the name of the particular TI, TE, or Taut if it is one of those with a name. But note that some steps represent "nameless" applications of TI, TE, or Taut.

1.

1	$A\to((B\vee\sim B)\to C)$	hyp
2	$\sim\sim A$	
3	$(B\vee\sim B)\to C$	
4	$B\vee\sim B$	
5	$C\,\&\,C$	
6	$C\,\&\,A$	
7	$\sim(\sim C\vee\sim A)$	
8	$(A\,\&\,(B\vee\sim B))\to C$	

2.

1	$(B\vee\sim A)\vee C$
2	$\sim(D\to B)\,\&\,\sim C$
3	$B\vee\sim A$
4	$D\,\&\,\sim B$
5	$A\to B$
6	$\sim A\,\&\,D$
7	$\sim(\sim\sim D\to A)$

▷ . . . . ◁

2C.3 Conditional proof (CP)

The following rule is powerful.

2C-11 RULE. *(Conditional proof* (CP)*)*

Let a proof have as one of its items a subproof, with hypotheses $A_1, \ldots, A_n$ and last step B. Then $(A_1\,\&\,\ldots\,\&\,A_n)\to B$ may be written as the next step, the vertical line of the subproof stopping just above $(A_1\,\&\ldots\&\,A_n)\to B$.

As reason, write →int or CP, together with "j–n," where j is the number of the first, and n of the last, step of the subproof. As a special case, the subproof can have a single hypothesis A and last step B; then the conclusion is just A→B. Tricky point: The last line can be the first line (that is, the subproof can have just one line).

Picture.

.	.	
.	.	
j	| <u>A</u>	hyp
.	| .	
.	| .	
.	| .	
n	| B	
n+1	A→B	j–n CP

Example.

1	A→(B∨E)	hyp
2	<u>~E</u>	hyp
3	| <u>A</u>	hyp (for CP)
4	| B∨E	1, 3 MP
5	| B	2, 4 DS
6	A→B	3–5 CP

Exercise 13 *(CP exercises)*

Complete and supply *all* justifications. Note that we have used a few dots in place of left parentheses, per Church's convention as described in §9A. (Don't use any "wholesale" rules.")

1.

1	| <u>P</u>	hyp
2	| | <u>Q</u>	
3	| | P	1, reit
4	| Q→P	
5	P→.Q→P	

2.

1	P	hyp
2	(P&Q)→R	
3	<u>Q</u>	
4		2, Exp
5		1, 4 MP
6		3, 5 MP

3.

1 $P \rightarrow Q$

2 $R \vee \sim Q$ hyp

3

4

5

6

7 $P \rightarrow R$

4.

$A \rightarrow ((B \vee B) \rightarrow C)$

B

$A \rightarrow C$

CP

5.

$A \rightarrow B$

$B \rightarrow C$

$A \rightarrow .D \rightarrow C$

Note, for (6), that $\vDash_{TF} P \rightarrow .Q \rightarrow P$.

6.

$A \rightarrow .(P \rightarrow .Q \rightarrow P) \rightarrow R$

$R \rightarrow S \rightarrow .A \rightarrow S$

▷ . . . . ◁

2C.4 Fitch proofs, both hypothetical and categorical

Before offering additional exercises on CP, we introduce a little more jargon and a point of ethics.

2C-12 DEFINITION. *(Fitch proof, hypothetical and categorical)*

A *Fitch proof (*Fi*-proof* or *a proof in* Fi*)* is a construction each line of which is in accord with the rules. (Because we are going to be adding rules as we go along, we consider this statement to have an "open texture" until we give the "final" definition of an Fi proof at the end of this chapter, §2C.9.) It is required that the last line of the proof exhibit a sentence on the *outermost* vertical line of the proof. There are two sorts of Fitch proofs, depending on whether or not there are hypotheses on the *outermost* vertical line of the proof.

- A proof in Fi may leave the sentence on the last line of the *outermost* vertical—its *conclusion*—within the scope of one or more hypotheses; such a proof is *hypothetical*. In other words, the outermost vertical line of a hypothetical proof of A will begin with some hypotheses and will end with A.

- In contrast, a *categorical proof* in Fi of A is a proof of a conclusion A whose outermost vertical line does not begin with any hypotheses. In other words, a proof proves A categorically iff in that proof, A is not within the scope of any hypothesis.[13]

To give a hypothetical proof of A in the system Fi that we have so far developed is to prove that the hypotheses tautologically imply A. To give a categorical proof of A in the system Fi that we have so far developed is to prove that A is a tautology. Here is the very cheapest possible example of a categorical proof.

1	| A	hyp
2	A→A	1–1, CP

Not only is the above proof categorical—since its last step does *not* lie in the scope of any hypothesis, but it also illustrates the tricky part of the definition of CP (Rule **2C-11** on p. 56): The last line of the subproof is also the first line.

2C-13 COMMUNICATIVE CONVENTION. *(Standards of politesse)*

In proofs designed for communication rather than for discovery or self-conviction, one should only use *obvious* tautologies for Taut, or *obvious* tautological implications for TI, or *obvious* tautological equivalences for TE. What counts as "obvious" is a matter of taste, and can vary from context to context, and from time to time.

2C.5 Conditional proof—pictures and exercises

CP is a **failsafe** rule: If there is any way at all of proving a conditional, you can do so by using CP. So how do you *use* conditional proof in order to prove a conditional? We are going to introduce a variety of rules in addition to CP that make

[13]This way of speaking presupposes that categorical proofs have a kind of "extra" vertical line on the very outside, a line with no hypotheses. It is much better to begin the construction of every categorical proof by drawing this vertical (with of course no hypotheses at its top). Let us mention, however, that such a line isn't strictly speaking a "scope marker," just because there are no hypotheses at its top. The use of the "extra" line on paper is to guide the eye in lining things up in a readable way.

reference to subproofs. The following advice on the order of doing things applies equally to *all* of them.

2C-14 ADVICE. *(Rules involving subproofs:* ADF*)*

You have decided to obtain a step A by a rule that involves subproofs. Just for a name, let us call the method we are about to suggest the ADF method: **A**nnotate, **D**raw, **F**ill in. It goes as follows. .

Before you start, there is always something or other already written at the top of your paper, and you have already written A toward the bottom of your paper as a goal to be proved.

1. **Annotate.** The very first thing you do is **annotate** your goal sentence A with the name of the rule that you are going to apply, e.g. CP. You do this *first* so that you do not later become confused. You are, in effect, providing a presumptive *justification* for your goal sentence.

2. **Draw.** There is some vertical space between what is already at the top of your paper and your goal A, which is located toward the bottom. The rule by which you plan to obtain A, for example CP, requires one or more subproofs as premiss(es). **Draw** the vertical line(s) needed as scope-marker(s) for these subproofs. In order to avoid becoming confused, *use up all the space*. Therefore your scope marker(s) will run from immediately below the steps already written at the top of your paper, and will stop exactly *just before* your goal A. Do this completely *before* moving on to the third step.

3. **Fill in.** Next, you execute the last part of the ADF method: You **fill in** *both* the top *and* the bottom of the scope-marker(s) you have just drawn.

   (a) **Fill in** the first line of the new subproof with the proper sentence (if, as in the case of CP, one is required), and also with the justification, often "hyp."

   (b) **Fill in** the last line of the new subproof with whatever new goal is appropriate, and also annotate that last line with a question mark, **?**, to indicate that you have work left to do in order to obtain your new goal (a subgoal).

Carry out these three steps in the order indicated: (1)First **annotate** your given goal with the rule you plan to use. Then (2) **draw** your scope-marker(s). Finally

(3) **fill in** both the top and the bottom of the newly drawn scope-marker(s). By doing things in this order you will avoid substantial confusion.

Let us turn to CP specifically. Suppose A and B are any sentences, perhaps complex, and suppose you want to derive a conditional, A→B:

```
Given: | ·
       | ·
       | ·        (last of already derived steps)
       |
       |
       |
       |
       |
       |
       | A→B   ?
```

What you do in this situation is to "get set to use conditional proof" as follows. You write:

```
| ·
| ·
| ·          (last of already derived steps)
| | A        hyp (or P)
| |--
| |
| |
| |
| | B        ?
| A→B        ? CP (or→I)
|            [Cross out the question-mark that was on A→B]
```

Study this pair of pictures. Taken together as signifying a transition from one stage of your proving to the next stage, they are a good guide to good logic. This picture, however, combines all three steps. Advice **2C-14** says that it is best *first* to **annotate** your conclusion with "CP," and *second* to **draw** the scope-marker all the way from top to bottom (!), and *third* to **fill in** the top *and* bottom of the new scope-marker, with the appropriate "hyp" at the top and the appropriate new goal at the bottom. Do get in the habit of carrying out these steps in the order **ADF**.

Note that the inserted vertical line begins immediately under the last step for which you already have a justification, and ends immediately above the questioned A→B. This is good policy to avoid mistakes. What will you do next? You will note that B is questioned, and you will attack the "subgoal" of deriving B (which, remember, may itself be complex).

As the picture indicates, it is a matter of working both ends against the middle: As in working a maze, you have to keep one eye on the "entrance" (available already-derived steps) and the other on the "exit" (the questioned conclusion).

Exercise 14 *(Conditional proof)*

To test your understanding of conditional proof, and how it is used with other rules, annotate the following proofs, and fill in any missing steps. An example is almost completely annotated for you.

```
1 | A                  hyp
2 |_(A&B)→(C∨∼A)      hyp
3 | |_B                hyp
4 | | A&B              1, 3 TI (&int)
5 | | C∨∼A
6 | | C                1, 5 DS
7 | B→C                3–6 CP (or →int)
```

1.
```
1 |_A→(B→C)
2 | |_A&B
3 | | B→C
4 | | C
5 | (A&B)→C
```

2.
```
1 |_(A&B)→C
2 | |_A
3 | | |_B
4 | | | A&B
5 | | | C
6 | | B→C
7 | A→(B→C)
```

3.
```
1 |_(A&B)→C
2 | |__
3 | | ∼(A&B)
4 | | ∼A∨∼B
5 | |
6 | (A&∼C)→∼B
```

4.
```
1 |_A→(B∨C)
2 | |_A&∼B
3 | | B∨C
4 | | C
5 |
```

▷ . . . . ◁

Exercise 15 *(More conditional proof)*

Up to (9) the problems are *exceedingly simple and easy*; they should be done in order to help you be sure you know how to construct subproofs. (If you find these troublesome, ask immediately for *one-on-one* instruction. It helps!) Most illustrate important principles of "pure" logic. Here and below, if the problem involves a trio of dots (∴), supply a hypothetical proof. If the problem exhibits a single formula, supply a categorical proof. For these problems, use *only* the following rules: hyp, reit, MP, CP, Simp, and Conj.

Problems from (10) onward go better if you mix the above-listed rules with wholesale rules, either TI rules or TE rules, (but do be polite, **2C-13**).

1. A→B ∴ (C→A)→(C→B)
2. A→(B→C) ∴ B→(A→C)
3. A→(B→A). (Don't forget reit; a categorical proof is wanted.)
4. A→(A→B) ∴ A→B.
5. A ∴ (A→(A→B))→B
6. S→P, Q→R ∴ (P→(S→Q))→(S→R)
7. P→(Q→R) ∴ (P&Q)→R
8. ((P&Q)→R) → (P→(Q→R))
9. If the enemy is fully prepared, we had better increase our strength. If we had better increase our strength, we must conserve our natural resources. Therefore, if the enemy is fully prepared, we must conserve our natural resources. (Use P, I, R.)
10. (∼A∨∼B)→∼C ∴ C→A
11. (A→B)→C, A→∼(E∨F), E∨B ∴ A→C
12. F↔∼(Z&Y), ∼(G∨Z)→∼H, ∼(F&H)∨Y ∴ F→(H→G)

▷ . . . . ◁

2C.6 Reductio ad absurdum (RAA⊥)

A second powerful rule is a form of *reductio ad absurdum*. In fact RAA⊥ is a **failsafe** rule for proving negations: If you can prove a negation ∼A in any way at all, you can always prove it using RAA⊥ . For this version of *reductio*, we rely on the presence in our language of the so-called "propositional constant," ⊥, for absurdity. Its interpretation is as a horrible falsehood, what in German is sometimes called *das Absurd*, "the absurd." (Recall from Convention **2B-5** that the table for ⊥ has an "F" in every row.) It is so false that the *only* way to get it as a conclusion is by that special and curious case of TI in which the premisses are themselves truth-functionally inconsistent. Since every inference with truth-functionally inconsistent premisses is an instance of TI, so is the special case in which the conclusion is awful ⊥. In symbols, we are saying that if

$$A_1, \ldots, A_n \vDash_{TF}$$

then

$$A_1, \ldots, A_n \vDash_{TF} \bot\,.$$

(The special case of TI in which we derive ⊥ from *explicitly* contradictory steps A and ∼A we called "⊥ introduction" or "⊥I"; see Examples **2B-16** on p. 40.) Now the rule RAA⊥ can be stated.

2C-15 RULE. *(Reductio ad absurdum* (RAA⊥)*)*

Let a proof have as one of its items a subproof with hypothesis A [variant: ∼A] and last step ⊥. Then ∼A [variant: A] can be written as the next step, the vertical line of the subproof stopping just above ∼A [or, on the variant, A]. As reason, write RAA⊥ , together with "m–n," where m is the number of the first, and n of the last, step of the proof. You are to be reminded that the only way to obtain ⊥ is via a truth-functionally inconsistent set, which is precisely why the hypothesis A [or, on the variant, ∼A] is said to be "reduced to absurdity."

Picture of RAA⊥ , both regular and variant forms.

```
  .  | .                                   .  | .
  .  | .                                   .  | .
  j  |  |_A    hyp                         j  |  |_∼A    hyp
  .  |  | .                                .  |  | .
  .  |  | .                                .  |  | .
  .  |  | .                                .  |  | .
  n  |  | ⊥                                n  |  | ⊥
n+1  | ∼A      j–n, RAA⊥  [regular]      n+1  | A         j–n, RAA⊥
                                              |             [variant]
```

Though it is good for you to observe the difference between the regular and variant forms, there is no need for you to mention it in your exercises unless specially asked to do so. Here is an example of the use of RAA⊥ .

1	∼A	hyp
2	A & B	hyp (for RAA⊥)
3	⊥	1–2, TI
4	∼(A & B)	2–3, RAA⊥

Exercise 16 *(RAA⊥ exercises)*

Complete and *fully annotate* the following, in each case using RAA⊥ at least once.

1.

1	A
2	∼B
	∼(A → B)

2.

1	∼A → B
2	∼B
	A

▷ . . . . ◁

How does one *use reductio* proofs? Suppose first that A is any sentence, perhaps complex, and suppose that, at a certain stage, you are trying to derive ∼A.

·	
·	
·	(last of already derived steps)
∼A	**?**

What you do in this situation is to "get set to use RAA⊥ " as follows, using **ADF** as in Advice **2C-14**. You write as follows.

```
|  ·
|  ·
|  ·        (last of already derived steps)
|  |_A      hyp
|  |
|  |
|  |
|  |
|  | ⊥      ?
|  ∼A       ? RAA⊥
|           [Cross out the question-mark that was on ∼A]
```

For the best time-order in which to carry out "getting set to use RAA⊥ ," use **ADF**, Advice **2C-14**: *First* **annotate** the desired conclusion with "RAA⊥ ," *second* **draw** your scope marker to use up the entire space, and *third* **fill in** both the top (the hyp) and the bottom (⊥) as appropriate.

Note that the inserted vertical line begins immediately under the last step for which you already have a justification, and ends immediately above the questioned ∼A.

What will you do next? You will note that ⊥ is questioned, and you will attack the "subgoal" of deriving ⊥. Recalling the awful truth table for ⊥ you will also recall that there is in general only one way of deriving it: by TI from an available set of truth-functionally inconsistent steps. These steps can include the new hypothesis, A, as well as any already derived available steps. There is an example to be annotated in the exercise below, but first observe that there is a variant form of RAA⊥ . Suppose you are trying to derive some perhaps complex sentence, A, and have run out of other ideas:

Given:

```
| ·
| ·
| ·     (last of already derived steps)
|
|
|
| A   ?
```

As a last resort, you can try to obtain A by the variant form of RAA⊥ :

You write:

```
| ·
| ·
| ·         (last of already derived steps)
| |_~A      hyp (or P)
| |
| |
| | ⊥       ?
| A         ? RAA⊥  [Cross out the question-mark that was on A.]
```

Proofs using this variant form are less "constructive" than those that avoid it, since A just pops up out of nowhere; but sometimes use of this form is essential.

Exercise 17 *(RAA⊥)*

Annotate the following proofs.

1.

```
1 |_⊥   hyp
2 | A
```

2.

```
1 | A     hyp
2 |_~A    hyp
3 | ⊥
```

3.

1	A→(B→C)	hyp
2	~C	hyp
3	│ A&B	
4	│ A	
5	│ B→C	
6	│ C	
7	│ ⊥	
8	~(A&B)	

4.

1	~A→~B	hyp
2	B&C	hyp
3	│ ~A	
4	│ ~B	
5	│ ⊥	
6	A	

▷ . . . . ◁

Exercise 18 *(More* RAA⊥ *)*

In proving the following, use RAA⊥ (regular or variant) in each problem.

1. ⊥ ∴ A
2. A ∴ ~(~A&B)
3. ~B, B∨C, ~A→~C ∴ A
4. A&~B ∴ ~(A→B)
5. ~(A→B) ∴ A&~B
6. A∨(A→B) [Huh?]
7. P→S, S→~(B&D), ~B→T, ~(D→T) ∴ ~P
8. (A→B)→(~C→D), ~(C∨E), (A∨F)→B ∴ D
9. A→(B→(C&D)), C→(E→(H &I)), F→~I ∴ (A&B)→(A →(F→~E))
10. If the laws of mathematics refer to reality, they are not certain. Hence, if they are certain, they do not refer to reality. (Use R. C)
11. If forces A and B are equal and are the sole forces acting on point p, the resultant force R will be either nil, or will have the direction of the bisector of the angle between A and B. But R does not have this direction. If R is nil, then A and B are equal and directly opposed. Hence, if A and B are equal but not directly opposed, then they are not the sole forces acting on p. (Use E, S, N, D, O.)

▷ . . . . ◁

We close this discussion by noting that sometimes—but not often—it is more convenient to use a form of RAA that does not involve ⊥.

2C-16 RULE. *(*RAA*)*

Here is a picture of RAA (sans ⊥), in both its regular and variant forms:[14]

```
    .  | .                               .  | .
    .  | .                               .  | .
    j  |  | A      hyp                   j  |  | ~A    hyp
       |  |-                                |  |-
    .  |  | .                            .  |  | .
    .  |  | .                            .  |  | .
    k  |  | B                            k  |  | B
    .  |  | .                            .  |  | .
    .  |  | .                            .  |  | .
    n  |  | ~B                           n  |  | ~B
  n+1  | ~A        j–n, RAA [regular]  n+1  | A          j–n, RAA
                                            |              [variant]
```

RAA has to be interchangeable with RAA⊥ for the following conclusive reason: If you have ⊥ in a subproof, you can always get a contradictory pair B and ∼B by ⊥elim, p. 40. And if you have two contradictory sentences B and ∼B, you can always get ⊥ by ⊥int. Our giving RAA⊥ primacy of place is for a low-level practical reason: With plain RAA you have nothing definite to put at the bottom of the subproof as a new subgoal, since you don't really know exactly what you are looking for. But with RAA⊥, you can always write ⊥ as your new subgoal. Therefore, although you may use RAA as part of any Fitch proof, we won't provide any exercises or problems that *require* its use, and you can if you wish ignore it entirely. (Oh, well: As a part of Exercise 18, prove (4) twice, the second time by using RAA instead of RAA⊥.)

2C.7 Case argument (CA) and biconditional proof (BCP)

The following rules are useful additions, as we find out in §2C.8.

2C-17 CHECK-LIST FOR MASTERY. *(Two more rules)*

- Case argument
- Biconditional proof

---

[14]Note that in contrast to e.g. Klenk (2002), you do *not* need to conjoin the contradictory steps to form B & ∼B. Conjunction really has nothing to do with it.

Case argument. Case argument (CA) is a **failsafe** rule for application to a disjunction A∨B as *premiss* (repeat: as premiss). The key is that *you are going somewhere*, that is, you have a desired conclusion, say C. The rule says that you can obtain this conclusion from three premisses: the disjunction A∨B, a subproof with hypothesis A and last step C, and a subproof with hypothesis B and last step C.

2C-18 RULE. *(Case argument* (CA)*)*

·	·		
·	·		
j	A∨B		
·	·		
·	·		
k		| A (hypothesis)	hyp
·		| ·	
·		| ·	
k′		| C	**?**
m		| B (hypothesis)	hyp
·		| ·	
·		| ·	
m′		| C	**?**
	C		j, k–k′, m–m′ CA

The confusing point is that the desired conclusion C occurs three times: once where you want it, but also separately as the last step of each of the two subproofs. None of these occurrences is redundant. Because the rule CA is **failsafe**, it is worth getting clear on exactly how it goes.

Observe in particular that unlike CP and RAA⊥, the rule CA requires setting up *two* subproofs.[15] That means that when you annotate the conclusion C, you must refer to *three* premisses: the disjunction and each of the two subproofs. When you "get set to use CA," fill up the entire available space with the two subproofs.

Go back and look at Advice **2C-14** on p. 60: use the **ADF** method. *First* **annotate** the conclusion C with "CA." *Second* **draw** your two subproofs, using up all of the available space.[16] *Third* **fill out** the top of each subproof (with the respective appropriate hypothesis) and the bottom of each subproof (with identical occurrences

[15] But CA has three *premisses*: two subproofs and the disjunction.

[16] Well, it is important to leave one line of space between the two subproofs so that there is no mistake as to which sentences belong to which subproofs.

of the desired conclusion C). Do all three **ADF** steps before doing anything else. Study the following example.

1	A∨B	
2	│ A	hyp
3	│ B∨A	2, Add
4	│ B	hyp
5	│ B∨A	4, Add
6	B∨A	1, 2–3, 4–5 CA

Don't go further until you can carry out the following exercise.

Exercise 19 *(Case argument)*

Annotate the first two, and use Case Argument to prove the rest.

1.

1	(A&B)∨(A&C)	hyp
2	│ A&B	
3	│ A	
4	│ A&C	
5	│ A	
6	A	

2.

1	A∨B	hyp
2	∼A	hyp
3	│ A	
4	│ ∼A	
5	│ ⊥	
6	│ B	
7	│ B	
8	│ B	
9	B	

3. (A&B)∨(A&C) ∴ A&(B∨C)

4. (A→B)∨(C→B) ∴ (A&C)→B

5. ∼A∨∼B ∴ ∼(A&B)

6. A→B, ∼B∨∼C ∴ ∼A∨∼B)

▷ . . . . ◁

Biconditional proof. To prove a biconditional it is always possible and usually convenient to use the rule of Biconditional Proof or BCP or ↔int, which says that you can prove A↔B from two subproofs: one with hypothesis A and last step B, and the other with hypothesis B and last step A. BCP is a **failsafe** rule.

2C-19 RULE. *(Biconditional proof)*

```
.  | .
.  | .
j  |  |_A        hyp
.  |  | .
.  |  | .
j′ |  | B

k  |  |_B        hyp
.  |  | .
.  |  | .
k′ |  | A
   | A↔B         j–j′, k–k′, BCP (or ↔int)
```

BCP is a two-premiss rule, where each premiss is itself a subproof. BCP is evidently a "double-barreled CP." The time-order in getting set to use BCP is exactly analogous to that for CP and RAA⊥ and CA, namely, **ADF** in accord with Advice **2C-14** on p. 60: *First* **annotate** the desired conclusion as "BCP," *second* **draw** the two required subproofs in such a way as to use up all the available space, and *third* **fill in** the respective tops and bottoms of the two subproofs as required. Leave no step untaken. Here is an example of a categorical proof using BCP.

```
1 |  |_A&B             hyp
2 |  | A                1, Simp
3 |  | B                1, Simp
4 |  | B&A              2, 3 Conj
  |
5 |  |_B&A             hyp
6 |  | A                5, Simp
7 |  | B                5, Simp
8 |  | A&B              6, 7 Conj
9 | (A&B) ↔ (B&A)      1–4, 5–8, BCP (or ↔int)
```

It seems easy to see the point of the rule BCP: In order to establish that A↔B, you must establish that you can obtain each from the other.

Do not proceed without carrying out the following exercise.

Exercise 20 *(Biconditional proof)*

Annotate the first two proofs and use biconditional proof to prove the rest.

1.

1	| $C \rightarrow A$	
2	| $\underline{B \& C}$	hyp
3	| | $\underline{A}$	
4	| | | $\underline{B}$	
5	| | | C	
6	| | $B \rightarrow C$	
7	| | $\underline{B \rightarrow C}$	
8	| | C	
9	| | A	
10	| $A \leftrightarrow (B \rightarrow C)$	

2.

1	| $\underline{A \& B}$	hyp
2	| | $\underline{A}$	
3	| | B	
4	| | $\underline{B}$	
5	| | A	
6	| $A \leftrightarrow B$	

3. $A \rightarrow ((A \leftrightarrow B) \leftrightarrow B)$

4. $C \therefore (A \vee B) \leftrightarrow ((C \rightarrow B) \vee (C \rightarrow A))$

5. $\sim A \& \sim B \therefore A \leftrightarrow B$

▷ . . . . ◁

2C.8 Some good strategies for truth functional logic

In constructing proofs in system Fi of Fitch proofs (Convention **2C-1**), always remember that you are working from both ends towards the middle. You must work stereoscopically, keeping one eye on the premisses you have that are available, and the other eye on the conclusions to which you wish to move.

Make a habit of putting a *question mark* on any conclusion for which you do not *yet* have a reason. (You should at any stage already have a reason for each of your premisses, and perhaps for some of your conclusions.)

Divide, then, the steps you have so far written down into those towards the top—your premisses—and those towards the bottom that are questioned—your desired conclusions. Then use the following strategies. Pay particular attention to those marked **failsafe**, since these have the feature that if the problem can be solved at all, then it can (not must) be solved by using the indicated failsafe strategy.

2C-20 STRATEGIES. *(For **conclusions** (questioned items) at bottom)*

When desired (not yet obtained) conclusion (bearing a question mark) is:

A&B. Try to prove A and B separately, and then use Conj, that is, &int. Your subgoals should be written as follows: Annotate your goal A&B with "Conj." In the empty space at your disposal, write B **?** just above A&B; and write A **?** about half way up your blank space. (The question mark on A is essential to remind you that A is a subgoal, not a premiss.) **Failsafe.**

A∨B. In some cases you will see that either A or B is easily available, and so you can use Addition, that is, ∨int. (This procedure is not failsafe!) Otherwise, it is always **failsafe** and usually efficient to rely on the fact that disjunctions and conditionals are "exchangeable." assume ∼A and try to obtain B. Then, as an intermediate step, derive ∼A→B by CP. Then obtain the desired A∨B by Conditional Exchange.

A→B. It is always **failsafe** and usually efficient to try to obtain B from A, and then use CP, that is, →int.

A↔B. Try to obtain each of A and B from the other, and then use BCP, that is, ↔int. **Failsafe.**

∼A. If A is *complex*, obtain from an equivalent formula with negation "driven inward" by the appropriate "Rule for Negated Compounds." If A is *simple*, try RAA⊥ (regular version): Assume A and try to derive ⊥. Both procedures are **failsafe**.

A. If nothing else works, try RAA⊥ in the variant version: Assume ∼A and try to reason to ⊥. This is in principle failsafe, but should be avoided unless absolutely necessary, since if you use this rule much your proofs will often involve long redundant detours.

⊥. Look for a contradiction among all the available steps, and use ⊥int. **Failsafe.**

2C-21 STRATEGIES. *(For **premisses** (already justified items) at top)*

When among the premisses (item has already been given a definite reason) you have

A&B. Derive A and B separately by Simp, that is, &elim—or anyhow look around to see how A or B or both can be used. **Failsafe.**

A→B. Look for a modus ponens (A) or a modus tollens (∼B). Sometimes you need to generate A as something you want to prove (at the bottom, with a question mark). This procedure is not failsafe.[17]

A↔B. Look for an MPBC (A or B) or an MTBC (∼A or ∼B). Not failsafe.

A∨B. Look for a disjunctive syllogism: a ∼A, a ∼B, or anyhow a formula differing from one of A or B by one negation. This will often work, but it is not failsafe. Suppose the "look for DS" doesn't work. Then consider that you not only have A∨B as a premiss; you also have a desired conclusion, say C. Get set to derive C from each of A and B, and use Case argument, CA. **Failsafe**.

∼A. If A is *complex*, "drive negation inward" by a Rule for Negated Compounds. **Failsafe.** If A is *simple*, hope for the best.

⊥. It doesn't often happen that you have awful ⊥ as a premiss. If you do, you can take yourself to have solved your problem, whatever it is; for *any* (every) desired conclusion is obtainable from ⊥ by ⊥elim (Examples **2B-16** on p. 40). **Failsafe.**

We are about to ask you to practice using these strategies, namely, in Exercise 21 on p. 78. In preparation for this exercise, we insert some extended advice and we run through an example.

2C-22 Exercise advice. *(Strategies for truth-functional logic)*

In each exercise problem, please imagine that the single letters stand for perhaps extremely complex sentences. The sentences listed to the left of ∴ stand for items that are available to you. The sentence given at the right of the ∴ stands for an item that you are trying to prove. Then for a given problem in Exercise 21, you should do the following.

- Draw a vertical "subproof" line. Go down perhaps a half page.
- Write the available items at its top.
- Number them.

[17]There is a failsafe procedure for A→B as premiss, but it leads to uninformative and even weird proofs.

- Draw a short horizontal line under them.
- Write the item to be proved at the very *bottom* of the vertical line.
- Write a question mark after it to show that it is something that you wish to prove (rather than something that is available).
- Now start applying as many strategies as you can think of. *Do not expect to be able to complete the proof.* You will be left with one or more items still marked with a question mark indicating that they are still "to be proved."

For example, suppose that you are given the problem, (A→B)→C, B ∴ C. When you write it down, it looks like this.

1	(A→B)→C	hyp
2	B	hyp
	(you would have more space here)	
	C	**?**

You look around for a modus ponens, and you see you would need (A→B). So you generate it as a wanted conclusion, *and* you indicate how you are going to use it:

1	(A→B)→C	hyp
2	B	hyp
	A→B	**?**
	C	**?** 1, _ MP

Resist the common temptation to write down A→B at the top, or to think of it as having a reason, or to think of it as available. You are instead generating it as a subgoal, something you wish to prove, but haven't yet proved.

Now the question mark is on the next-to-last step, A→B. Use the strategy for A →B as conclusion, i.e., hypothesize A and try to prove B:

1	(A→B)→C	hyp
2	_B	hyp
3		_A	hyp
		B	?
	A→B	?
	C	?1, _MP

Now complete the proof without additional steps: The remaining questioned item can now have "reit" as its reason, and numbers can be filled in:

1	(A→B)→C	hyp
2	_B	hyp
3		_A	hyp
4		B	? 2 reit
5	A→B	? 2–4 CP
6	C	? 1, 5 MP

Some of your proofs for the upcoming exercises will be incomplete. That is, after having employed the appropriate strategies, you will be left with one or more items still marked with question marks indicating they remain to be proved.

Here is another example. You are given the problem D ∴ (B→C)→∼A. When you set it up, it looks like this.

1	_D	hyp
	(B→C)→∼A	?

Employ the strategy for a conditional as conclusion.

```
1 │ D                hyp
2 │ │ B→C            hyp
  │ │
  │ │
  │ │
  │ │ ~A             ?
  │ (B→C)→~A         ? 2–, CP
```

Next use the strategy for ∼A as conclusion, i.e., hypothesize A and try to prove ⊥.

```
1 │ D                hyp
2 │ │ B→C            hyp
3 │ │ │ A            hyp
  │ │ │
  │ │ │
  │ │ │ ⊥            ?
  │ │ ~A             ? 3–, RAA⊥
  │ (B→C)→~A         ? 2, CP
```

When you have exhausted the strategies for a given problem, move on to the next one. (Do not waste your time with successive applications of RAA⊥ .)

Exercise 21 *(Strategy problems)*

Here are some "strategy" problems. Follow Exercise advice **2C-22**. We made these so that, with one exception, it is *not possible* to finish them with complete proofs—just sketch a beginning. Use *only* simple failsafe strategies stopping, when there are none left to apply; and for this exercise do *not* use RAA⊥ [variant].

1. C ∴ A&B
2. C ∴ A→B
3. C ∴ A∨B
4. C ∴ ∼A
5. C ∴ A↔B
6. A&B ∴ C

7. $A \lor B \therefore C$ (use Case Argument)
8. $\sim(A \to B) \therefore \sim C$
9. $E \therefore (A \& B) \to (C \& D)$
10. $D \therefore (A \lor B) \to (C \to D)$
11. $\bot \therefore (A \& (B \lor C))$

▷ . . . . ◁

Exercise 22 *(More strategy problems)*

Use the strategies to the full. Prove:

1. $(\sim A \lor \sim B) \to \sim C \therefore C \to A$.
2. $A \to (B \to C)$, $(C \& D) \to E$, $F \to \sim(D \to E) \therefore A \to (B \to \sim F)$
3. $\sim A \lor C$, $T \to (S \& \sim B)$, $S \leftrightarrow \sim(D \lor C)$, $\sim A \to (E \to (B \lor C))$ $\therefore T \to \sim(D \lor E)$. Use Case Argument.
4. $B \to ((F \& G) \leftrightarrow D)$, $(C \to D) \to \sim A$, $\sim F \to (D \& E)$, $\sim G \to \sim B$ $\therefore \sim(A \& B)$. Use RAA$\bot$.
5. $(\sim(A \lor B) \lor \sim(B \lor C)) \to \sim(A \& C)$. This is a theorem. Use Case Argument.
6. $((P \to Q) \to P) \to P$
7. $(P \leftrightarrow Q) \leftrightarrow R \therefore P \leftrightarrow (Q \leftrightarrow R)$

Exercise 22(6) is "Peirce's Law," a tautology and therefore a theorem. It is surely unintuitive! Try starting out with CP, and then, inside the CP, using the variant form of RAA$\bot$. Exercise 22(7) states that the truth-functional biconditional is "associative" (parentheses can be regrouped) in exactly the same sense as conjunction and disjunction (and in contrast to the conditional). The property holds only for the two-valued version of the biconditional. You can easily verify (7) by truth tables, and therefore you know that its conclusion follows from its premiss by TI. To prove it *strategically*, however, is tedious and also uninformative; for this reason, problem (7) should be *skipped* unless you wish to see just how tedious a strategic proof of a simple fact can be.

▷ . . . . ◁

Exercise 23 *(Simple set theory with truth functions)*

Later we will come to use certain symbols of set theory in addition to "$x \in X$," which means that x is a member of the set X, and $\{a, b\}$, which is the set containing exactly a and b (Symbolism **1B-10**). These are some of the additional symbols that we will be using, with an indication of what they mean.

- "$\subseteq$" stands for the subset relation; $X \subseteq Y$ iff every member of X is also a member of Y. This includes the possibility that X = Y.
- "$\subset$" stands for the proper subset relation; $X \subset Y$ iff $X \subseteq Y$ but not $Y \subseteq X$.
- $\varnothing$ is *the empty set*; $\varnothing$ is the one and only set that has no members.
- $X \cup Y$ is the *union* of sets X and Y; $X \cup Y$ is a set that contains anything that belongs to either X or Y (or both).
- $X \cap Y$ is the *intersection* of sets X and Y; $X \cap Y$ is a set that contains anything that belongs to both X and Y.
- $X - Y$ is the *set difference between* X and Y; $X - Y$ is a set that contains everything in X that is not in Y.

We ask you to prove some principles involving this notation. In so doing, use anything on the following list as a premiss.

S-1 $c \in \{a, b\} \leftrightarrow (c = a \vee c = b)$

S-2 $\sim(a \in \varnothing)$

S-3a $a \in (X \cup Y) \leftrightarrow (a \in X \vee a \in Y)$

S-3c $c \in (X \cup Y) \leftrightarrow (c \in X \vee c \in Y)$

S-3a$\varnothing$ $a \in (X \cup \varnothing) \leftrightarrow (a \in X \vee a \in \varnothing)$

S-4a $a \in (X \cap Y) \leftrightarrow (a \in X \& a \in Y)$

S-4b $a \in (X \cap \varnothing) \leftrightarrow (a \in X \& a \in \varnothing)$

S-5 $a \in (X - Y) \leftrightarrow (a \in X \& \sim a \in Y)$

S-6a $X \subseteq Y \rightarrow (a \in X \rightarrow a \in Y)$

S-6b $Y \subseteq Z \rightarrow (a \in Y \rightarrow a \in Z)$

When you use one of the premisses listed above as a premiss for some rule, you do not need to copy it. Count each of them as "available," and just cite any one of them as if it existed as an earlier line in your proof, as indicated in the following example, which works out one problem.

Problem: $a \in X \ \therefore\ a \in X \cup Y$.

Proof:

1	$a \in X$	hyp
2	$a \in X \vee a \in Y$	1, Add
3	$a \in X \cup Y$	2, S-3a, MPBC

1. $a \in X \cup Y, X \subseteq Y \ \therefore\ a \in Y$.
2. $a \in X, X \subseteq Y, Y \subseteq Z \ \therefore\ a \in Z$.
3. $Z \in Z \leftrightarrow ((Z \in Z) \rightarrow p) \ \therefore\ p$.
4. $c \in \{a, b\}, c = a \rightarrow c \in X, c = b \rightarrow c \in Y \ \therefore\ c \in (X \cup Y)$.
5. $a \in (X \cup \varnothing) \ \therefore\ a \in X$.

▷ . . . . ◁

Later on we are going to introduce some proof techniques for some of these set-theoretical concepts. You should *not* refer back to Exercise 23 in that connection. The point here is only to improve your facility with truth-functional connectives when they are mixed up with other notation.

2C.9 Three proof systems

A "logical system" can be one of a number of things. Here we mean a *proof system*, that is, a collection of rules of proof. We describe three such systems.

- All three concern sentences made from our six favorite logical connectives, $\perp$, $\sim$, $\rightarrow$, &, $\vee$, and $\leftrightarrow$.
- All three are constructed by a *selection* from among the following rules. We *first* list all the rules, dividing them into "families." *Later* we define the systems. The word "intelim" used in classifying some of the rules is explained below. (If you are not fully in command of these rules and their names, now is the time to become so. Otherwise the following will make little sense.)

2C-23 DEFINITION. *(Families of rules)*

Structural rules. hyp (Rule **2C-3**) and reit (Rule **2C-5**). Also the concept of Availability, Definition **2C-6**.

Intelim & and ∨ rules. &int (or Conj, Examples **2B-16**) and &elim (or Simp, Examples **2B-16**) and ∨int (or Add, Examples **2B-16**) and CA (or Case Argument, Rule **2C-18**).

Intelim → and ↔ rules. →int (or CP, Rule **2C-11**), →elim (or MP, Examples **2B-16**), ↔int (or BCP, Rule **2C-19**), and ↔elim (or MPBC, Rule **2C-19**).

Intelim ∼-and-⊥ rules . ⊥int and ⊥elim (Examples **2B-16**), and RAA⊥ (in either regular or variant forms; Rule **2C-15**).

Intelim *pure* negation rules (no ⊥). RAA (in either regular or variant forms), Rule **2C-16**; and XAQ (or ∼elim), Examples **2B-16**.

Mixed rules. CE (Examples **2B-18**), MT (or Modus tollens, Examples **2B-16**), MTBC (or Modus tollens for the biconditional, Examples **2B-16**), and DS (or disjunctive syllogism, Examples **2B-16**). These rules all mix at least two connectives.

Negated-compound rules. ∼& (or DeM), ∼∼ (or DN or Double negation), ∼∨ (or DeM), ∼→ (or Negated conditional), ∼↔ (or Negated biconditional); see Examples **2B-19** and Definition **2C-9** for all of these. And ∼⊥ (or Negated *absurdity*, Examples **2B-21**).

Wholesale rules. TI (Tautological implication, Rule **2C-7**), TE (Tautological equivalence, Rule **2C-8**), Taut (Tautology, Rule **2C-10**).

That completes the list of rules on which we will rely. Here are the three proof systems.

2C-24 DEFINITION. *(Three proof systems)*

Fitch proof or proof in Fi**.** Can use anything. Especially the *wholesale* rules TI, TE, and Taut. This is the most useful system of logic. It combines delicate strategic considerations with the possibility of relaxed bursts of horsepower. (Sometimes we use "wholesale proof" as a synonym for "proof in Fi.")

Strategic Fitch proof. In a "strategic proof in Fi," you can use any rule mentioned in §2C.8 on strategies, which is precisely: all *except* the unnamed Wholesale rules, which cannot be used in a "strategic proof." That is, you can use the Structural rules, the Intelim rules, the Mixed rules, and the Negated-compound rules—but not the unnamed Wholesale rules.

Intelim Fitch proof. In an "intelim proof in Fi," you can use only the structural rules and the "intelim" rules).

These systems are of importance for different reasons.

In the long run, the *wholesale* system Fi is of the greatest practical use. When you are in the midst of serious logical analysis, you should feel free to use anything your mind can manage, constrained by rigor and clarity, but unconstrained by considerations of elegance.

But the *strategic* Fitch system is also of great use in guiding you. It is good to know that you do not *need* to step outside it (although you are encouraged to do so whenever you feel impatient with its limitations). An efficient and sometimes elegant proof can usually be constructed using only the strategic rules.

The system of *intelim* Fitch proofs is of theoretical (philosophical and mathematical and historical) interest rather than practical interest. The historical connections are these.

When natural deduction was first introduced (independently by Gentzen (1934) and Jaśkowski (1934)), brute-force rules like TI and TE were not part of the apparatus. In contrast, there were postulated for each connective a pair of rules, one for operation on *premisses* with that connective as principal, the other for obtaining *conclusions* with that connective as principal. Only "pure" strategies were allowed.

Fitch's book (1952) followed this plan, calling the former sorts of rules *elimination* rules and the latter sort *introduction* rules. (In our opinion, "premiss-rules" and "conclusion-rules" might have been better choices because they suggest more accurately what is going on.) Proofs that used *only* these introduction and elimination rules Fitch called *intelim proofs*. And though our exact choice of rules is a little different, the terminology remains as apt as it ever was.[18]

2C-25 FACT. *(Soundness of each system)*

Each of these three systems is "sound": If there is a proof with hypotheses G and last line A, then in fact G tautologically implies A ($G \vDash_{TF} A$).

[18]The choice of words here is intended to suggest that the aptness was and is more Procrustean than not.

Indeed, take any step of any proof in one of the three systems, and let A be written at that step. Let G be the set of hypotheses that are available at that step. Then G $\vDash_{\text{TF}}$ A.[19]

2C-26 FACT. *(Completeness of intelim proofs)*

The system of intelim proofs is "complete": If an argument is truth-functionally valid, there is an intelim proof that shows it to be so. That is, if a set G of sentences tautologically implies a sentence A (i.e., if G $\vDash_{\text{TF}}$ A), then there is an intelim proof all of whose hypotheses are in G and whose last line is A. (This is intended to include the case when G is empty, so that there is a categorical proof of A.) All the other rules—TI, TE, and the like—are therefore redundant.

But mixed, negated-compound and wholesale rules are nevertheless a useful sort of redundancy, shortening the work and making complex proofs (with quantifiers and special theoretical machinery) vastly more perspicuous.

Exercise 24 *(Intelim proofs)*

This is an important exercise. *Do not proceed without success in carrying it out.*

1. In defining "strategic proofs" we relied on the Mixed rules (CE, MT, MTBC, and DS) and the Negated-compound rules (DN, DeM, Negated conditional, Negated Biconditional, Negated $\perp$) as defined in Definition **2C-23** on p. 82. Establish a *each* of these by intelim proofs. If a rule is a TI, a hypothetical proof is wanted; for example, give an intelim proof from hypotheses A$\rightarrow$B and $\sim$B to conclusion $\sim$A. If a rule is a TE, you should supply two hypothetical proofs, which is of course enough to prove the biconditional; for example, give an intelim proof from hypothesis A$\vee$B to conclusion $\sim$($\sim$A & $\sim$B), and another intelim proof from hypothesis $\sim$($\sim$A & $\sim$B) to A$\vee$B. Here is a list of the rules—one version of each (except both versions of DeM)—for you to check off as you go:

[19] For this to be true, if the step in question is itself a hypothesis, then it itself must be counted as one of the hypotheses that are "available" at that step. We must, that is, use the fact that A $\vDash_{\text{TF}}$ A.

☐	CE	$(A \to B) \approx_{TF} \sim A \vee B$
☐	MT	$A \to B, \sim B \vDash_{TF} \sim A$
☐	MTBC	$A \leftrightarrow B, \sim A \vDash_{TF} \sim B$
☐	DS	$A \vee B, \sim A \vDash_{TF} B$
☐	DN	$\sim\sim A \approx_{TF} A$
☐	DeM	$\sim(A \& B) \approx_{TF} \sim A \vee \sim B$
☐	DeM	$\sim(A \vee B) \approx_{TF} \sim A \& \sim B$
☐	NCond	$\sim(A \to B) \approx_{TF} A \& \sim B$
☐	NBC	$\sim(A \leftrightarrow B) \approx_{TF} A \leftrightarrow \sim B$
☐	Negated ⊥	$\vDash_{TF} \sim\sim\bot$

2. A rule is *redundant* in a logical system provided the following holds: Anything you can prove *with* the rule can be proved *without* the rule. The system of intelim proofs itself contains redundancies. For example, the intelim rules RAA and XAQ for negation are redundant because anything you can prove with them can be proved instead by using the intelim rules involving ⊥. As an example, show that XAQ (a.k.a. ∼ elim, p. 40) is redundant.

▷ . . . . ◁

Chapter 3

The art of proof in the logic of predicates, operators, and quantifiers

In this chapter we survey the use in proofs of the universal and existential quantifiers and the identity predicate in the context of predicates and operators and truth functions. The following chapter considers the arts of symbolization and counterexample for quantifiers.

The language is often called "first order" because, as we noted in Symbolism **1B-13**, quantification is not permitted at the higher order represented by predicates and operators. We shall tag various grammatical ideas with "Q" instead of with "TF" to mark the transition from the study of purely truth-functional ideas to the study of ideas involving quantification. The reader should review §1B in general, and Symbolism **1B-2** in particular for what we intend when we speak of "variables," "constants," and "parameters."

3A Grammar for predicates and quantifiers

We begin—as always—with (symbolic) grammar. Our discussion will presuppose that you have followed our advice to review §1B starting on p. 11. As noted, we will use "Q"—for "quantification"—as a tag, in strict analogy to our use of "TF" as a tag for truth-functional logic. Q has the following apparatus.

Variables of quantification. General-purpose variables x, y, z, x_1, y', etc. There must be infinitely many individual variables available for use with quantifiers. When we come to using our logic in order to understand special topics such as the theory of sets and the theory of truth-functional logic, we will also want variables with restricted ranges such as we have from time to time introduced by convention; for example, A for sentences, G for sets of sentences, and **i** for TF interpretations.

Quantifiers. "∀" and "∃" are called *quantifiers*, *universal* and *existential* respectively. In addition, we also label as *quantifiers* the combinations "∀v" and "∃v," where v is any variable. In this case, we say that the quantifier *uses* the variable v.

Individual constants. Including both "real" constants and parameters a, b, c, etc. as needed. There must be infinitely many individual constants available for use as parameters.

Predicate constants. Including both "real" predicate constants and parameters. F, G , etc.; how many "places" is left to context.

Operator constants. Including both "real" operator constants and parameters. f, g, sometimes ∘, +, etc., as needed.

Terms. Individual variables and constants are Q-*atomic terms*, and of course Q-atomic terms are terms. New terms can be made from old ones by means of operators. The story for Q-atomic terms parallels the story for TF-atoms in truth-functional logic: Atomicity is a "negative" idea. See §2A.1. Parentheses are used as needed or wanted for communication. For example, when f is a one-place operator, we use "fx" and "f(x)" interchangeably, and similarly for a two-place operator: "hxy," "h(x)(y)," and "h(x, y)" all come to the same thing.

Q-atoms. The Q-*atoms* are exactly (1) the individual variables, and (2) the individual, predicate, and operator constants. We will need this concept only later, for semantics.

Sentences. A predicate constant followed by a suitable number of terms (a "predication" in the sense of Symbolism **1B-8** on p. 15) is a *sentence*, and others can be made either with the truth-functional connectives from the preceding chapter, or with the help of the quantifiers and variables: The result of prefixing a quantifier and a variable (in that order) to a sentence is itself a sentence.[1]

Scope. When a sentence ∀xA or ∃xA results from prefixing a quantifier, the entire sentence is said to lie within the *scope* of the quantifier.

Free and bound. An occurrence of a variable is *free* or *bound* in a term or sentence according as it does not or does lie within the scope of a quantifier using that variable. The variable itself is *free* in a term or sentence if each of its occurrences (if there are any) is free therein, and is *bound* therein if some occurrence is bound therein.

3A-1 REMINDER. *(Restricted variables)*

When discussing the logic the art of which is the topic of these notes, as largely explained in Symbolism **1B-21**, we use letters in the following way.

A, B, C range over sentences (perhaps containing free variables, perhaps not).

G ranges over sets of sentences.

i ranges over TF interpretations.

X, Y, Z range over sets.

t, u range over terms (perhaps containing or being free individual variables, perhaps not).

x, y, z, v range over variables (they include themselves in their range—for they *are* variables—but there are others, too).

T and F for Truth and Falsity. Formally there is ambiguity between this use of "F" and our use of "F" as a predicate constant; but in practice there is never a problem.

For compact expression of the quantifier rules, we need a convention allowing compact expression of substitution of terms for variables.

---

[1] We might label as "Q-atomic" any sentence that involves neither one of the six truth-functional connectives listed in Symbolism **1B-12**, nor a quantifier connective ∀x_ or ∃x_ (for any variable x) in accord with Symbolism **1B-1**, so that predications would be Q-atomic sentences. This frequently-given account of "Q-atomic sentence," though entirely understandable, is parochial and misleading: So-called Q-"atomic" sentences obviously have structure, namely, predicative structure.

3A-2 CONVENTION. *(*Ax *and* At*)*

First we use "Ax" as standing for *any* sentence, without commitment as to whether x occurs in it. Second, *provided* t is a term not containing (or being) a variable bound in Ax, we declare that At is the result of substituting t for *every free* occurrence of x in Ax. As an extension of this convention, we use "$Ax_1 \ldots x_n$" and "$At_1 \ldots t_n$" in the same way, additionally assuming all the x's are distinct. It is important that the various terms t_i need not be distinct.[2]

Examples of Ax and At and Axy and Atu.

Source Ax	*Result* At	*Source* Axy	*Result* Atu
Fx & Gx	Ft & Gt	Fx →Gy	Ft →Gu
∀xFx & Gx	∀xFx & Gt	Fxu →Fty	Ftu →Ftu
∀x(Fx & Gx)	∀x(Fx & Gx)		
Fx & Gt	Ft & Gt		

The direction of substitution is always from Ax to At, never the other way. We sometimes call the motion from Ax to At "instantiation"; then we say that At is an *instance* of Ax.

Warning. When we use this convention in stating a rule, the direction of substitution is from Ax to At regardless of which of Ax and At is in premiss and which in conclusion. Later on (in UG and EG) you will be substituting "uphill" from conclusion to premiss.

Exercise 25 *(*Ax *and* At*)*

1. Given a sentence Ax and a term t, you should be able to compute At; likewise, given Axy, t, and u, you can compute Atu. This is illustrated in the following.

Ax	At	Axy	Atu
Gx	?	Gxy	?
Gx ∨ Hx	?	Gxy ∨ Fxy	?
Gx ∨ Hc	?	Gxt ∨ Fxu	?
∃Gx ∨ Fx	?	Gxu ∨ Fty	?
∀x(Gx ∨ Fx)	?	Gvv	?

[2]This convention is in lieu of using an explicit notation for substitution, e.g., using "[t/x](A)" to name the result of substituting t for every free occurrence of x in A.

2. For each row below: Given that the formulas in the second and third columns are respectively Aa and Ab, fill in at least one candidate for what Ax must be.

	Ax	Aa	Ab
a.	?	Raa	Rbb
b.	?	Raa	Rab
c.	?	Raa	Rba
d.	?	Raa	Raa

▷ . . . . ◁

3B Flagging and universal quantifier

We introduce the proof theory in separated pieces.

3B.1 Flagging restrictions

Our quantifier rules depend on so-called "flagged" terms that must satisfy certain "flagging restrictions"; the idea is that a "flagged" term can be taken (within its scope) as standing for an "arbitrarily chosen entity," while the restrictions guarantee the propriety of this construal. Since we need to begin somewhere, we begin by giving these flagging restrictions, though you cannot see how they work until you come to the rules of Universal Generalization and Existential Instantiation. Observe that although the flagging restrictions are a little complicated, so that you must learn them with explicit effort, the rest of the rules for the quantifiers are easy. And in fact if you think of the intuitive point of the restrictions—to guarantee that flagged terms can be regarded (within their scope) as standing for arbitrarily chosen entities—then they make immediate sense.[3]

3B-1 DEFINITION. *(Flagging)*

A term t is *flagged* by writing "flag t" as a step or as part of a step.[4] The term that

[3]Other logical systems use "flagging" with just an opposite meaning: Variables are "flagged" when they can*not* be looked on as standing for arbitrarily chosen entities.

[4]The choice of notation here is of no consequence, as long as the term t is marked in some way or other. Elsewhere we have sometimes used the imperative, "choose t," and sometimes the declarative, "t arbitrary." Fitch just writes the term t itself—but his geometry is a little different than ours, and that policy might be just slightly confusing in the present context. To repeat: Nothing hangs on these choices.

is flagged must be Q-atomic, because complex terms (for example, "x^2") have too much structure to stand for an "arbitrarily chosen entity" (for example, "x^2" cannot stand for an arbitrary entity because whatever it stands for is constrained to be a perfect square). It is understood that t *occurs* in "flag t" just as in any other step, and that if t is a variable, it *occurs free* therein.

For absolute beginners in logic, it is best to use only *constants* (parameters, not "real" constants) as flags, so that variables are used only with quantifiers. There is that way less risk of confusion. After a while, however, the rigidity of this rule is counterproductive; so our discussion envisages that *flagged terms can be either constants or variables*. To compensate, we will use the phrase "occurs free" in such a way that any occurrence of any *constant* is automatically free: The word "free" draws a distinction only among *variables*.[5] Therefore, if all your flagged terms are constants, you can ignore the word "free."

The flagging of a term has something in common with the assumption of a hypothesis: The flagged term t is not taken as standing for an "arbitrarily chosen entity" for all time, but only for a while, just as hypotheses are assumed only for a while, but not for all time. This feature of hypotheses is usually discussed under the heading of "scope," and we shall use the same language for flaggings. A scope is defined by its beginning and its ending.

3B-2 DEFINITION. *(Scope of a flagging)*

Beginning. The step where a term is flagged with "flag t" *begins* the *scope* of this flagging.

Ending. The scope of a flagging always *ends* with the end of the subproof immediately containing the flagging step. Pictorially, the scope of a flagging ends with the vertical line against which the flagging step sits. (Fussy point: If a flagging step occurs in an *outermost* or main proof, the scope ends with the last step of the proof in which the flagged term occurs free.)

Picture of the scope of a flagging:

[5]Fitch himself certainly did not restrict application of the word "free" to variables. When he phoned along about eleven of a morning, he would sometimes ask, "Are you free—or bound—for lunch?"

```
| .
| .
| | .
| | flag t    [beginning of scope]
| | .
| | .
| | | ·       [possibly beginning some other subproof]
| | | .
| | .
| | ·         [end of scope]
| .
```

Since quite universally it does not make sense to permit scopes to cross, we avoid considerable trouble by forcing the scope of a flagging to stop at the end of its innermost subproof. You can see by geometrical intuition that it is a result of our convention about scopes that (1) the scopes of different flaggings never cross or overlap, and (2) the scope of a flagging never crosses the scope of a hypothesis. Scopes of either kind are, then, always either properly nested one within the other, or else totally separate. We must now fasten one restriction or another onto flagged terms so that they genuinely deserve to be construed as standing for "arbitrarily chosen entities." The essential idea is that we cannot move information about a flagged term either into or out of its scope; the beginning and the ending of the scope of a flagging of t stand as barriers to all flow of information concerning t. There are, however, some restrictions that although stronger than this essential idea, and accordingly stronger than necessary, are much easier to check in practice. We state both an easy and an intermediate version for your practical use.

3B-3 DEFINITION. *(Flagging restrictions)*

Easy flagging restriction. (1) The flagged term must be a constant (a parameter, not a "real" constant) rather than a variable. (2) Part of the meaning of (1) is that the flagged term cannot be complex. (3) The flagged term cannot occur outside of its scope, nor in any event in the last line of an entire main proof.

Intermediate flagging restriction. (1) The flagged term can be either a parameter or a variable. (2) Part of the meaning of (1) is that the flagged term cannot be complex. (3) The flagged term cannot occur free in any step above its scope that is available inside its scope. Further, and more or less symmetrically, (4) the flagged term cannot occur free in any step below its scope that depends on subproofs that include its scope, nor in any event in the last step of an entire main proof.

Beginning students should learn only the easy flagging restriction, since anything that can be proved at all has a proof satisfying this easy restriction. The intermediate version is given for more advanced students to learn if they like, because it makes some proofs a little easier to follow (see §3E.1 for an example).[6] It is obvious that if the easy flagging restriction is satisfied, then so is the intermediate flagging restriction.[7] That is, these are not independent restrictions, but just different versions of the same idea.

Flagging made easy. Given the picture of building up a proof from both ends toward the middle, you can guarantee satisfying the easy flagging restriction, and therefore the others as well, simply by flagging a parameter (when required by the rules below) that is absolutely new to what you have so far written down (at either the top *or* bottom); you will not have to worry. (Later, in §3E.1, we give examples of the intermediate version of the restrictions.)

3B.2 Universal Instantiation (UI)

The first quantifier rule is Universal Instantiation (UI). This rule is easy (maybe too easy).

[6]The "**essential flagging restriction**," which is of only theoretical interest, might be put as follows. There are two parts. (I) A term flagged once may not be flagged again within the scope of the first flagging. (II) No application of a rule that "crosses the scope barrier" is allowed to contain free occurrences of the flagged term on both sides of the scope barrier. It is awkward to say exactly what is meant by "crossing the scope barrier" without becoming involved with the details of the various rules; rather than try, we just note that there are two cases. (1) An application of a rule with one or more premisses above the scope barrier and a conclusion within the scope is said to "cross the scope barrier." In this case the restriction requires that the flagged term must be absent either from all the premisses lying above the scope or from the conclusion that lies within the scope. The flagged term may not use the rule to cross over the scope barrier. (2) An application of a rule that has as one of its premisses the subproof containing a flagging step—its conclusion will certainly be below the scope—is also said to "cross the scope barrier." In this case, the restriction requires that the flagged term must be absent from the conclusion (since it will certainly occur free in the premiss containing the scope). Again, what is forbidden is that the flagged term use a rule to cross over the scope barrier. And as usual we must add that a flagged term may not occur in the last line of an entire main proof. (The essential flagging restriction is obviously Too Hard for any but theoretical use.)

[7]And if either the intermediate or the easy flagging restriction is satisfied, then so is the essential flagging restriction.

3B-4 RULE. *(*UI*)*

If x is any variable and $\forall x Ax$ is available, you may enter At for *any* term t *not containing a variable that is itself bound in* Ax. Write premiss number, "UI" and "t/x" to show what is substituted for what, as in the following.

	.	
	.	
j	$\forall x Ax$	
	.	
	.	
j+k	At	j, UI t/x

It is understood that line j need only be "available" at line j+k.[8]

Further, if $x_1, \ldots, x_n$ are distinct variables and $\forall x_1 \ldots \forall x_n A x_1 \ldots x_n$ is available, you may enter $At_1 \ldots t_n$ for *any* terms $t_1, \ldots, t_n$ (whether distinct or not and whether complex or not) not containing a variable bound in $Ax_1 \ldots x_n$. Write premiss number, "UI," and "$t_1/x_1, \ldots, t_n/x_n$" to show what is substituted for what, as in the following.

	.	
	.	
j	$\forall x_1 x_2 \ldots x_n A x_1 x_2 \ldots x_n$	
	.	
	.	
j+k	$At_1 t_2 \ldots t_n$	j, UI t_1/x_1, t_2/x_2, …, t_n/x_n

3B-5 EXAMPLE. *(Universal Instantiation)*

1	$\forall x(x=x)$	
2	$3+4=3+4$	1, UI, $3+4$/x
3	$b=b$	1, UI, b/x

1	$\forall x \forall y(Fxy \to Gyxx)$	
2	$Fab \to Gbaa$	1, UI a/x, b/y
3	$Faa \to Gaaa$	1, UI a/x, a/y [OK: a for both x, y]

[8] Students often make up their own restrictions on this rule; don't you. *Any* term t, simple or complex, *whether used before in the proof or not*. We repeat: *Any* term (not containing a variable bound in Ax).

1	∀x(∀xFx→Gxx)	
2	∀xFx→Gf(d)f(d)	1, UI, f(d)/x

1	∀x∃yFxy	
2	∃yFyy	*Wrong* UI, since y is bound in Ax of premiss.

1	∀xFxbx	
2	Fubt	*Wrong* UI; *same* term must be put for all free x.
3	Fubx	*Wrong* UI; term must be put for *all* free x.

You can (and sometimes must) use UI twice on the same premiss. See first example above.

In using Convention **3A-2**, we substitute for all free x in Ax, *not* in ∀xAx. (Of course *no* x is free in ∀xAx, but plenty may be free in Ax.)

3B-6 ADVICE. *(Common errors in applying* UI*)*

1	∀xFx→Gu	
2	Ft→Gu	*wrong* UI.

This is *wrong* because the premiss does not have the form ∀xAx. It appears to have that form because "∀x" is on the left, but actually the scope of that "∀x" is not the whole formula: It only covers "Fx," whereas you can only use UI when the scope of "∀x" is the whole step.

1	Gu→∀xFx	
2	Gu→Ft	*wrong* UI.

This is wrong for about the same reason: UI can only be used when the scope of "∀x" is the *whole* step.

3B-7 ADVICE. *(Advice on* UI*)*

Postpone the use of UI until you *know* how you want to substitute or instantiate.

3B.3 Universal Generalization (UG)

We are now ready for the second rule governing the universal quantifier, the one that relies on the flagging restrictions Definition **3B-3**. Universal Generalization is like CP (→int) and RAA⊥ in requiring a vertical line, but this vertical line will begin with a "flagging step" instead of a hypothesis. It is best to annotate this step "for UG." The rule is as follows.

3B-8 RULE. *(*UG*)*

The one-variable case of universal generalization:

```
        | .
        | .
j       | | flag c    for UG
        | | .
        | | .
j+k     | | .
j+k+1   | | Ac
        | ∀xAx        UG, j–(j+k+1), c/x
```

provided that c satisfies the flagging restrictions, including the part that says that c must be an individual constant (not complex). You may of course use a or any other constant you like in place of c as long as it does not violate the flagging restrictions. The many-variable case of universal generalization:

```
        | .
        | .
j       | | flag c1, ..., cn       for UG
        | | .
        | | .
        | | .
j+k+1   | | Ac1 ... cn
        | ∀x1...∀xnAx1...xn    UG, j–(j+k+1), c1/x1, ..., cn/xn
        |
```

provided that $c_1, \ldots, c_n$ satisfy all the flagging restrictions. There must be no repetitions in the list $c_1, \ldots, c_n$.

The subproof in the one-variable case, beginning with the flag and stopping just over the universal generalization ∀xAx, is called a *proof general with respect to* the flagged constant, c, since we may think of c as arbitrarily chosen within its scope.

The subproof in the many-variable case is called a "*proof general with respect to* $c_1, \ldots, c_n$," since we may think of $c_1, \ldots, c_n$ as *arbitrarily chosen* within their scope. Observe that by Convention **3A-2**, the direction of substitution is "uphill" when using UG.

It is logically crucial that the premiss for UG is *not* a single step. To infer from the single step Fc to ∀xFx is the worst possible logic. The premiss for UG is instead the *entire subproof*, including the flagging step. It is the fact that we can get to Ac for arbitrarily chosen c that warrants the inference to ∀xAx; nothing less will do, and nothing less than the entire subproof expresses this information.

3B-9 ADVICE. *(Advice on* UG *I)*

UG is a **failsafe** rule: Given that you want to show ∀xAx, *always* "set up to use UG." That is, suppose that you are given:

```
| ·
| ·
| ·        (last of already derived steps)
|
|
|
| ∀xAx   ? you want this
```

Then you add material to make your proof look like the following:

```
| ·
| ·
| ·          (last of already derived steps)
| | flag b   for UG
| | ·
| | ·
| | ·
| | Ab       ?
| ∀xAx       UG b/x
```

Be sure to extend the vertical line for UG as far up as possible.

This picture conceals the best time-order for carrying out the parts of "getting set for UG." Go back and look at the **ADF** Advice **2C-14** on p. 60: *First* **annotate** your conclusion as coming by UG, *second* **draw** your vertical scope-marker to fill up the entire space, and *third* **fill in** both the top (the flagging step) and the bottom (the new subgoal).

Follow the above advice to "get set to use UG" *before* anything else, especially before using any UI. In other words, postpone UI until after you are set to use UG. Otherwise, the flagging restrictions will get in your way.

3B-10 ADVICE. *(Advice on* UG *II)*

In "getting set to use UG," you need to choose a term to flag. At the moment of choice, you are in the *middle* of a proof, with some already-derived steps above, and also some "desired" steps below (including your desired conclusion).

- Choose a (new!) parameter.[9]
- By choosing something brand new, you will be sure to satisfy the easy flagging restriction, Definition **3B-3**.
- **Warning.** Do not be tempted by "wish-fulfillment" to flag a constant already appearing in what you have so far written, either above or below. Look both up and down: The constant should simply not occur at all. Especially not in your desired conclusion.

3B-11 EXAMPLE. *(*UG *and* UI*)*

All artists are mad; all humans are artists; so all humans are mad.

[9]In easy cases, when you have become familiar with proofs involving quantifiers, it is all right to choose the very variable of the quantifier "$\forall x$." Even so, it is necessary to check that you do not violate the intermediate flagging restriction, Definition **3B-3**. (You are all too likely to run into confusion if you ever flag with any variable other than the very variable used in the conclusion of your UG, especially if it occurs anywhere else in the problem. So play safe: Flag only parameters, never variables.)

1	$\forall x(Ax \to Mx)$	hyp
2	$\forall x(Hx \to Ax)$	hyp
3	flag a	for UG
4	$Aa \to Ma$	1 UI a/x
5	$Ha \to Aa$	2 UI a/x
6	$Ha \to Ma$	4, 5 TI (HS)
7	$\forall x(Hx \to Mx)$	3–6 UG a/x

3B-12 EXAMPLE. *(*UI *and* UG*)*

None of Sam's friends wear glasses. But oculists always wear glasses. And every known gavagai is one of Sam's friends. So no known gavagai is an oculist.

1	$\forall x(Sx \to \sim Wx)$	hyp
2	$\forall y(Oy \to Wy)$	hyp
3	$\forall z(Gz \to Sz)$	hyp
4	flag b	for UG
5	Gb	hyp
6	$Sb \to \sim Wb$	UI b/x
7	$Ob \to Wb$	2 UI b/y
8	$Gb \to Sb$	3 UI b/z
9	Sb	5, 8 TI (MP)
10	$\sim Wb$	6, 9 TI (MP)
11	$\sim Ob$	7, 10 TI (MT)
12	$Gb \to \sim Ob$	CP 5–11
13	$\forall x(Gx \to \sim Ox)$	4–12 UG b/x

3B-13 EXAMPLE. *(*UI *and* UG*)*

Extortionists and gamblers are uncongenial. Alberta is congenial. So Alberta is no extortionist.

1	$\forall x[(Ex \vee Gx) \to \sim Cx]$	hyp
2	Ca	hyp
3	$(Ea \vee Ga) \to \sim Ca$	1, UI a/x
4	$\sim(Ea \vee Ga)$	2, 3 TI (MT)
5	$\sim Ea \,\&\, \sim Ga$	4, TE (DM)
6	$\sim Ea$	5, TI (Simp)

3B-14 EXAMPLE. *(*UI *and* UG*)*

The subproof-example below illustrates two significant points.

- There is the use of *the many-variable case of* UI on line 3.
- In choosing a flag, you must look both above *and below*. At line 2, it would be totally wrong to choose a as the flag letter, since a occurs in the last line of the proof.

1	$\forall x \forall y(Fy \vee Gx)$	hyp
2	flag b	for UG
3	$Fb \vee Ga$	1 UI a/x, b/y
4	$\forall x(Fx \vee Ga)$	2–3, UG b/x

It's worth pausing over Line 4, since it seems to have no thoroughly idiomatic reading that is also faithful to its symbolic grammar. Here are three English candidates, each to be compared with Line 4.

1. Everything is either F or Ga.

   Is (1) bad English grammar? Or is OK, but ambiguous? It would take us too far afield to sort out the many intricacies of this candidate. For example, if we are thinking of F as corresponding to an English predicate constant such as "is foxy," we come up with "Everything is either foxy or Alfred is a giraffe," which, although good enough English, harbors a sad "scope ambiguity" as between "everything" and "either-or."

2. Either everything is F or Ga.

   (2) is not faithful to the symbolic grammar of Line 4; but its symbolism, $\forall xFx \vee Ga$, is indeed logically equivalent to Line 4. Optional: Try to prove their logical equivalence in five minutes, and *without* flagging errors; but stop after the five minutes. We will return to this equivalence in due time.

3. Everything is such that either it is F or Ga.

   (3) is grammatically correct English that is faithful to Line 4. It is, however, clumsy, ugly, and in poor taste.

Exercise 26 *(*UI *and* UG*)*

1. The set-theoretical principles listed in Exercise 23 on p. 80 for the purpose of providing exercises in truth-functional proof theory are of course more general than their forms there let on. Here we reproduce some of them in fuller (but not fullest) generality, and ask you to prove three of them from the others. As before, consider the following S-principles "available" for use in proofs; you need cite only their names when you use them.

S′-1	$\forall x(x\in\{a, b\} \leftrightarrow (x=a\vee x=b))$
S′-2	$\forall y{\sim}(y\in\varnothing)$
S′-3a	$\forall x(x\in(X\cup Y) \leftrightarrow (x\in X)\vee(x\in Y))$
S′-3b	$\forall z(z \in(X\cup\varnothing) \leftrightarrow (z\in X)\vee(z\in\varnothing))$
S′-6a	$\forall y((X\subseteq Y) \rightarrow (y \in X \rightarrow y \in Y))$

Prove:

a. $X\subseteq Y \;\therefore \forall x(x\in(X\cup Y) \rightarrow x\in Y)$
b. $\forall y(y=a \rightarrow y\in X), \forall z(z=b \rightarrow z\in Y)$
$\quad\therefore \forall x(x\in\{a, b\} \rightarrow x\in(X\cup Y))$
c. $\forall z(z\in(X\cup\varnothing) \rightarrow z\in X)$

2. Explain what is illustrated by each proof among the following that is marked as *Wrong*. To say what is *Wrong* with a figure is to say *why* that figure is not a correct instance of UG, *as described* in §3B.3. So either the figure has the wrong form (Convention **3A-2** is relevant) or the figure violates the flagging restrictions Definition **3B-3**. (This is a tedious exercise that most readers should dip into only lightly if at all. *Do not hand in your work.*) The first five of the following illustrate correct uses of UG.

```
| | flag a, b   for UG
| | .
| | .
| | .
| | Fab
| ∀x∀yFxy      UG a/x, b/y
```

```
| | flag b   for UG
| | .
| | .
| | .
| | Fba
| ∀xFxa      UG b/x
```

```
| | flag a   for UG
| | .
| | .
| | .
| | Faa
| ∀xFxx      UG a/x
```

```
| | flag a   for UG
| | .
| | .
| | .
| | Fax
| ∀yFyx      UG a/y
```

```
| | flag a          for UG
| | .
| | .
| | .
| | ∀xFx→Fa
| ∀x(∀xFx→Fx)   UG a/x

| | flag a   for UG
| | .
| | .
| | .
| | Fax
| ∀xFxx     Wrong UG a/x

| | flag a   for UG
| | .
| | .
| | .
| | Faa
| ∀xFax     Wrong UG a/x
```

```
| Fa&Gb
| ___
| | flag a   for UG
| | .
| | .
| | .
| | Gac
| ∀xGxc     Wrong UG a/x

| | flag x      for UG
| | .
| | .
| | .
| | Fx→Gu
| ∀xFx→Gu
|       Wrong UG x/x
```

▷ . . . . ◁

Exercise 27 *(Short and positive* UQ *Proofs)*

Prove the following. Except for the first, categorical proofs are wanted! Use only positive rules (no negation, no ⊥).

1. ∀x(Fx→Gx)→B, ∀x(Gx&Hx) ∴ B
2. ∀x(Fx→Gx)→(∀xFx→∀xGx)
3. ∀xFxx→∀yFyy
4. ∀x∀yF(x, y)→F(a, g(a))
5. (∀xFx∨B)→∀x(Fx∨B)[10]
6. ∀x∀yFxy→∀y∀xFxy
7. (B→∀xFx)↔∀x(B→Fx)
8. ∀x(Fx&Gx)→(∀xFx&∀xGx)

Double check your work to be sure that you treated any sentence "∀vA→B" as a conditional rather than as a universal quantification. This mistake is both serious

[10]The converse is also valid, but requires an awkward detour through negation. Try it if you feel like it.

and common. Consult Advice **3B-6**.

▷ . . . . ◁

Definitions are often universal generalized biconditionals; the following is a typical use of such a definition. (Such definitions are treated in detail in 8B.1.)

3B-15 EXAMPLE. *(*UI *and* UG*)*

Let (*) be the following truth, which in English we express by saying that A is tautologically equivalent to B iff A tautologically implies B and B tautologically implies A.

$$\forall x \forall y[x \approx_{TF} y \leftrightarrow (x \vDash_{TF} y \ \&\ y \vDash_{TF} x)]. \qquad (*)$$

It follows from (*) alone that tautological equivalence is symmetrical: If A is tautologically equivalent to B then B is tautologically equivalent to A. Since this is a "complicated" case, we are especially careful to flag *constants* for UG. In particular, it is good to avoid having to annotate line 7 below with "UI x/y, y/x," a point well worth remembering.

1	$\vert\ \forall x \forall y(x \approx_{TF} y \leftrightarrow (x \vDash_{TF} y \,\&\, y \vDash_{TF} x))$	hyp (namely, (*))
2	$\vert\ \vert\ $ flag a, b,	for UG
3	$\vert\ \vert\ \vert\ a \approx_{TF} b$	hyp
4	$\vert\ \vert\ \vert\ a \approx_{TF} b \leftrightarrow (a \vDash_{TF} b \,\&\, b \vDash_{TF} a)$	1, UI a/x b/y
5	$\vert\ \vert\ \vert\ a \vDash_{TF} b \,\&\, b \vDash_{TF} a$	3, 4 TI (MPBC)
6	$\vert\ \vert\ \vert\ b \vDash_{TF} a \,\&\, a \vDash_{TF} b$	5, TE (Comm)
7	$\vert\ \vert\ \vert\ b \approx_{TF} a \leftrightarrow (b \vDash_{TF} a \,\&\, a \vDash_{TF} b)$	1 UI b/x a/y
8	$\vert\ \vert\ \vert\ b \approx_{TF} a$	6, 7 TI (MPBC)
9	$\vert\ \vert\ a \approx_{TF} b \rightarrow b \approx_{TF} a$	3–8 CP
10	$\vert\ \forall x \forall y(x \approx_{TF} y \rightarrow y \approx_{TF} x)$	2–9 UG a/x b/y

Note how we *used* (*) in the proof: (*) is a universally generalized biconditional, which we used first to take us from line 3 to line 5 *by a UI followed by a* MPBC, and second to take us from line 6 to line 8 *by another UI followed by a* MPBC.

3B-16 ADVICE. *(*UI+MPBC*)*

You should look for a chance to use the UI+MPBC pattern whenever you have a universally quantified biconditional among your premisses. That will happen often, since definitions are frequently expressed in this way, and definitions are essential to organized thought. See, for example, Definition **4A-5** and other set-theory definitions in §4A.

In fact it happens so often that below we advise shortening the work by use of two new rules dealing explicitly with definitions: Def. introduction and Def. elimination (Rule **3E-8** and Rule **3E-9**). As a preview, here is the short form of Example **3B-15**; it assumes that (*) is given *as a definition* of $\approx_{TF}$ before the start of the subproof.

1	| | flag a, b,	for UG
2	| | | $\underline{a \approx_{TF} b}$	hyp
3	| | | $a \vDash_{TF} b \,\&\, b \vDash_{TF} a$	2, $\approx_{TF}$ elim, a/x b/y
4	| | | $b \vDash_{TF} a \,\&\, a \vDash_{TF} b$	3, TE (Comm)
5	| | | $b \approx_{TF} a$	4, $\approx_{TF}$ int, b/x, a/y
6	| | $a \approx_{TF} b \rightarrow b \approx_{TF} a$	2–5 CP
7	| $\forall x \forall y (x \approx_{TF} y \rightarrow y \approx_{TF} x)$	1–6 UG a/x b/y

To use this annotation of lines 3 and 5, you must be treating (*)[11] as a definition of $\approx_{TF}$.

It is essential to be prepared to use UI with complex terms; here is an easy example.

3B-17 EXAMPLE. *(*UI *and* UG *with complex terms)*

Any parent of anyone is their ancestor. The father of anyone is a parent of him or her. Therefore, the father of anyone is their ancestor. (We represent parenthood and ancestorhood with relations, and fatherhood with an operator: fx = the father of x.)

1	| $\forall x \forall y (Pxy \rightarrow Axy)$	
2	| $\underline{\forall x P(fx)x}$	
3	| | flag b	for UG
4	| | $P(fb)b$	2 UI b/x
5	| | $P(fb)b \rightarrow A(fb)b$	1 UI fb/x b/y
6	| | $A(fb)b$	4, 5 MP
7	| $\forall x A(fx)x$	3–6 UG b/x

---

[11] *Instead* of Definition **2B-17**—it won't do to define the same symbol in two different ways at once. You need to pick. See §8B.

Exercise 28 *(UI and UG)*

1. Take the following as a definition, called "Def. $\subseteq$": $\forall x \forall y(x \subseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y))$. Prove that each of the following is a consequence.

   a. $\forall x(x \subseteq x)$.

   b. $\forall x \forall y \forall z((x \subseteq y \;\&\; y \subseteq z) \rightarrow x \subseteq z)$.

2. (Inspired by Rule **3D-1**.) Prove that, since someone is a person's parent if and only if that person is their child, and since someone is a person's child if an only if that person is either their mother or their father, it follows that each person's mother is their parent.[12] That is, you should be able to prove the following.

   $\forall x \forall y(Pxy \leftrightarrow Cyx)$, $\forall x \forall y(Cxy \leftrightarrow (y = mx \vee y = fx))$ $\therefore$ $\forall z P(mz)z$. You will need an additional premiss, soon to be licensed as an inference rule (Rule **3D-1**), that $\forall x(x = x)$.

3. Symbolize and prove. By definition,[13] a bachelor is an unmarried male. Sally is a female. No male is a female. Sally's father is married. Therefore, neither Sally nor her father is a bachelor.

4. Problem with operators. Read "hx" as "half of x" and "dx" as "the double of x." As usual, "$x < y$" means that x is less than y. Symbolize (1), (2), and (3), and prove that (3) follows from (1) and (2). (1) For all x and y, if x is less than y, then half of x is less than half of y. (2) For all z and w, half of the double of z is less than w iff z is less than w. Therefore, (3) for all x and y, if the double of y is less than x, then y is less than half of x.

5. Optional (skippable). You may wish to figure out why (given our Convention **3A-2**) you could not obtain an analog to line 7 of Example **3B-15** by way of two successive uses of UI annotated "x/y" and "y/x."

[12]This sentence fails the test of grammatical excellence, since "their" is supposed to be plural, whereas here it is used as singular. You could substitute "his or her," which is singular; in this sentence, however, the substitution would yield awkwardities. We are fortunate to be logicians rather than English grammarians! Still, when bad English grammar seems *essential* to someone's philosophical point, you should suspect that he or she has made a substantive conceptual error.

[13]A definition in these notes,is always an "if and only if," even though there are many other useful forms of definition.

6. Suppose our domain consists of color shades, and suppose that Sxy represents the relationship of *similarity* of colors: Sxy $\leftrightarrow$ x and y are similar in shade of color. We would expect the relationship of similarity to be *reflexive* and *symmetric*; that is, we would expect that

   $\forall$xSxx; and $\forall$x$\forall$y(Sxy$\rightarrow$Syx).

   However, we would not expect the relationship of similarity to be *transitive*; that is, we would not expect that

   $\forall$x$\forall$y$\forall$z((Sxy & Syz)$\rightarrow$Sxz).

   We might *define* Exy, the relationship of *exact similarity*, by the formula

   (Def. E) $\forall$x$\forall$y(Exy$\leftrightarrow$$\forall$z(Sxz$\leftrightarrow$Syz)).

   In other "words," two colors are exactly similar iff they are similar to exactly the same colors. Using the above definition, (Def. E), as your only premiss, prove that the relation of exact similarity has the following three features that make it what is called an "equivalence relation."

   Reflexivity: $\forall$xExx.
   Symmetry: $\forall$x$\forall$y(Exy$\rightarrow$Eyx).
   Transitivity: $\forall$x$\forall$y$\forall$z((Exy & Eyz)$\rightarrow$Exz).

   You may well need help on this exercise, and if so, you should ask for it immediately, and even repeatedly. *Do not, however, continue your study of logic without being able to complete all parts of this exercise—without peeking. That is, go through it over and over with peeking, until finally you can do it without peeking.* Big hint: Your premiss, (Def. E), is a universally quantified biconditional, which is the very form discussed in Example **3B-15**. In each problem, you will need to use the UI+MPBC procedure on Def. E once per occurrence of "E" in the problem. Thus, transitivity will require using UI+MPBC on Def. E three times.

▷ . . . . ◁

3B.4 Vertical lines in subproofs

There are only five reasons for drawing a new vertical line as an indicator of scope: rules CP, BCP, RAA$\perp$, CA, and UG. These have something in common beyond the obvious fact that they are all failsafe rules: In each case, you are working at least partly *from the bottom up*!!! In each case, that is, you are generating a "sub-goal," a new sentence to try to prove; and the vertical line tells you the *scope* within which you must prove it. The **ADF** Advice **2C-14** on p. 60 applies in each and every case: *First* **annotate** the desired conclusion, *second* **draw** the scope-markers for the needed subproof(s), and *third* **fill in** *both* top *and* bottom of the subproofs. The five rules are these:

- Rule CP: You want to prove A→B. You must prove B, "under" A as new hypothesis. The new vertical line guarantees that B is proved "under" A by providing a *scope* for that hypothesis.
- Rule BCP: You want to prove A↔B. You must prove B "under" A, and A "under" B. The new vertical lines provide *scope*-markers guaranteeing that B is proved "under" A, and A "under" B.
- Rule RAA⊥: You want to prove A [or ∼A]. You must prove ⊥, "under" ∼A [or "under" A] as new hypothesis. The new vertical line guarantees that ⊥ is proved "under" A [or ∼A] by providing a *scope* for that hypothesis.
- Rule CA: You want to prove C, and you have A∨B available. You must prove C "under" A, and also you must prove C "under" B. The new vertical lines guarantee that C is proved "under" each of these hypotheses by providing appropriate *scopes*.
- Rule UG: You want to prove ∀xAx. You must prove an arbitrary instance of Ax, with flagging guaranteeing that it is indeed arbitrary. The new vertical line gives the *scope* of the flagging.

In these cases, the new vertical lines mark the *scope* inside which you must generate a subgoal as you work from the bottom up. So never introduce a new vertical line for any other reason. In particular, do *not* introduce vertical lines when using rules which do *not* generate sub-goals—rules such as UI or modus ponens.

3C Quantifier proof theory: Existential quantifier and DMQ

In this section we continue with proof theory, first adding to our battery of techniques a pair of rules governing reasoning with the existential quantifier.

3C.1 Existential quantifier rules (EG and EI)

We rely on Convention **3A-2**, defining At as the result of putting t for all free x in Ax (*proviso*: t must not contain—or be—a variable bound in Ax). We use Ac instead of At when the substituted term must be an individual constant (not complex). We use $Ax_1 \dots x_n$ and $At_1 \dots t_n$ similarly.

The rule of existential generalization, EG, says that given any instance At of Ax, you may derive ∃xAx. EG, like UI, is *not* a failsafe rule.

3C-1 RULE. *(Existential generalization—*EG*)*

```
j | _At
. | .
. | .
k | ∃xAx    j, EG t/x
```

Also the many-variable case:

```
j | _At1...tn
. | .
. | .
k | ∃x1 ...∃xnAx1... xn    j, EG t1/x1, ... , tn/xn
```

Naturally, the premiss for EG does not need to be just above the conclusion; it only needs to be "available" in the technical sense of Definition **2C-6**.

The direction of substitution is, as always, substituting the *constant* for the quantified *variable*. Therefore, in EG the direction of substitution is "uphill." Furthermore, in the many-variable case, the same constant can be substituted for several variables, just as for UI. Here are some examples

```
| _Sab
| ∃xSxb    EG a/x

| _Saa
| ∃xSax    EG a/x

| _Saa
| ∃xSxx    EG a/x

| _Saa
| ∃xSxa    EG a/x

| _F(a+b)((c+d)−e)
| ∃xF(a+b)(x−e)
|             EG (c+d)/x
```

$$\begin{array}{|l} \underline{X \cap Y = (Y \cap X) \cap X} \\ \exists X_1[X \cap Y = X_1 \cap X] \\ \qquad\qquad \text{EG } (Y \cap X)/X_1 \end{array}$$

Observe that from "Alfred shaves Alfred" any one of the following may be correctly deduced by EG: "Someone shaves himself," "Someone shaves Alfred," and "Alfred shaves someone." *That* is why it is critical that the substitution of a for x be "uphill."

The rule of Existential instantiation, EI, is considerably tighter in its requirements, in strict analogy to UG. Furthermore, like UG, EI is a **failsafe** rule. Suppose an existentially quantified sentence ∃xAx is available. For any individual constant c *that meets all the flagging restrictions* (Definition **3B-3** on p. 92), you may write "Ac EI, c/x, flag c."

3C-2 RULE. *(Existential instantiation—*EI*)*

$$\begin{array}{l|l} j & \underline{\exists x A x} \\ \cdot & \quad | \; \cdot \\ \cdot & \quad | \; \cdot \\ k & \text{Ac} \qquad \text{j, EI, c/x, flag c} \end{array}$$

There is also a many-variable version.

$$\begin{array}{l|l} j & \underline{\exists x_1 \ldots \exists x_n A x_1 \ldots x_n} \\ \cdot & \quad | \; \cdot \\ \cdot & \quad | \; \cdot \\ k & Ac_1 \ldots c_n \qquad \text{j, EI } c_1/x_1, \ldots, c_n/x_n, \text{flag } c_1, \ldots, c_n \end{array}$$

It is part of the flagging restrictions that there must be no repetition among the flagged constants $c_1, \ldots, c_n$.

The idea is that we first choose c as entirely arbitrary, and then go on to postulate of it that Ac. *Writing "flag* c*" is part of the rule, and must not be omitted;* otherwise your choice of c as "entirely arbitrary" would be a mere metaphor.

The use of EI in formal proofs answers to the following. "By hypothesis, we know there is an F (i.e., we know that ∃xFx); let us pick one out arbitrarily, say c, about which we will suppose nothing is known except that it is an F (i.e., we suppose that Fc). Now we will push our reasoning further, by whatever means we can. If we can reason to some conclusion that does not involve the constant c, we can be sure that such a conclusion legitimately follows from our hypothesis ∃xFx."

Note. The "Ac" exhibited in EI is not really a conclusion from ∃xAx, for to reason from ∃xFx to Fc is a blunder. Just because there is a fox (∃xFx), it does not follow that Charlie is a fox (Fc). In our system the step "Ac" represents not a conclusion, but rather an auxiliary hypothesis that the arbitrarily chosen c is such that Ac. It is the flagging restrictions that keep us out of trouble when using EI in this way.

As a matter of fact, EI could be replaced by the following rule, ∃elim, which is the rule that Fitch himself preferred:

j	∃xAx		
j+1		Ac	hyp, flag c for ∃elim
.		.	
.		.	
j+n		B	
j+n+1	B		∃elim, j, (j+1)–(j+n) (*provided* c does not occur in B)

Our EI collapses the two vertical lines and the two occurrences of B of ∃elim, which makes it easier to use than Fitch's own preferred rule; and that is why we adopt it in spite of the fact that some Trained Logicians find it not quite "natural."

3C.2 One-place-predicate examples and exercises

Interesting quantifier problems are generally based on predicates with two or more places. Just for a warm-up, however, here are some one-place-predicate examples of proofs involving EG and EI.

3C-3 EXAMPLE. *(Bad answers)*

Bad answers are answers; so, since there is a bad answer, there must be an answer. Dictionary: obvious.

1	∀x(Bx→Ax)	hyp
2	∃xBx	hyp
3	Ba	2 EI a/x, flag a
4	Ba→Aa	1, UI a/x
5	Aa	3, 4,TI
6	∃xAx	5, EG a/x

Note that it was *essential* to do the EI before the UI, on pain of violating the flagging restriction Definition **3B-3**, which says that for "flag a," a must be *new*.

This illustration leads to the following.

3C-4 ADVICE. *(Use* EI *early)*

If you have a premiss ∃xAx, use EI on it *immediately*. It cannot hurt, and it is likely to help you find the best possible proof. EI, like e.g. CP, and UG, is a **failsafe** rule: If *any* proof will get you to your conclusion, then a proof that uses EI correctly (obeying the flagging restrictions) is bound to get you there.

Here are several more examples. Although these problems are all worked out for you, it will improve your mind to go over them sufficiently often that you can do each one yourself, *without peeking*.

3C-5 EXAMPLE. *(Terrible answers)*

Bad answers are answers, and some of them are terrible, so some answers are terrible. Dictionary: obvious.

1	∀x(Bx→Ax)	hyp
2	∃x(Bx&Tx)	hyp
3	Ba&Ta	2 EI a/x, flag a
4	Ba→Aa	1 UI a/x
5	Aa&Ta	3, 4 TI
6	∃x(Ax&Tx)	5 EG a/x

3C-6 EXAMPLE. *(Astronomy)*

Any asteroid is smaller than any quasar. Any asteroid is smaller than no imploded star. There are some asteroids. So no quasar is an imploded star. Dictionary: obvious.

1	∀x(Ax→∀y(Qy→Sxy))	hyp
2	∀x(Ax→∀z(Iz→∼Sxz))	hyp
3	∃xAx	hyp
4	flag b	for UG
5	Aa	3 EI a/x, flag a
6	∀y(Qy→Say)	1 UI a/x, 5 MP
7	∀z(Iz→∼Saz)	2 UI a/x, 5 MP
8	Qb→Sab	6 UI b/y
9	Ib→∼Sab	7 UI b/z
10	Qb→∼Ib	8, 9 TI
11	∀y(Qy→∼Iy)	4, 10 UG b/y

Two steps are collapsed on line 6, and another two on line 7. This is good practice, provided you know what you are doing.

3C-7 EXAMPLE. *(*EI*, etc.)*

1	∃x(Fx & Sx)→∀y(My→Wy)	hyp
2	∃y(My & ∼Wy)	hyp
3	flag a	for UG
4	Mb & ∼Wb	2 EI b/y, flag b
5	∀y(My→Wy)	hyp
6	Mb→Wb	5 UI b/y
7	⊥	4, 6 TI (⊥intro)
8	∼∀y(My→Wy)	5–7 RAA⊥
9	∼∃x(Fx & Sx)	1, 8 TI (MT)
10	∼(Fa→∼Sa)	hyp
11	Fa & Sa	10 TE
12	∃x(Fx & Sx)	11 EG a/x
13	⊥	9, 12 TI (⊥intro)
14	Fa→∼Sa	10–13 RAA⊥
15	∀x(Fx→∼Sx)	UG a/x 3–14

1	$\forall$x(Px→Ax)	hyp
2	flag a	for UG
3	$\exists$y(Py & Hay)	hyp
4	Pb & Hab	3 EI b/y, flag b
5	Pb→Ab	1 UI b/x
6	Ab & Hab	4, 5 TI
7	$\exists$y(Ay & Hay)	6 EG b/y
8	$\exists$y(Py & Hay)→$\exists$y(Ay & Hay)	3–7 CP
9	$\forall$x[$\exists$y(Py & Hxy)→$\exists$y(Ay & Hxy)]	UG a/x 2–8

1	$\forall$x[Ox→$\forall$y(Ry→∼Lxy)]	hyp
2	$\forall$x[Ox→$\exists$y(Hy & Lxy)]	hyp
3	$\exists$xOx	hyp
4	Oa	3 EI a/x, flag a
5	$\forall$y(Ry→∼Lay)	1 UI a/x 4 TI
6	$\exists$y(Hy & Lay)	2 UI a/x 4 TI
7	Hb & Lab	6 EI b/y, flag b
8	Rb→∼Lab	5 UI b/y
9	Hb & ∼Rb	7, 8 TI
10	$\exists$x(Hx & ∼Rx)	9 EG b/x

Exercise 29 *(Quantifier proofs with one-place predicates)*

1. Prove the following.

   (a) $\forall$y(Ay→(By ∨ Cy)), $\forall$x∼Cx, $\exists$xAx ∴ $\exists$xBx

   (b) $\forall$x(Fx→Gx), ∼($\exists$xGx ∨ $\exists$xHx) ∴ ∼$\exists$xFx

   (c) ∼($\exists$xDx ∨ $\exists$x∼Hx)→$\forall$x(Dx→ Hx)

   (d) ∼$\forall$x(Sx→∼Bx), $\forall$x(Sx→(Cx & Dx)) ∴ ∼$\forall$x(Dx→∼Bx)

   (e) ∼$\exists$x(Ax & Bx), $\forall$x(Fx→∼Bx), $\exists$x(Bx & (∼Dx→(Fx ∨ Ax))) ∴ $\exists$x(Dx & ∼Ax)

2. The following forms of argument are counted as valid syllogisms in traditional Aristotelian logic. Prove their validity by constructing a derivation in our natural deduction system. (For Felapton, you will need to add an extra

premise to the effect that there are M's. This expresses the existential import of the explicit premises as understood in traditional logic.)[14]

(a) *Darii*. All M are L, Some S is M $\therefore$ Some S is L.
(b) *Cesare*. No L is M, All S are M $\therefore$ No S is L.
(c) *Felapton*. No M is L, All M are S, (There are M's) $\therefore$ Some S is not L.

3. Suppose we were to enlarge our stable of rules by declaring that $\exists$ is interchangeable in every context with $\sim\forall\sim$. Show that EI and EG would then be redundant.

▷ .... ◁

3C.3 De Morgan's laws for quantifiers—DMQ

In using quantifier rules UI, UG, EI, and EG, it is essential that they be applied only to quantifiers $\forall$x and $\exists$x that occur on the *absolute outside* of a step. All of our rules—especially those concerning flagging—are geared to this expectation. A common and deeply serious error is to think that they might apply when the quantifier is inside a negation as in "$\sim\forall$x" or "$\sim\exists$x." To forestall this error, you need to learn that $\sim\forall$xFx (for instance) is Q-logically equivalent to $\exists$x$\sim$A, and that $\sim\exists$xFx is Q-logically equivalent to $\forall$x$\sim$Fx. We call these equivalences "De Morgan's laws for quantifiers," or "DMQ." You will learn them best if you yourself prove that they follow from UI, UG, EI, and EG together with our truth-functional rules for negation and $\bot$. We show you how the strangest part of the proof goes, but leave you to fill in the justifications. The remaining parts are left as an exercise.

3C-8 EXAMPLE. *(One part of* DMQ*)*

```
 1 | ~∀xFx
 2 | | ~∃x~Fx
 3 | | | flag a        for UG
 4 | | | | ~Fa
 5 | | | | ∃x~Fx
 6 | | | | ⊥
 7 | | | Fa
 8 | | ∀xFx
 9 | | ⊥
10 | ∃x~Fx
```

[14] We explicitly work on symbolization later, in chapter 5. Exercise 29, however, assumes that you have picked up a bit of symbolization from your earlier study of logic. Still, just in case, we mention here that "Some S is not L" in Felapton symbolizes as $\exists$x(Sx & $\sim$Lx).

Exercise 30 *(DMQ)*

1. Fill in the justifications for Example **3C-8**.
2. Prove the following categorically, using only UI, UG, EI, EG, and ⊥/negation rules, and of course BCP.
   (a) $\sim\forall xFx \leftrightarrow \exists x\sim Fx$
   (b) $\sim\exists xFx \leftrightarrow \forall x\sim Fx$

▷ . . . . ◁

After you have established these equivalences, you are entitled to use them:

3C-9 RULE. *(DMQ)*

You can interchange "$\sim\exists$x" with "$\forall$x$\sim$," and you can interchange "$\sim\forall$x" with "$\exists$x$\sim$" in any context. Cite "DMQ" as your reason. Versions with built-in DN, namely, rules permitting the interchange of $\sim\exists\sim$with $\forall$ and of $\sim\forall\sim$with $\exists$, are also called DMQ.

The principal use of the rules DMQ is to move quantifiers to the outside, where you may be guided by the rules UI, UG, EI, and EG, and where you understand exactly when to use flagging—and when *not* to use flagging—without confusion. (The presentation of this rule is repeated below by putting Definition **3E-2** together with Rule **3E-7**.)

3C.4 Two-place (relational) problems

The following problems, chiefly two-place, are more typical of those that you will encounter.

Exercise 31 *(Two-place quantifier proofs)*

Prove the following.

1. $\forall x\forall y Syx \therefore \sim\forall x\exists y\sim Syx$

2. $\forall x\forall y Rxy \therefore \exists y\exists x Rxy$

3. $\exists x(Fx \,\&\, Gx), \sim\exists x(Fx \,\&\, \sim Cxx) \therefore \exists x\exists y Cxy$

4. Use DMQ. $\sim\exists x(Fx \,\&\, Gx), \forall x((Fx \,\&\, \sim Gx)\to\sim\exists y(Tyx \,\&\, Hxy)), \forall x(\forall y(Hxy\to Zxy)\to Gx) \therefore \forall x(Fx\to\sim\forall y(Tyx\vee Zxy))$

5. If there is some (one) dog that all persons love, then each person loves some dog (or other): $\exists x(Dx \,\&\, \forall y(Py\to Lyx)) \to \forall y(Py\to\exists x(Dx \,\&\, Lyx))$

6. $\forall x(Fx\to\exists y(Gy \,\&\, Hyx)), \sim\exists x\exists y Hxy \therefore \sim\exists x Fx$

7. $\sim\exists x(\sim Gx\vee\sim\exists y Fy) \therefore \exists z Fz$

8. $\forall x(Ux\to\exists y(By \,\&\, \sim Cxy)), \forall y(By\to Cay) \therefore \sim Ua$

9. $\forall x(Fx\to\exists y Bxy) \therefore \sim\exists x(Fx \,\&\, \forall y\sim Bxy)$

10. $\forall x(Ux\to Bx), \sim\exists x(Bx \,\&\, Cxx) \therefore \sim\exists x(Ux \,\&\, \forall y Cyx)$

11. $\forall x(Fx\to(\exists y Jxy\to\exists z Jzx)), \forall x(\exists z Jzx\to Jxx), \sim\exists x Jxx \therefore \forall x(Fx\to\forall y\sim Jxy)$

13. You should also be able to prove that if there is a woman whom all male Greeks love (e.g., Helen of Troy), then each male Greek loves some woman (e.g., his mother). Why? Because it has exactly the same form as (5) above.

14. Some people don't own any goats, all kids are goats, so some people don't own any kids. (Use P, O, G, K. O, of course, must be two-place.)

15. A relation, R, is *irreflexive* iff $\forall x\sim Rxx$, and is *asymmetric* iff $\forall x\forall y(Rxy \to \sim Ryx)$. Prove that if R is asymmetric, then it is irreflexive.

3D Grammar and proof theory for identity

From at least one theoretical point of view, identity is just another two-place predicate, but the practice of the art of logic finds identity indispensable. In this section we present its (trivial) grammar and a little proof theory.

The grammar of identity. If t and u are terms, then $t=u$ is a sentence. That is, "_=_" is a predicate. That's it.

Proof theory. There is, as usual, an introduction and an elimination rule. (Most people think that identity is the *only* predicate for which there is a complete set of rules. It is in this misguided sense the only "logical" predicate.)

3D-1 RULE. *(Identity introduction)*

```
 . | .
 . | .
 n | t=t    =int
```

Note that =int, like the rule Taut, involves no premisses at all. Also like Taut it is seldom of use. (But not never!)

3D-2 RULE. *(Identity elimination)*

```
   . | .
   . | .
   j | At
   . | .
   . | .
 j+k | t=u
   . | .
   . | .
   n | Au     j, j+k, =elim
```

or

```
   . | .
   . | .
   j | Au
   . | .
   . | .
 j+k | t=u
   . | .
   . | .
   n | At     j, j+k, =elim
```

where At and Au are exactly alike except that Au contains one or more occurrences of u where At contains t.

More accurately but somewhat more obscurely, we have in mind Convention **3A-2** according to which At must be obtained from Ax by putting t for all free x, and Au from that same Ax by putting u for all free x. For example, let Ax be Fxuxt; then At is Ftutt and Au is Fuuut, which in fact must have the property expressed in the preceding "where"-clause. It is also important that the Convention **3A-2** guarantees that no variable in either t or u becomes bound in the process of substituting for x. (What makes this particular application of the Convention hard to process is the fact that there is no explicit occurrence of Ax in the vicinity.) Another picture of (the first statement of) the rule of identity elimination is this:

·	·
·	·
j	(... t ...)
·	·
·	·
j+k	t=u
·	·
·	·
n	(... u ...) j, j+k, =elim

The easiest way to tell if you have a correct use of =elim is as follows. One of your premisses is the identity t=u. You also have another premiss, (... t ...), and a conclusion, (... u ...). Now see if you can find a way to underline some (or all) of the t's in (... t ...), and some (or all) of the u's in (... u ...), so that "everything else" in (... t ...) and (... u ...) looks exactly the same. This is illustrated below.

At: $F\underline{t}u\underline{t}t$

Au: $F\underline{u}u\underline{u}t$

The idea is that the conclusion must say *exactly* about u what one of the premisses says about t. The circling or underlining makes this manifest.

3D-3 EXAMPLE. *(Identity examples)*

Charlie accepted Mary's offer. Whoever accepted Mary's offer caused a turmoil. None of Mary's friends caused a turmoil, and the famous author, Barbie Dolly, is certainly one of Mary's friends. So Charlie isn't Barbie Dolly.

1	Ac	hyp
2	$\forall$x(Ax$\rightarrow$Tx)	hyp
3	$\forall$x(Mx$\rightarrow\sim$Tx)	hyp
4	Mb	hyp
5	b$=$c	hyp (for RAA$\perp$)
6	Mb$\rightarrow\sim$Tb	3 UI b/x
7	$\sim$Tb	4, 6, TI
8	Ab$\rightarrow$Tb	2, UI b/x
9	$\sim$Ab	7, 8, TI
10	$\sim$Ac	5, 9, =elim
11	$\perp$	1, 10, TI ($\perp$ I
12	$\sim$(b$=$c)	5–11, RAA$\perp$

In the following example, which is a piece of real mathematics, "$\circ$" is used as a two-place operator and a prime is used as a one-place operator (in the sense of §1A). Also "e" is an individual constant.

1	$\forall$x$\forall$y$\forall$z[x$\circ$(y$\circ$z) = (x$\circ$y)$\circ$z]	hyp (associativity)
2	$\forall$x(x$\circ$e$=$x)	hyp (e is an identity)
3	$\forall$x(x$\circ$x$'$ = e)	hyp ($'$ is an inverse)
4	a$\circ$c$=$b$\circ$c	hyp, flag a, b, c, for CP+UG
5	(a$\circ$c)$\circ$c$'$ = (a$\circ$c)$\circ$c$'$	=int
6	(a$\circ$c)$\circ$c$'$ = (b$\circ$c)$\circ$c$'$	4, 5, =elim
7	a$\circ$(c$\circ$c$'$) = (a$\circ$c)$\circ$c$'$	1, UI a/x, c/y, c$'$/z
8	b$\circ$(c$\circ$c$'$) = (b$\circ$c)$\circ$c$'$	1, UI b/x, c/y, c$'$/z
9	a$\circ$(c$\circ$c$'$) = (b$\circ$c)$\circ$c$'$	6, 7, =elim
10	a$\circ$(c$\circ$c$'$) = b$\circ$(c$\circ$c$'$)	8, 9, =elim
11	c$\circ$c$'$$=$e	3, UI c/x
12	a$\circ$e$=$b$\circ$e	10, 11 =elim
13	a$\circ$e$=$a	2, UI a/x
14	a$=$b$\circ$e	12, 13, =elim
15	b$\circ$e$=$b	2 UI, b/x
16	a$=$b	14, 15, =elim
17	$\forall$x$\forall$y$\forall$z[(x$\circ$z = y$\circ$z) $\rightarrow$ (x$=$y)]	5–16, CP+UG x/x, y/y, z/z

Line 17 is called the Cancellation Law. Try this proof tomorrow, without peeking.

The next example outlines what we suppose is one of the most famous of all uses of RAA (here, RAA$\perp$): the ancient Greek proof that $\sqrt{2}$ is irrational. *Note the interplay of arithmetic and logic.* All letters range over positive integers.

1	$\sqrt{2}$ is rational	hyp, for RAA
2	$((m/n \times m/n) = 2))$ & (m, n have no common divisor (except 1))]	1, Def., EI, flag m, n
3	$m^2/n^2 = 2$	2, arith.
4	$m^2 = (2 \times n^2)$	3, arith.
5	2 divides m^2 evenly	4, EG on n^2, Def.
6	2 divides m evenly	4, arith.
7	$(a \times 2) = m$	6, Def., flag a
8	$(a \times 2)^2 = (2 \times n^2)$	4, 7, =
9	$(a \times 2 \times a \times 2) = (2 \times n^2)$	8, arith.
10	$(a \times 2 \times a) = n^2$	9, arith.
11	2 divides n^2 evenly	10, arith., Def.
12	2 divides n evenly	11, arith.
13	$2 \neq 1$ & 2 divides m & 2 divides n	6, 12, arith., Conj
14	m and n have a common divisor (not 1)	14, EG on 2, Def.
15	$\perp$	2, Simp, 14, $\perp$int
16	$\sqrt{2}$ is irrational	1–15, RAA$\perp$

More Rules. The following rules involving identity are all redundant;[15] but they crop up with fair frequency, and their use shortens proofs. You can cite them all as just "=" if you like.

$$\begin{array}{|l} \underline{t = u} \\ u = t \end{array} \quad \text{(i.e., symmetry of identity)}$$

$$\begin{array}{|l} t = u \\ \underline{u = w} \\ t = w \end{array} \quad \text{(i.e., transitivity of identity)}$$

Versions of the above in which the terms are reversed, in either premiss or in the conclusion, are also acceptable. The following equally useful (redundant) identity rules should be seen as the closest of cousins to each other. We'll give them a special name, "Added context," to indicate how they work.

3D-4 RULE. *(Added context)*

[15] Optional: Show this. That is, argue from premisses to conclusion using =int and =elim.

$$
\begin{array}{l|ll}
\vdots & \vdots & \\
1 & \underline{t = u} & \\
\vdots & \vdots & \\
2 & (\ldots t \ldots) = (\ldots u \ldots) & 1, = \text{(Added context)}
\end{array}
$$

$$
\begin{array}{l|ll}
\vdots & \vdots & \\
1 & \underline{t = u} & \\
\vdots & \vdots & \\
2 & (\ldots t \ldots) \leftrightarrow (\ldots u \ldots) & 1, = \text{(Added context)}
\end{array}
$$

Grammar requires that in the first case (. . .t. . .) be a term, but in the second case, (. . .t. . .) must be a sentence. Why? If an answer isn't on the tip of your tongue, see Definition **1A-3** on p. 6. With the grammatical distinction in mind, we might have called the first form "Added operator context," and the second form "Added predicate context." But we choose to leave well enough alone.

Examples.

$$
\begin{array}{l|ll}
1 & \underline{3 = (1+2)} & \text{hyp} \\
2 & 4+3 = 4+(1+2) & 1, = \text{(Added context)}
\end{array}
$$

$$
\begin{array}{l|ll}
1 & \underline{3 = (1+2)} & \text{hyp} \\
2 & ((3 \text{ is odd}) \rightarrow a = b) \leftrightarrow (((1+2) \text{ is odd}) \rightarrow a = b) & 1, = \text{(Added context)}
\end{array}
$$

In the "piece of real mathematics" of Example **3D-3**, the transition from line 4 to line 6 is a case of Added Context. If we had annotated line 6 of Example **3D-3** with "4, Added context," we could have omitted line 5. (Check this.)

Exercise 32 *(Identity proofs)*

Identity is deeply intertwined with operators. In the following exercises, be prepared to use UI and EG with *complex* terms, not just individual constants.

1. Prove that it is a law of logic (not just arithmetic) that adding equals to equals yields equals:

   $$\forall x\forall y\forall w\forall z[(x=y \;\&\; w=z) \rightarrow (x+w=y+z)]$$

   (You will need to use =elim and either =int or Added context.)

2. Prove that it is a law of logic that every two numbers have a sum:

   $$\forall x\forall y\exists z(x+y=z)$$

3. Prove: Mark Twain wrote *Life on the Mississippi*, and he was really Samuel Clemens. Samuel Clemens was once a river pilot. Therefore, a one-time river pilot wrote *Life on the Mississippi*. (Use m, W, l, s, R. W is of course two-place.)

4. Each of the following is a way of saying that there is exactly one F. This is easy to see for (4a), which evidently says "there is at least one F and there is at most one F." Show that (4b) and (4c) say the same thing by means of proof. You can show that all three are equivalent by means of only three (instead of six) hypothetical proofs that "go round in a circle," for example, from (4a) to (4c) to (4b) to (4a). (The order is up to you; any circle will do.) Note: It took Russell to figure out the impenetrable version (4c).

   (a) $\exists xFx \;\&\; \forall x\forall y[(Fx \& Fy)\rightarrow x=y]$
   (b) $\exists x(Fx \& \forall y(Fy\rightarrow x=y))$
   (c) $\exists x\forall y(Fy\leftrightarrow x=y)$

5. Prove the following as theorems.

   (a) $\forall x(f(gx)=x) \leftrightarrow \forall x\forall y(gx=y\rightarrow fy=x)$.
   (b) $\forall x\exists y(fx=y \& \forall z(fx=z\rightarrow z=y))$.

6. Two of the following equivalences are Q-logically valid and two are not. *Prove* two that you believe to be Q-valid.

   (a) $Fa \leftrightarrow \forall x(x=a \;\&\; Fx)$
   (b) $Fa \leftrightarrow \exists x(x=a \;\&\; Fx)$
   (c) $Fa \leftrightarrow \forall x(x=a \rightarrow Fx)$
   (d) $Fa \leftrightarrow \exists x(x=a \rightarrow Fx)$

7. Prove the following.

   (a) $\forall x(g(fx)=x) \;\therefore\; \forall x\forall y(fx=fy\rightarrow x=y)$.

(b) $\forall y(\exists x(fx = y) \rightarrow fy = y), \forall x \forall y(fx = fy \rightarrow x = y) \therefore \forall x(fx = x)$.

8. Optional—but don't count yourself as having mastered quantifier theory unless you can do it. You have a domain *partially ordered* by a binary relation $\leqslant$ when $\leqslant$ satisfies the following three conditions. *Reflexivity*: $\forall x(x \leqslant x)$. *Transitivity*: $\forall x \forall y \forall z((x \leqslant y \,\&\, y \leqslant z) \rightarrow x \leqslant z)$. *Anti-symmetry*: $\forall x \forall y((x \leqslant y \,\&\, y \leqslant x) \rightarrow x = y)$. You have a *lower semi-lattice* when, in addition to the partial order, there is a binary operator $\circ$ on the domain that, for any two elements of the domain, gives their *greatest lower bound* $(x \circ y)$. That is, when the following also hold. *Lower bound for* x *and* y: $\forall x \forall y((x \circ y \leqslant x) \,\&\, (x \circ y \leqslant y))$. *Greatest among such lower bounds*: $\forall x \forall y \forall z((z \leqslant x \,\&\, z \leqslant y) \rightarrow (z \leqslant x \circ y))$. Your job, should you agree to undertake it, is to prove *uniqueness* of the greatest lower bound of x and y in any lower semi-lattice: Given as premisses that your domain is partially ordered by $\leqslant$ and that $(x \circ y)$ is a greatest lower bound of x and y, prove that if anything w is also a greatest lower bound of x and y, then $w = (x \circ y)$.

9. Optional. Prove the following result from group theory. Note the interesting transition from "$\forall\exists$" in the premisses to "$\exists\forall$" in the conclusion. We give an annotated empty "proof form" as an elaborate hint. The least obvious step is step 7.

   (1) $\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$, (2) $\forall x \forall y \exists z(x = y + z)$,
   (3) $\forall x \forall z \exists y(x = y + z) \therefore \exists y \forall x(x + y = x)$.

1		hyp
2		hyp
3		hyp
4		2, UI a/x, a/y
5		4, EI e/z, flag e
6	flag b	for UG
7		3, UI _/x, _/z
8		7, EI c/y, flag c
9		5, 8, =elim
10		1, UI _/x, _/y, _/z
11		9, 10, =elim
12		8, 11, =elim
13		UG 6–12, b/x
14		13, EG e/y
15		14, = (symmetry)

▷ . . . . ◁

3E Comments, rules, and strategies

We collect together some comments, some additional (redundant) rules, and some advice.

3E.1 Flagging—cautions and relaxations

We begin with some cautions, and then indicate how the flagging restrictions can be relaxed a bit. In the first place, observe that there are only two occasions on which to flag:

1. When starting a new vertical line for later use with UG:

```
| .
| .
| .
| | flag a    for UG
| | .
| | .
| | .
```

2. When using EI:

```
| .
| .
| .
| ∃xAx
| Ab      EI b/x, flag b
```

Never flag on any other occasion. In particular, do *not* flag when using UI or EG. The second "caution" is to remember the "flagging made easy" remark in §3B.1: If every time you flag you choose a brand new term that you haven't yet written down *anywhere*, you cannot go wrong. The third "caution" is to keep in mind that the four quantifier rules are *whole line* rules, and cannot be used "inside." Experience teaches the importance of this caution. Sometimes you can move a quantifier to the outside by a Swoosh rule, which you will learn in §3E.3, but this must be inserted as an extra step, and only if you *know* what you are doing. Using quantifier rules on the "inside" is a common mistake, and must be consciously avoided; *use quantifier rules only on the outside*. So much for cautions; now for relaxations.

The easy flagging restriction is stricter than it needs to be. For example, there is really nothing "wrong" with the following proof, even though step 6 violates the easy flagging restriction Definition **3B-3**. (Be sure you can say why, looking up Definition **3B-3** if you do not recall it *exactly*.)

1	$\forall x(Fx \& Gx)$	hyp
2	$\quad$ flag a	for UG
3	$\quad Fa \& Ga$	
4	$\quad Fa$	
5	$\forall xFx$	UG a/x
6	$\quad$ flag a	for UG
7	$\quad Fa \& Ga$	
8	$\quad Ga$	
9	$\forall xGx$	UG a/x

But the above proof satisfies the intermediate flagging restriction, Definition **3B-3**. Usually you will find the original (more strict) easy version easier to remember and easier to check; but feel free to use the more relaxed intermediate version if you like.

3E.2 Universally generalized conditionals

So many interesting sentences have the form $\forall x(Ax \rightarrow Bx)$, or maybe with more universal quantifiers up front, that it is convenient to allow introduction or elimination of such steps with single rules, CP+UG and UI+MP, instead of by means of rules for the conditional combined with rules for the universal quantifier.

3E-1 RULE. *(*CP+UG—*universally generalized conditional)*

The introduction part CP+UG combines CP and UG as follows (only the one-variable case is illustrated, but the many-variable case is also allowed):

$\quad Ab$	hyp, flag b, for CP+UG
$\quad .$	
$\quad .$	
$\quad .$	
$\quad Cb$	
$\forall x(Ax \rightarrow Cx)$	CP+UG b/x

The elimination part UI+MP combines UI and MP:

1	∀x(Ax→Cx)	
2	_At	
3	Ct	1, 2 UI+MP t/x

The elimination part, UI+MP, simply gives a name to the combination of UI and MP. In the introduction part, the vertical lines for the UG and for the CP are collapsed into a single vertical line, with the UG flag directly on the line of the hypothesis. The proof using CP+UG above then replaces the one below.

```
|  | flag b        for UG
|  |  |_Ab         hyp
|  |  |  .
|  |  |  .
|  |  |  .
|  |  |  Cb
|  | Ab→Cb         CP
| ∀x(Ax→Bx)        UG
```

Example using CP+UG:

1	_∀x(Hx→Ax)	hyp
2	| _∃x(Hx&Tax)	hyp, flag a for CP+UG
3	| Hb&Tab	2 EI b/x, flag b
4	| Hb→Ab	1 UI b/x
5	| Ab&Tab	3, 4 TI
6	| ∃z(Az&Taz)	5 EG b/z
7	∀y[∃x(Hx&Tyx)→∃z(Az&Tyz)]	2–6, CP+UG a/y

Having established line 7, one could use it with UI+MP as follows:

1	∃x(Hx&Tcx)	premiss
2	∃z(Az&Tcz)	1, line 7 above, UI+MP c/y

One might well consider adding also special rules for "universally generalized *bi*conditionals," namely BCP+UG and UI+MPBC, since they crop up frequently indeed. Definitions usually have this form. If you wish to use such a rule, by all means do so. Here is how the introduction part BCP+UG could go:

```
|  | Ac            hyp. flag c for BCP+UG
|  | .
|  | .
|  | Bc
|
|  | Bc            hyp. flag c for BCP+UG
|  | .
|  | .
|  | Ac
| ∀x(Ax↔Bx)      BCP+UG c/x
```

We spell out the details of the "elimination" half separately below since a version of "UI+MPBC" that we call "UI+RE" (UI + replacement of equivalents) can be used not only as a "whole line" rule, but as a rule permitting *replacement* in the middle of sentences. See Rule **3E-7**. Even more important are the special cases to be used in connection with definitions, Def. int and Def. elim (Rule **3E-8** and Rule **3E-9**).

3E.3 Replacement rules

There are five (and *only* five) rules that allow manipulation of the *insides* of formulas. One of them is TE (tautological equivalence), which we have already had, in Rule **2C-8**. We call three more, all together, "the quantifier equivalences." Here they are, one by one.

3E-2 DEFINITION. *(*DMQ *equivalence)*

We say that formulas on the same line below are DMQ-*equivalent* (equivalent by "De Morgan's laws for quantifiers"):

$\sim\forall xA$	$\exists x\sim A$
$\sim\exists xA$	$\forall x\sim A$
$\sim\forall x\sim A$	$\exists xA$
$\sim\exists x\sim A$	$\forall xA$

These are generalizations of De Morgan's Laws if the universal quantifier is thought of as a generalized conjunction, and the existential quantifier as a generalized disjunction. They are used to "move quantifiers to the outside," where we can work with them by the generalization and instantiation rules UI, UG, EI, and EG. We presented them earlier as Rule **3C-9** so that you could start using them right away; here we merely re-present them in a suitable context.

3E-3 DEFINITION. *(Swoosh equivalence)*

We also say that formulas on the same line below are *swoosh-equivalent*, given the important stated proviso.

Proviso: B must not contain x free![16]

$\forall x Ax \& B$	$\forall x(Ax \& B)$
$\exists x Ax \& B$	$\exists x(Ax \& B)$
$\forall x Ax \vee B$	$\forall x(Ax \vee B)$
$\exists x Ax \vee B$	$\exists x(Ax \vee B)$

The above represent various connections between the quantifiers and conjunction and disjunction. They are used to "move quantifiers to the outside" where they can be worked on by UI, UG, EI, and EG.

Also, pairs of formulas that are like the above, but with left-right of the & or $\vee$ interchanged, are *swoosh-equivalent*.

Furthermore, the following, which represent connections between quantifiers and the conditional, are also *swoosh-equivalent*, with the same proviso: B must not contain x free.

$\forall x Ax \rightarrow B$	$\exists x(Ax \rightarrow B)$
$\exists x Ax \rightarrow B$	$\forall x(Ax \rightarrow B)$
$B \rightarrow \forall x Ax$	$\forall x(B \rightarrow Ax)$
$B \rightarrow \exists x Ax$	$\exists x(B \rightarrow Ax)$

Notice and remember that when the quantified formula is in the *antecedent* of the conditional, the quantifier must be *changed*—just as in DMQ; furthermore, the reason is the same here as it is for DMQ: The antecedent of a conditional is really a "negative" position, as can be inferred from the equivalence between $A \rightarrow B$ and $\sim A \vee B$. Swoosh equivalences are more memorable than you might think. In *every* case one starts with a quantified formula to either the left or right of a connective,

[16]The proviso is particularly relevant when the displayed formulas are subformulas lying inside quantifiers that might bind variables free in B.

and then moves the quantifier to the outside—*changing* it if moving from the left of an arrow.

The last equivalence is only seldom useful—it allows us to exchange one bound variable for another, provided we are careful. Its chief use is to prepare for application of swoosh-equivalence.

3E-4 DEFINITION. *(*CBV *equivalence)*

Formulas on the same line below are defined as CBV-*equivalent* ("equivalent by change of bound variable") provided that x occurs free in Ax *precisely* where y occurs free in Ay:

$\forall$xAx $\forall$yAy
$\exists$xAx $\exists$yAy

Given Convention **3A-2**, this means not only that Ay is the result of putting y for all free x in Ax, without any such y becoming bound in the process (so much is part of the Convention), but also that y is not free in Ax. It follows that x is free in Ax just where y is free in Ay, so that (and this is the intuitive point) Ax says about x precisely what Ay says about y; so that the quantified formulas must say the same thing.

Now for the quantifier equivalence rules stated altogether.

3E-5 RULE. *(Rules* DMQ*,* Swoosh*, and* CBV*)*

If (. . .A. . .) is "available" at a given step, and if A and B are

- DMQ-equivalent (DMQ), or
- swoosh-equivalent (Swoosh), or
- equivalent by change of bound variable (CBV)

then (. . .B. . .) may be inferred—giving as a reason one of DMQ, Swoosh, or CBV, together with the step-number of (. . .A. . .).

You are reminded that the purpose of all these quantifier equivalences is to permit "moving the quantifiers to the outside" where they can be worked on by UI, UG, EI, and EG.

3E-6 EXAMPLE. *(Swoosh and* CBV*)*

The following illustrates why you must sometimes precede swoosh by change of bound variable (CBV), and why there is no simple swoosh equivalence for equivalence ($\leftrightarrow$).

1	$\forall xFx \leftrightarrow B$	
2	$(\forall xFx \rightarrow B) \& (B \rightarrow \forall xFx)$	1, TE
3	$\exists x(Fx \rightarrow B) \& \forall x(B \rightarrow Fx)$	2, swoosh (twice)
4	$\exists x[(Fx \rightarrow B) \& \forall x(B \rightarrow Fx)]$	3, swoosh
5	$\exists y[(Fy \rightarrow B) \& \forall x(B \rightarrow Fx)]$	4, CBV
6	$\exists y \forall x[(Fy \rightarrow B) \& (B \rightarrow Fx)]$	5, swoosh

Exercise 33 *(*DMQ, Swoosh*, and* CBV *)*

With rules DMQ (in full generality), Swoosh, and CBV newly at your disposal, prove the following.

1. $\exists x(Gx \rightarrow Haa)$, $\sim\exists xFx$, $\forall x(Gx \vee Fx)$ $\therefore$ $\exists yHyy$
2. $\forall x((Fx \& Gx) \& \exists y(Ryy \& Gb))$ $\therefore$ $\sim\forall x\sim Rxx$

▷ . . . . ◁

The fifth rule is called "replacement of equivalents."

3E-7 RULE. *(Replacement of equivalents)*

There are two versions.

1. **Replacement of equivalents** (or **RE).** The plain-biconditional based version is this: If (. . .A. . .) and $A \leftrightarrow B$ are each "available" at a given step, then (. . .B. . .) may be inferred, giving "RE" or "repl. equiv." as the reason. Observe that whereas in the other of the five replacement rules, A and B are known to be Q-*logically* equivalent in one of four special ways, here in this fifth rule it is only a matter of $A \leftrightarrow B$ being available.

2. **UI+RE.** The second form of replacement of equivalents permits the available equivalence to be universally quantified, and so we call it "UI+RE." If $(\text{—}At_1\ldots t_n\text{—})$ and $\forall x_1\ldots\forall x_n(Ax_1\ldots x_n\leftrightarrow Bx_1\ldots x_n)$ are each "available," then you are entitled to infer $(\text{—}Bt_1\ldots t_n\text{—})$; recalling that the At/Ax Convention **3A-2** forbids certain troublesome cases. The rule is also intended to permit replacement of $Bt_1\ldots t_n$ by $At_1\ldots t_n$. Cite the rule UI+RE.

The rule UI+MPBC (Advice **3B-16** on p. 103) is merely a "whole line" special case of UI+RE. Each is worth its weight in horseshoes when your inferences are being carried out in the context of a perhaps large number of available definitions in the sense of §8B below, which nearly always have the form of universally quantified biconditionals. Whenever such a definition is available as a premiss, then *in any context* you can replace an instance of the left side, which will be headed by the predicate constant serving as the *definiendum*, by a matching instance of the *definiens* (right side), and vice versa. (See Definition **8B-1**.)

In fact, this is so important that we list a third and fourth form of the rule. The underlying assumption is that definitions falling under these forms have the form

$$\forall x_1\ldots\forall x_n(Rx_1\ldots x_n\leftrightarrow Ax_1\ldots x_n), \qquad (*)$$

with the left side being headed by the *definiendum* (the defined) and the right side being the *definiens* (the defining). Note that the left side must be a predication, not an arbitrary sentence, in order for this to count as a definition. The predicate constant, R, that heads the left side is called the *definiendum* because it is what is being defined.

Then the two "new" rules are "Def. introduction" and "Def. elimination."

3E-8 RULE. *(Def. introduction)*

Given that the definition (*) is available, if a *definiens*-instance is available in some context, as in

$$\text{—}At_1\ldots t_n\text{—},$$

then you may infer the matching left-side-instance (which will be headed by the *definiendum*, R) in exactly the same context, as in

—$Rt_1 \ldots t_n$—,

by the rule Def. introduction. The proper annotation on the conclusion is "R introduction" or "R int," where R is the *definiendum* (the predicate letter being defined).

3E-9 RULE. *(Def. elimination)*

Given that the definition (*) is available, if a left-side-instance (which will be headed by the *definiendum*, R) is available in some context, as in

—$Rt_1 \ldots t_n$—,

then you may infer the matching *definiens*-instance in exactly the same context, as in

—$At_1 \ldots t_n$—,

by the rule Def. elimination. The proper annotation on the conclusion is "R elimination" or "R elim," where R is the *definiendum*.

You will avoid considerable muddle if in applying these rules you take care to insist that all of the terms t_i are *closed*. Otherwise, errant binding of variables may lead you far astray. Also your definitions (*) must satisfy the conditions 8B.1. It is a good idea to peek at these now.

3E-10 EXAMPLE. *(Def. introduction and Def. elimination)*

Given the two-place addition *operation*, $x+y$, you might define a three-place sum *relation*, to be read "x and y sum to z," as follows.

$$\forall x \forall y \forall z[\Sigma(x, y, z) \leftrightarrow (x+y)=z]. \qquad \text{(Def. } \Sigma\text{)}$$

Suppose you have commutativity for + as an axiom. Then a proof that Σ inherits this property might go as follows.

1	$\forall x \forall y(x+y = y+x)$	hyp
2	$\Sigma(a, b, c)$	hyp, flag a, b, c for UGC
3	$a+b = c$	2, Σ elim, a/x, b/y, c/z
4	$b+a = c$	1, UI, a/x, b/y, 3, =
5	$\Sigma(b, a, c)$	4, Σ int, b/x, a/y, c/z
6	$\forall x \forall y \forall z[\Sigma(x, y, z) \rightarrow \Sigma(y, x, z)$	2–5, UGC, a/x, b/y, c/z

Exercise 34

(Simple Def. int and Def. elim)

Using Def. Σ from Example **3E-10**, and the arithmetic fact that $\forall x(x+0 = x)$. prove that $\Sigma(3, 0, 3)$.

▷ . . . . ◁

3E.4 Strategies again

This section is a repetition of section §2C.8, with strategies for the quantifiers and Definition introduction and elimination added in their proper places. In constructing proofs in system Fi, always remember that you are working from both ends towards the middle. You must work stereoscopically, keeping one eye on the premisses you have that are available, and the other eye on the conclusions to which you wish to move. Make a habit of putting a *question mark* on any conclusion for which you do not *yet* have a reason. (You should at any stage already have a reason for each of your premisses, and perhaps for some of your conclusions.) Divide, then, the steps you have so far written down into those towards the top—your premisses—and those towards the bottom which are questioned—your desired conclusions. Then use the following strategies.

3E-11 STRATEGIES. *(For **conclusions** (questioned items) at bottom)*

When your desired (not yet obtained) conclusion (bearing a question mark) is:

$Rt_1 \ldots t_n$, where R is a defined predicate constant; let **D** be the universally quantified biconditional that is announced as the definition of R. Try to prove the appropriate matching instance of the right side (the *definiens* side) of **D**, and then obtain from this the desired *definiendum*, $Rt_1 \ldots t_n$, by Def. int (you annotate with "R int"). **Failsafe.**

A & B. Try to prove A and B separately, and then use Conj, that is, &int. **Failsafe.**

A ∨ B. In some cases you will see that either A or B is easily available, and so you can use Addition, that is, ∨int. (Not failsafe.) Otherwise, it is always safe and usually efficient to rely on the fact that disjunctions and conditionals are "exchangeable." First assume ∼A and try to obtain B. Then, as an intermediate step, derive ∼A → B by CP. Then obtain the desired A ∨ B by Conditional Exchange. **Failsafe.**

A → B. It is always safe and usually efficient to try to obtain B from A, and then use CP, that is, →int. **Failsafe.**

A ↔ B. Try to obtain each of A and B from the other, and then use BCP, that is, ↔I. **Failsafe.**

∀xAx. Obtain by UG. That is, try to obtain At within the scope of a "flag t," where you pick t so as to adhere to the flagging restrictions. **Failsafe.**

∃xAx. *No* very nice strategy is available. First, look around for some At on which to use EG. Not failsafe. If this doesn't seem to get you anywhere, try obtaining ∃xAx by RAA⊥ as indicated just below.

∼A. If A is *complex*, obtain from an equivalent formula with negation "driven inward" by the appropriate "Rule for Negated Compounds." If A is *simple*, try RAA⊥ (regular version): Assume A and try to derive ⊥. **Failsafe.**

A. If nothing else works, try RAA⊥ : Assume ∼A and try to reason to ⊥. Failsafe, but desperate and to be avoided if possible.

⊥. About the only way to get ⊥ is to find some (any) kind of tautological inconsistency in your available steps, and use the special case ⊥int of TI. So if your desired conclusion is ⊥, turn your attention immediately to your premisses, as below. **Failsafe.**

3E-12 STRATEGIES. *(For **premisses** (already justified items) at top)*

When among your premisses (item has already been given a definite reason) you have

$Rt_1 \ldots t_n$, where R is a defined predicate constant; let **D** be the universally quantified biconditional that is announced as the definition of R. Use Def. elim (you annotate with "R elim") to obtain the appropriate matching instance of the right side (*definiens*) of **D**. **Failsafe.**

$A \& B$. Derive A and B separately by Simp, that is, &elim—or anyhow look around to see how A or B or both can be used. **Failsafe.**

$A \to B$. Look for a modus ponens (A) or a modus tollens ($\sim$B). Sometimes you need to generate A as something you want to prove (at the bottom, with a question mark). Not failsafe (but a really good idea).

$A \leftrightarrow B$. Look for an MPBC (A or B) or an MTBC ($\sim$A or $\sim$B). Not failsafe (but a good idea).

$A \vee B$. Look for a disjunctive syllogism: a $\sim$A, a $\sim$B, or anyhow a formula differing from one of A or B by one negation. This will often work. Not failsafe. But suppose it doesn't. Then consider that you not only have $A \vee B$ as a premiss; you also have a desired conclusion, say C. Try to derive C from each of A and B, and use Case argument, CA. **Failsafe.**

$\exists x Ax$. Use EI. This is always a good idea. When you use EI, you must flag, choosing t so as to adhere to the flagging restrictions. Be sure to use EI before using UI in order to avoid foolish violation of the flagging restrictions. **Failsafe.**

$\forall x Ax$. Of course you will look around for appropriate UI. Not failsafe. The important thing is to *delay* using UI until after the relevant EI's and getting-set-for-UG's, because of the flagging restrictions. After thus delaying, look around for plausible instantiations, remembering that *any* are allowed, *without* restriction—except of course the restriction built into Convention **3A-2**.

$\sim$A. If A is *complex*, "drive negation inward" by a Rule for Negated Compounds. **Failsafe.** If A is *simple*, hope for the best.

$\perp$. It doesn't often happen that you have awful $\perp$ as a premiss. If you do, you can take yourself to have solved your problem, since *any* (every) desired conclusion is obtainable from $\perp$ by the special case $\perp$E of TI. **Failsafe.**

3E-13 EXAMPLE. *(Strategy example)*

To prove: $\forall x(Gx \leftrightarrow Hx)$ $\therefore$ $\exists x(Fx \& Gx) \leftrightarrow \exists x(Fx \& Hx)$. You should begin by employing the strategy for a biconditional as conclusion.

```
1 | ∀x(Gx↔Hx)                         hyp
  | ----------
2 | | ∃x(Fx&Gx)                       hyp
  | | ----------
  | |
  | |
  | | ∃x(Fx&Hx) ?
  |
  | | ∃x(Fx&Hx)                       hyp
  | | ----------
  | |
  | |
  | | ∃x(Fx&Gx) ?
  | ∃x(Fx&Gx)↔∃x(Fx&Hx) ?            BCP
```

Now you have both universal and existential claims as premisses. You should delay using UI until after you have used EI.

```
1 | ∀x(Gx↔Hx)                         hyp
  | ----------
2 | | ∃x(Fx&Gx)                       hyp
  | | ----------
3 | | Fa&Ga                           2, EI a/x, flag a
  | |
  | |
  | | ∃x(Fx&Hx) ?
  |
  | | ∃x(Fx&Hx)                       hyp
  | | ----------
  | | Fb&Hb                           EI b/x, flag b
  | |
  | |
  | | ∃x(Fx&Gx) ?
  | ∃x(Fx&Gx)↔∃x(Fx&Hx) ?            BCP
```

At this point, you should be able to do appropriate universal instantiations and finish the proof.

1	$\forall x(Gx \leftrightarrow Hx)$	hyp
2	$\exists x(Fx \& Gx)$	hyp
3	$Fa \& Ga$	2, EI a/x, flag a
4	$Ga \leftrightarrow Ha$	1, UI a/x
5	Ha	3, 4 MPBC
6	$Fa \& Ha$	3, 5 Conj
7	$\exists x(Fx \& Hx)$?	6, EG a/x
8	$\exists x(Fx \& Hx)$	hyp
9	$Fb \& Hb$	8, EI b/x, flag b
10	$Gb \leftrightarrow Hb$	1, UI, b/x
11	Gb	9, 10 MPBC
12	$Fb \& Gb$	9, 11 Conj
13	$\exists x(Fx \& Gx)$?	12, EG b/x
14	$\exists x(Fx \& Gx) \leftrightarrow \exists x(Fx \& Hx)$?	BCP(2–7, 8–13)

Exercise 35 *(Strategy exercises)*

Use the strategies in solving the following problems. (Be self-conscious; the point is to *use the strategies*.) [Apology: These exercises are not well chosen for their purpose. Request: Make up one or more problems that illustrate the use of strategies *not* naturally used in the following. Also there should be a problem or two involving identity.]

1. $\forall xGx \& Fa \therefore \exists yFy \& Ga$
2. $\sim\exists y(\sim Fy \vee \sim\exists zGz) \therefore Fa \& \exists zGz$
3. $\sim\forall x(Fx \rightarrow (Gx \vee Hx)) \therefore \sim\forall y(Gy \leftrightarrow Fy)$
4. $\forall x(((Gx \rightarrow Fx) \rightarrow Gx) \rightarrow Gx)$
5. $\exists xFx \vee (\sim\forall xPx \vee \forall xQx), \exists xFx \rightarrow \sim\exists xGx \therefore \forall xPx \rightarrow (\exists xGx \rightarrow \forall xQx)$
6. $\exists x\sim(Gx \leftrightarrow Hx) \therefore \exists y(Gy \vee Hy)$
7. $\exists xFx \rightarrow \forall y(Gy \rightarrow \sim Hy), \exists y(\sim\forall zJz \rightarrow Fy) \therefore \sim Jc \rightarrow \forall y(\sim Gy \vee \sim Hy)$

▷ . . . . ◁

3E.5 Three proof systems again

Since we have added the universal and the existential quantifiers to our grammar, you would expect that we would be adding four rules to each of the three systems described in §2C.9, and of course we do.

- UI is the intelim rule for universally quantified sentences as premisses.
- UG is the intelim rule for universally quantified sentences as conclusions.
- EI is the intelim rule for existentially quantified sentences as premisses.
- EG is the intelim rule for existentially quantified sentences as conclusions.

Thus, the system of intelim proofs is enriched by these four rules, and so accordingly is the system of strategic proofs and the system Fi of wholesale proofs. Where should we put the quantifier replacement rules DMQ, Swoosh, CBV, and replacement of equivalents of §3E.3? Evidently they aren't intelim, but are they strategic or wholesale? We think of them as "strategic"; so these three rules can and ought to be used in strategic proofs. That leaves TI, TE, and Taut as the only difference between strategic proofs and (wholesale) proofs in Fi. The upshot is then this.

3E-14 DEFINITION. *(Proof systems for quantifiers)*

Proof in Fi**.** All rules are usable here. This means that you can use:

- the structural rules hyp and reit (and availability),
- any truth-functional rule (especially the *wholesale* rules TI, TE, and Taut),
- the four intelim quantifier rules UI, UG, EI, EG,
- the identity rules, and
- the quantifier equivalences DMQ, Swoosh, CBV, and replacement of equivalents, including UI+RE.

This is the most useful system of logic. It combines delicate strategic considerations with the possibility of relaxed bursts of horsepower. (Sometimes we use "wholesale proof" as a synonym for "proof in Fi.")

Strategic proof. You can employ any rule mentioned in §3E.4 on strategies *except* the wholesale rules. More precisely, you can use:

- the structural rules
- the intelim rules (including UI, UG, EI, EG),
- the mixed rules,
- the negated-compound rules,
- and the quantifier equivalences DMQ, Swoosh, CBV, and replacement of equivalences

You cannot, however, use the wholesale rules.

Intelim proof. Here only the "intelim" rules may be employed. (The mixed rules, negated-compound rules, wholesale rules, and quantifier equivalences are off limits.) In addition to the truth-functional intelim rules described in §2C.9 on p. 81, this means only UI, UG, EI, EG, =int, =elim; and of course the structural rules.

Exercise 36 *(Intelim and Replacement Rules)*

You already established half of the DMQ equivalences in connection with Exercise 30 on p. 115. Show that the rest of the DMQ equivalences (those involving two signs of negation) and the Swoosh equivalences laid down in Definition **3E-3** on p. 128 as the basis for the respective replacement rules can be established *by using intelim rules alone*; this will give you the best possible feel for how the various quantifier intelim rules interact; that is, give intelim proofs of the following, assuming that B does not contain x free. You really should do all of these; but, at a minimum, you should do those marked with an asterisk (*) or a double asterisk (**). Those with a double asterisk are of most philosophical-logical importance.

1. ** $\sim\forall x{\sim}Ax \leftrightarrow \exists xAx$
2. $\sim\exists x{\sim}Ax \leftrightarrow \forall xAx$
3. * $(\forall xAx \,\&\, B) \leftrightarrow \forall x(Ax \,\&\, B)$
4. $(\exists xAx \,\&\, B) \leftrightarrow \exists x(Ax \,\&\, B)$
5. **Tricky**
   $(\forall xAx \vee B) \leftrightarrow \forall x(Ax \vee B)$
6. * $(\exists xAx \vee B) \leftrightarrow \exists x(Ax \vee B)$
7. **Tricky**
   $(\forall xAx \to B) \leftrightarrow \exists x(Ax \to B)$
8. ** $(\exists xAx \to B) \leftrightarrow \forall x(Ax \to B)$
9. $(B \to \forall xAx) \leftrightarrow \forall x(B \to Ax)$
10. **Tricky**
    * $(B \to \exists xAx) \leftrightarrow \exists x(B \to Ax)$

▷ . . . . ◁

Exercise 37 *(Quantifier exercises)*

You should be ready for the following exercises. In each case, construct a proof showing Q-validity of the indicated argument. [Apology: These exercises are ill-chosen. You can improve them by insisting that you use the Swoosh and DMQ rules as much as possible.]

1. $\exists x\forall y(\exists zFyz\rightarrow Fyx)$, $\forall y\exists zFyz$ $\therefore$ $\exists x\forall yFyx$
2. $\forall x(\exists yGyx\rightarrow\forall zGxz)$ $\therefore$ $\forall y\forall z(Gyz\rightarrow Gzy)$
3. $\forall x(Hax\rightarrow Ixb)$, $\exists xIxb\rightarrow\exists yIby$ $\therefore$ $\exists xHax\rightarrow\exists yIby$
4. $\forall x(Ax\rightarrow\forall y(By\rightarrow Gxy))$, $\exists x(Ax\,\&\,\exists y{\sim}Gxy)$ $\therefore$ $\exists x{\sim}Bx$
5. $\exists x(Jx\,\&\,\forall y(Ky\rightarrow Lxy))$ $\therefore$ $\forall x(Jx\rightarrow Kx)\rightarrow\exists y(Ky\,\&\,Lyy)$
6. $\forall x((Bx\,\&\,\exists y(Cy\,\&\,Dyx\,\&\,\exists z(Ez\,\&\,Fxz)))\rightarrow\exists wGxwx)$, $\forall x\forall y(Hxy\rightarrow Dyx)$, $\forall x\forall y(Fxy\rightarrow Fyx)$, $\forall x(Ix\rightarrow Ex)$
   $\therefore$ $\forall x(Bx\rightarrow((\exists y(Cy\,\&\,Hxy)\,\&\,\exists z(Iz\,\&\,Fxz))\rightarrow\exists u\exists wGxwu))$

▷ . . . . ◁

Chapter 4

A modicum of set theory

We explain a portion of the theory of sets, going a little beyond Symbolism **1B-10**.

4A Easy set theory

We call this section "Easy set theory," or "EST." One purpose is simply to give you some practice with quantifier proofs in a serious setting ("real" constants, not mere parameters). We use truth functions and quantifiers to explain the empty set $\varnothing$, the subset relation $\subseteq$, the proper-subset relation $\subset$, and three set-set operators, intersection $\cap$, union $\cup$, and set difference $-$. We will return to this topic from time to time, as need dictates. What we shall do in this section is to set down the precise axioms and definitions required to give proofs, in quantifier logic, using the ideas of set theory that we introduce. The following two conventions now become relevant.

4A-1 CONVENTION. *(Types of numbered statement)*

In addition to exercises, examples, facts, conventions and definitions, we will now have the following types of numbered statements:

Axiom. An axiom is a clear-cut, formalizable postulate that is to be *used* in proving something. Though most are stated in (middle) English, *all* are to be thought of as cast in the language of quantifiers and truth functions. Refer to an axiom by name or number.

Variant. A variant is a definition that is so trivial that it doesn't even need reference.

Theorem. Theorems are statements that follow from axioms and definitions (sometimes via conventions and variants). Unless we explicitly mark the proof of the theorem otherwise, "follow" means: follows by the elementary truth-functional and quantifier techniques you already know from axioms and definitions explicitly declared in these notes.

Fact. A fact is a theorem that isn't so important.

Corollary. A corollary is a theorem that can be proved in an easy way from what closely precedes it; for this reason it usually bears no indication of proof.

4A-2 CONVENTION. *(Omitted universal quantifiers)*

We repeat Convention **1B-20**: We feel free to drop *outermost* universal quantifiers from what we say. So if we use a variable without binding it, you are to supply a universal quantifier whose scope is our entire statement.

Fundamental primitives of set theory: sethood and membership. The primitives we require from set theory are numerous. If our task were to understand the concepts of set theory itself instead of those of logic, we should start with just one or two and show how the rest could be defined in terms of those; as it is, we just permit ourselves whatever seems best—making, however, an exhaustive effort to keep track of precisely to which concepts with which properties we are helping ourselves. The initial and most fundamental primitives we require from set theory are two, both of which we introduced in Symbolism **1B-10**. The first of these is the one-place predicate, "_ is a set."

4A-3 CONVENTION. *(X and Y for sets)*

Let us repeat the content of Convention **1B-11**: We will use X, Y, etc., for sets. In contrast, we will use x, y, etc. as general variables. Thus, when we say or imply (by Convention **4A-2**) "for all X," we mean "for all X, if X is a set then"; and when we say "for some X" we mean "there is an X such that X is a set and."

The second most fundamental primitive is set membership, carried by the two place predicate, "_ $\in$ _," read "_ is a member of (the set) _." We often use $\notin$ for its negation; that is,

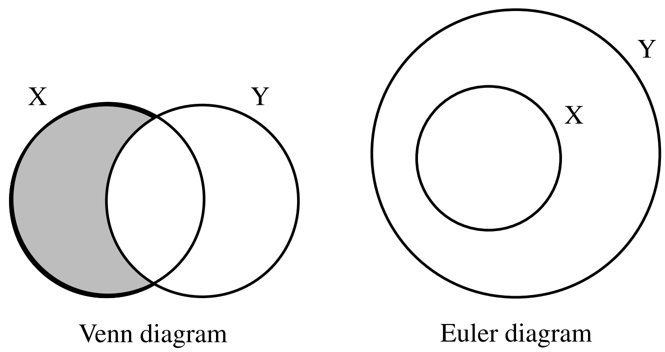


Figure 4.1: Euler and Venn diagrams of the subset relation

4A-4 VARIANT. *(Nonmembership)*

$z \notin X \leftrightarrow_{df} \sim(z \in X)$.

Subset. A good deal of set theory is definitional. The first two definitions are particularly straightforward. One set is a *subset* of another if all the members of the set are also members of the second, and is a *proper subset* if in addition the second set contains something that the first does not. There are two standard pictures of the subset relation, as indicated in Figure 4.1, a Euler diagram and a Venn diagram. The Euler diagram shows the subset X nested inside the superset Y, whereas the Venn diagram shows that the region of X outside of Y is empty, so that *a fortiori* every member of X is bound to be in Y.

4A-5 DEFINITION. *(Subset)*

$\forall X_1 \forall X_2[X_1 \subseteq X_2 \leftrightarrow \forall z[z \in X_1 \rightarrow z \in X_2]]$ (Def. $\subseteq$)

Strictly speaking this definition is not quite "proper," since it involves our conventional use of "X_1" and "X_2" as ranging over sets, but it is better at this stage if you ignore this delicacy and just treat Definition **4A-5** as a definition having the form of a universally quantified biconditional.

Exercise 38 *(Subsets)*

Show how to use Definition **4A-5** to prove the following theorems of set theory. Remark: You do not need to write down the definition of the subset relation as a step in your proof. Just refer to it by number (namely, "Definition **4A-5**") or by name (namely, "Def. $\subseteq$").

1. $\forall X_1[X_1 \subseteq X_1]$ (*reflexivity* of subset relation)
2. $\forall X_1 \forall X_2 \forall X_3[(X_1 \subseteq X_2 \;\&\; X_2 \subseteq X_3) \rightarrow X_1 \subseteq X_3]$ (*transitivity* of subset relation)

▷ . . . . ◁

Using definitions. Here is a "hint" for carrying out Exercise 38. It expands on the discussion of Example **3B-15** and of the rules Rule **3E-8** and Rule **3E-9** on p. 132.

4A-6 HINT. *(Using definitions)*

Whenever you are given a problem involving a defined expression, you *must* be ready to use the definition in the proof.[1] Since the definition will be a universally quantified biconditional, how to use the definition is automatic: You instantiate on those outermost universal quantifiers, and then, having obtained the instance of the definition that you want, you make a trade between the left side and the right side of the biconditional. That's hard to say in words, but it is so important that it is worth your while becoming crystal clear on the matter. Here are two key examples.

Suppose first you have a proof in which a premiss or some other already-justified line is "$X \subseteq Z$":

.	.	
n	$X \subseteq Z$	justified somehow

[1]Must? Yes, if you are thinking straight. Euclid was not being perfectly clear-headed when giving his famous "definition" of a point as *that which is without parts and without magnitude*. You yourself can tell that something has gone awry when you learn the striking fact that *nowhere* does Euclid appeal to this "definition" in proving his geometrical theorems! Similarly, if you come across a piece of philosophy or sociology or physics in which some "definition" is never *used*, you can be confident that something is not quite right. (It might be only that a helpful partial explanation of meaning has been mislabeled as a "definition." But it could be much worse.)

You continue as follows:

.	.	
n	$X \subseteq Z$	justified somehow
n+1	$\forall z[z \in X \rightarrow z \in Z]$	n, $\subseteq$ elim, X/X_1, Z/X_2

Be careful to observe, incidentally, that you must *not* instantiate the quantifier "$\forall z$" as part of step n+1. Go back and look at Definition **4A-5**: The quantifiers "$\forall X_1$" and "$\forall X_2$" govern the *entire* biconditional, whereas, in contrast, "$\forall z$" is just part of the right side (the *definiens*) of the biconditional. (Of course you can *now* use step n+1 in any way you like, but you need to record that use as a further step.)

Second, suppose that "$Z \subseteq W$" is a goal or subgoal marked with a question mark, so that it is something that you want to prove (rather than something already justified that can be used as a premiss).

m	·	justified somehow
	$Z \subseteq W$	**?**

You just work the definition in reverse, writing down the right side (the *definiens*) of the instantiated biconditional as what you now want to prove:

m	·	justified somehow
k	$\forall z[z \in Z \rightarrow z \in W]$	**?**
k+1	$Z \subseteq W$	k, $\subseteq$ int, Z/X_1, W/X_2

(The question mark on line k indicates that you still have work to do.) The application of this "hint" to Exercise 38 is straightforward: Every time you have $X \subseteq Y$ available, you apply $\subseteq$ elim, going on from there. And every time you want to infer to $X \subseteq Y$, you set up to obtain it by $\subseteq$ elim; in other words, write down its *definiens* as a new goal, from which you will infer the desired $X \subseteq Y$ by $\subseteq$ int. (We think that the picture is easier than the words.)

Extensionality. *The* most distinctive principle of set theory is known as the "axiom of extensionality":

4A-7 AXIOM. *(Axiom of extensionality)*

If X_1 is a set and Y_1 is a set, then:

$$\forall z[z \in X_1 \leftrightarrow z \in Y_1] \rightarrow (X_1 = Y_1) \qquad \text{(Ax. Ext.)}$$

The axiom of extensionality says that sets are identical just in case they have exactly the same members.[2] Observe that the converse (i.e., from right to left) is already *logically* true by the identity rule of Added context, **3D-4**, together with UG, so that there is no point in changing the conditional to a biconditional. Partly to enforce this insight, it is good to employ the *rule* of extensionality, which simply converts the conditional into a rule of inference:

4A-8 RULE. *(Rule of extensionality)*

Suppose that X and Y are sets. If $\forall z[z \in X \leftrightarrow z \in Y]$ is available, then you may infer $X = Y$, giving "Extensionality" (or "Ext.") as justification. Picture:

	·	
j	$\forall z[z \in X \leftrightarrow z \in Y]$	**?**
j+1	$X = Y$	j, Ext.

4A-9 ADVICE. *(Proving identity of sets)*

Just about always, if you wish to prove two sets identical, do so by appealing to the rule of extensionality. Be careful to keep in mind that the premiss of the rule is a universal quantification.

Exercise 39 *(Antisymmetry of ⊆)*

A relation is said to be a *partial order* if it is reflexive, transitive, and (new word) antisymmetric. You established reflexivity and transitivity in Exercise 38. A relation is said to be "antisymmetric" provided that its "going both ways" happens only in the case of identity. Prove, using the rule of extensionality, that the subset relation is antisymmetric and therefore a partial order:

$$(X \subseteq Y \;\&\; Y \subseteq X) \rightarrow X = Y.$$

▷ . . . . ◁

[2] It is essential that the application of this axiom be limited to sets; for example, your pencil has the same members as your pen (namely, none), but they are not thereby identical to each other.

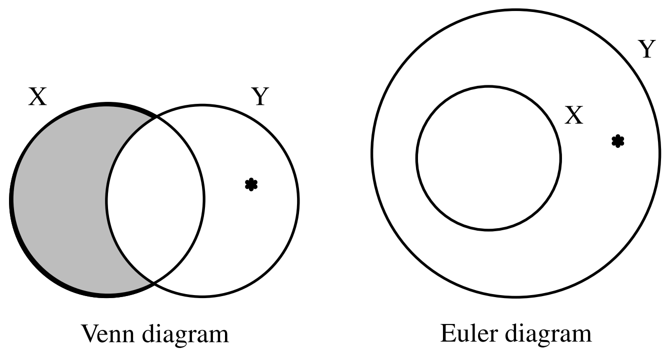


Figure 4.2: Euler and Venn diagrams of the proper subset relation

Proper subset. Moving on, X is said to be a *proper* subset of Y iff X is a subset of Y, but Y has a member that is not a member of X. We may indicate this pictorially in either a Venn or a Euler diagram by putting say a star in the portion of Y that is outside of X as indicated in Figure 4.2.

4A-10 DEFINITION. *(Proper subset)*

$$\forall X_1 \forall X_2 [X_1 \subset X_2 \leftrightarrow (X_1 \subseteq X_2 \;\&\; \exists z [z \in X_2 \,\&\, z \notin X_1])] \qquad \text{(Def. } \subset \text{)}$$

Exercise 40 *(Proper subset)*

The proper subset relation is a *strict partial ordering* of the sets (irreflexive, transitive, asymmetric).

1. Prove irreflexivity: $\forall X_1 {\sim}(X_1 \subset X_1)$
2. Prove asymmetry: $\forall X_1 \forall X_2 [X_1 \subset X_2 \rightarrow {\sim}(X_2 \subset X_1)]$
3. Optional. Prove transitivity: $\forall X_1 \forall X_2 \forall X_3 [(X_1 \subset X_2 \;\&\; X_2 \subset X_3) \rightarrow X_1 \subset X_3]$ (*transitivity* of proper subset)

4. Prove the following obvious relation between the subset and the proper subset relation: $X \subseteq Y \leftrightarrow (X \subset Y \vee X = Y)$.

▷ . . . . ◁

Lots of non-sets, such as gibbons and planets, have no members. There is, however, special terminology for *sets* according as to whether they have members or not.

4A-11 DEFINITION. *(Nonempty/empty)*

A set X *is nonempty* if it has members: $\exists y(y \in X)$.

A set X *is empty* if it has no members: $\forall y(y \notin X)$.

It turns out that there is only one *set* that is empty, and we give it a name: $\varnothing$, pronounced "the empty set." Its axiom describes it perfectly.

4A-12 AXIOM. *(Empty set)*

$\varnothing$ is a set, and

$$\forall x(x \notin \varnothing) \qquad \text{(Ax. } \varnothing\text{)}$$

Exercise 41 *(Empty set)*

Prove that Axiom **4A-12** (together with previous axioms or definitions) logically implies some facts about the empty set.

1. $\forall X_1[\varnothing \subseteq X_1]$
2. $\sim(\varnothing \in \varnothing)$
3. $\varnothing \subseteq \varnothing$
4. $\forall X_1 \sim (X_1 \subset \varnothing)$
5. $\forall y(y \notin X) \rightarrow X = \varnothing$ (uniqueness of the empty set; use extensionality)

▷ . . . . ◁

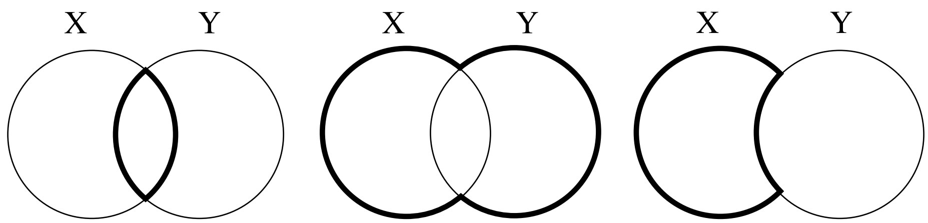


Figure 4.3: A picture of the intersection $X \cap Y$, union $X \cup Y$, and set difference $X - Y$

Intersection, union, and set difference. Next we add three operators, set intersection ($\cap$), set union ($\cup$), and set difference ($-$). Each takes two sets as inputs and delivers a set as output. The "Venn" diagram of Figure 4.3 makes clear to your right brain which things belong to the set that is output. The axioms speak instead to your left brain. Each axiom has two parts; you can ignore the first part of each, which simply says that the output of the operator always names a set.

4A-13 AXIOM. *(Intersection, or meet)*

$$\forall X_1 \forall X_2 [X_1 \cap X_2 \text{ is a set}]$$
$$\forall X_1 \forall X_2 \forall z [z \in (X_1 \cap X_2) \leftrightarrow (z \in X_1 \,\&\, z \in X_2)] \qquad (\text{Ax. } \cap)$$

4A-14 AXIOM. *(Union, or join)*

$$\forall X_1 \forall X_2 [X_1 \cup X_2 \text{ is a set}]$$
$$\forall X_1 \forall X_2 \forall z [z \in (X_1 \cup X_2) \leftrightarrow (z \in X_1 \vee z \in X_2)] \qquad (\text{Ax. } \cup)$$

4A-15 AXIOM. *(Set difference)*

$$\forall X_1 \forall X_2 [X_1 - X_2 \text{ is a set}]$$
$$\forall X_1 \forall X_2 \forall z [z \in (X_1 - X_2) \leftrightarrow (z \in X_1 \,\&\, z \notin X_2)] \qquad (\text{Ax. } -)$$

Observe that $\forall z$ is "on the outside," and so may be instantiated (with UI) along with $\forall X_1$ and $\forall X_2$.

Exercise 42 *(Set operations and subset)*

There are dozens of useful facts about the set-theoretical concepts so far introduced. Here is a family of such of some interest. Prove them. (In the presence of the convention that outermost universal quantifiers may be omitted (Convention **4A-2**), you are in effect proving the truth of some universally quantified statements. But nothing will go wrong if you just treat X and Y as constants (parameters) in proving the following—constants that *could* be flagged if you wished to continue on to prove the universal quantification.)

1. $X \cap Y$ is the "greatest lower bound" of X and Y with respect to the subset relation:
   (a) $X \cap Y \subseteq X$
   (b) Optional. $X \cap Y \subseteq Y$
   (c) Optional. $(Z \subseteq X \text{ and } Z \subseteq Y) \rightarrow Z \subseteq X \cap Y$
2. $X \cup Y$ is the "least upper bound" of X and Y with respect to the subset relation:
   (a) Optional. $X \subseteq (X \cup Y)$
   (b) Optional. $Y \subseteq (X \cup Y)$
   (c) $(X \subseteq Z \text{ and } Y \subseteq Z) \rightarrow (X \cup Y) \subseteq Z$
3. $X - Y$ is the set difference of X and Y, or the complement of Y relative to X:
   (a) Optional. $Y \cap (X - Y) \subseteq \varnothing$
   (b) Optional. $X - Y \subseteq X$
   (c) $X \subseteq (Y \cup (X - Y))$
4. Prove:
   (a) $X \subset Y \leftrightarrow (X \subseteq Y \;\&\; \exists z[z \in (Y - X)])$.
   (b) $\sim(X \subseteq \varnothing) \leftrightarrow \exists z[z \in X]$.

▷ . . . . ◁

We need identity to state the basic idea of our notation, Symbolism **1B-10**, for finite collections such as {a}, {a, b}, {a, b, c}, etc.

4A-16 AXIOM. *(Unit sets {a}, pair sets {a, b}, etc.)*

You can make the name of a set by enclosing a comma-separated list of entities in curly braces:

- {y} is a set, and

  $\forall x \forall y[x \in \{y\} \leftrightarrow x = y]$ (Ax. {})

- $\{y, z\}$ is a set, and

$$\forall x \forall y \forall z[x \in \{y, z\} \leftrightarrow (x = y \vee x = z)] \qquad \text{(Ax. } \{\}\text{)}$$

- $\{y_1, y_2, y_3\}$ is a set, and

$$\forall x \forall y_1 \forall y_2 \forall y_3[x \in \{y_1, y_2, y_3\} \leftrightarrow (x = y_1 \vee x = y_2 \vee x = y_3)] \qquad \text{(Ax. } \{\}\text{)}$$

- Etc.

Exercise 43 *(Finite collections)*

The first two exercises turn out, in *Notes on the science of logic*, to be essential principles for thinking about what it means for a set to be finite.

1. Prove: $y \notin X \rightarrow X = (X - \{y\})$
2. Prove: $y \in X \rightarrow X = (X - \{y\}) \cup \{y\})$
3. Prove: $\{a, b\} = \{b, a\}$.
4. Optional (not difficult but tedious, with much use of CA, case argument). Prove that if $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ then $a = c$ and $b = d$. You might start by showing that $\{a\}$ belongs to the left side, and that therefore $\{a\}$ also belongs to the right side. If you add three similarly justified facts, e.g. that $\{c, d\}$ belongs to the left side since it belongs to the right side, you will have quite a lot of information to begin your tedious investigation. (This fact is required in order to justify a standard set-theoretical construction of so-called ordered pairs $\langle a, b\rangle$, as we note in §4B.)

▷ . . . . ◁

Exercise 44 *(Sets with identity)*

In the following, it is given that X, Y, and Z are sets.

1. Prove that in the following very special case, it doesn't matter where you put the parentheses: $X \cap (Y \cup X) = (X \cap Y) \cup X$.
2. $X \subseteq \varnothing \rightarrow X = \varnothing$.

3. $(X-Y)-Z = X-(Y\cup Z)$
4. For each of the following, either prove it, or prove its negation.
   (a) $\{\varnothing\}=\varnothing$.
   (b) $\{\{\varnothing\}, \varnothing\}=\{\varnothing\}$
   (c) $\forall X(X\cap\varnothing=\varnothing)$
   (d) $\forall X(X\cup\varnothing=X)$

▷ . . . . ◁

We briefly indicate another standard axiom of set theory: separation. Once you have some sets to start you off, this "wholesale" axiom guarantees the existence of many additional sets, such as $\varnothing$ and $X\cap Y$, that we earlier introduced in a retail fashion (see Axiom **4A-12** and Axiom **4A-13**):

4A-17 AXIOM. *(Separation)*

$\forall X\exists Y\forall z[z\in Y \leftrightarrow (z\in X \,\&\, Az)]$.

Here Az can be any sentence, so long as its only free variable is z: Az is interpreted as a *condition* on z. So the axiom says that if you start with a set X and put some condition on its members, you are sure of the existence of the set of members of X that satisfy that condition. Important as it is, however, it would take us too far afield to put this axiom to use.

That's all the easy set theory (EST) there is. We explain a little more set theory in the next section, §4B. From now on, you are entitled to use any item of easy set theory in proofs, writing "EST" as a kind of wholesale justification.

4B More set theory: relations, functions, etc.

We next go through the set-theoretical background required for the art of the use of relations and functions, adding some set-theoretical lagniappe at the end. "Required" has a special meaning here. There is nothing intrinsic to the idea of relations or functions that demands their treatment within set theory instead of independently; but the set-theoretical treatment seems on the whole simpler in practice. First we go over the informal definition of a function. We want to define a function f from X to Y as a rule associating a member $y\in Y$ with each $x\in X$. The notation $y=fx$ or $y=f(x)$ will mean that f associates y with x.[3] Suppose $X=\{1, 2, 3, 4, 5\}$ and $Y=\{a, b, c, d\}$. Figure 4.4 pictures a function f defined on every member

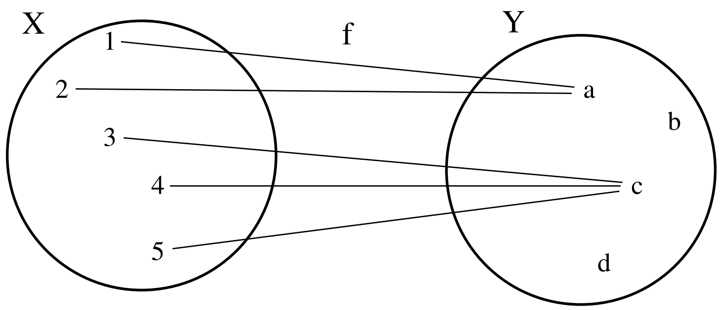


Figure 4.4: A picture of a function f from X into Y

of X and taking values in Y. The table [1] is another equally useful picture that displays almost the same information:

x	1	2	3	4	5
fx	a	a	c	c	c

[1]

More precisely, f is a function from X into Y if

1. X and Y are sets.

2. $\forall x[x \in X \to \exists y[y \in Y \,\&\, y = f(x)]]$. For every $x \in X$ there is an associated f(x) $\in Y$.

3. $\forall y \forall z[(y \in Y \,\&\, z \in Y \,\&\, \exists x[x \in X \,\&\, z = f(x) \,\&\, y = f(x)]) \to z = y]$.

The function f associates a unique y with each x. As the picture shows, functions can be "many-one," mapping different elements of X to the same element of Y. They cannot be one-many or many-many, by clause (3). It is not part of the definition that each member of Y is used as a target (in these cases the function is said

[3]This cannot help but be confusing since up to this point we have taken the notation "f(x)" as a given: a categorematic expression (namely, a term) that is governed by first-order logic. Now, however, we are being foundational: We are interested in analyzing the very notion of a function that lies behind the "f(x)" notation. For this purpose, we temporarily forbid ourselves to use "f(x)" as categorematic. Instead, we use it only syncategorematically to make up the predication "y = f(x)." Only when we have finished the foundational analysis and come to Definition **4B-9** will we entitle ourselves once more to use "f(x)" categorematically as a term.

to be "*on*to" Y). Note that Y could be X, or even a subset of X, as in the following example:

> "Father-of" is a function from the set of humans into the set of humans. For each $x \in X$, father-of(x) is x's father. In particular, father-of(Jacob) = Isaac, and father-of(Isaac) = Abraham.

A convenient and perspicuous way to form these intuitions into a rigorous theory is to define a function as a kind of relation. So what is a relation? In the same spirit of convenience and perspicuity, a relation will be defined as a set (we know what those are) of ordered pairs. What near earth is an ordered pair? Some people define "ordered pair" in terms of "$\{\}$." Using "$\langle x, y\rangle$" for the ordered pair the first member of which is x, and the second member of which is y, they give the following definition:

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}; \qquad [2]$$

But we do not know of anyone who claims in a philosophical tone of voice that is what an ordered pair "really is": The proposed definition [2] (which we are *not* going to adopt) is thought of as convenient but not perspicuous. In any event, it is necessary and sufficient (for any purposes anyone has ever thought of) that ordered pairs have the following properties:[4]

4B-1 AXIOM. *(Ordered pair)*

$\langle x, y\rangle = \langle u, v\rangle \rightarrow (x = u \;\&\; y = v)$. (Ax. $\langle\rangle$)

$\langle x, y\rangle \neq x$; $\langle x, y\rangle \neq y$.

The general attitude toward relations appears to be different from that toward ordered pairs: Many people seem to think the following definition tells us what a relation "really is": The definition below (which we do adopt) is thought of as not only convenient but also perspicuous.

4B-2 DEFINITION. *(Relation)*

A *relation* is a set of ordered pairs. That is,

R is a relation $\leftrightarrow$ R is a set and $\forall x[x \in R \rightarrow \exists y \exists z(x = \langle y, z\rangle)]$

[4] In Exercise 44(4) you were given the option of undertaking the tedious task of proving these properties for the proposed definition [2].

The following items introduce some indispensable language in the theory of relations.

4B-3 CONVENTION. *(Relation notation)*

When R is a relation,

$xRy \leftrightarrow Rxy \leftrightarrow R(x, y) \leftrightarrow \langle x, y\rangle \in R$

4B-4 AXIOM. *(Domain and range)*

The *domain*[5] of a relation is the set of its left elements, and its *range* is the set of its right elements: *Dom*(R) and *Rng*(R) are sets, and

$$x \in Dom(R) \leftrightarrow \exists y(\langle x, y\rangle \in R) \qquad \text{(Ax. } Dom\text{)}$$

$$x \in Rng(R) \leftrightarrow \exists y(\langle y, x\rangle \in R) \qquad \text{(Ax. } Rng\text{)}$$

4B-5 AXIOM. *(Cartesian product)*

$X \times Y$ is the set of all ordered pairs with left member in X and right member in Y: $X \times Y$ is a set, and

$$\forall z[z \in (X \times Y) \leftrightarrow \exists x \exists y(z = \langle x, y\rangle \text{ and } x \in X \text{ and } y \in Y)] \qquad \text{(Ax. } \times\text{)}$$

4B-6 COROLLARY. *(Cartesian product)*

$x \in X$ and $y \in Y \leftrightarrow \langle x, y\rangle \in (X \times Y)$

PROOF. Trivial. □

It is good to think of $X \times Y$ as the "universal relation" on X and Y. Cartesian products generalize to cartesian powers, where we think of a "power" as arrived at by repeated "multiplication" by the same thing. When X is a set, we define the set cartesian-powers-of(X) of the cartesian powers of X—that is, the results of repeatedly taking products of X. We do not give the definition rigorously, and we serve notice that, accordingly, we will not be proving anything involving it until and unless that lack is made good.

[5] In these notes "domain" has two uses: $Domain_{\mathbf{j}}$ is the domain of the Q-interpretation **j**, and *Dom*(R) is the domain of the relation R.

4B-7 DEFINITION. *(Cartesian power of X; casual version)*

$X \in$ cartesian-powers-of(X).

$(X \times \ldots \times X) \in$ cartesian-powers-of(X) (where the dots are to be interpreted with charity).

We later want cartesian powers to be the domains of definition of certain functions; but what is a function? We repeat the earlier account, but now with full rigor, as a definition.

Functions.

4B-8 DEFINITION. *(Function)*

A *function* is a relation such that there is a unique right member for each left member: f is a function $\leftrightarrow$ f is a relation and

$$\forall x \forall y \forall z[(\langle x, y\rangle \in f \;\&\; \langle x, z\rangle \in f) \rightarrow y = z].$$

The formal details of this definition are not important for our purposes, but the idea of a function is decidedly so: There is a set on which the function f is defined, its domain *Dom*(f) (see Axiom **4B-4**). For each thing x in *Dom*(f), f yields a unique thing y, namely, the unique member y of *Rng*(f) corresponding to x.

It is essential to have a compact notation for application of a function to one of its arguments.

4B-9 DEFINITION. *(Function notation)*

Provided f is a function and $x \in Dom(f)$: $f(x) = y \leftrightarrow \langle x, y\rangle \in f$.

When f is a function and x is a member of *Dom*(f), the notation is wholly transparent and easy to use, permitting you to make transitions with your eye that would otherwise call into play the higher faculties of your intellect (Whitehead); but when f is not known to be a function, or if x is not known to be in its domain, then it is pointless to use the notation because the definition, being conditional in form, gives no sense to its use (see §12A.4).[6] In many cases below we label as a "definition" some condition on a function-symbol. Showing that the "definition" really is a definition (eliminable and conservative as explained in §12A) is generally Too Much, and we omit it. In the same spirit, although we never use notations like "f(x)" when we shouldn't, we do not include verification of this as part of the development.

4B-10 CONVENTION. *(Function notation alternatives)*

We will (for example) use "fx" or "(f)x" or even "(f)(x)" or "((f)(x))" as meaning the same as "f(x)"—whichever is in a given context convenient. That is, we are wholly relaxed about parentheses in our use-language, adopting whatever communicative strategies seem to us best.

Also we nearly always use "f(x, y)" or "fxy" for "f(⟨x, y⟩)"; or sometimes it is convenient to write "x f y" or "(x f y)."

Any ambiguities caused by collating this convention with Convention **4B-3** are to be resolved in a seemly fashion.

4B-11 VARIANT. *(Defined on)*

A function f is *defined on* (an entity) x if $x \in Dom(f)$. And f is *defined on* (a set) X if $X \subseteq Dom(f)$.

Don't use "f(x)" unless f is defined on x; it would be a silly thing to do, since then nothing, not even Definition **4B-9**, gives us a hint as to the identity of f(x).

The next comes in handy from time to time.

4B-12 FACT. *(Range of functions)*

For functions f, $y \in Rng(f) \leftrightarrow \exists x(x \in Dom(f) \text{ and } fx = y)$.

PROOF. Tedious. □

The following is equally useful:

[6] The trickiest point is this: Use of the notation cannot itself make f a function or put x in its domain! In constructing proofs, things must go the other way: First you must verify that f is a function and that x is in its domain; only then is it sensible to use the notation f(x).

4B-13 DEFINITION. *(One-one)*

A function f is one-one $\leftrightarrow$ for every x, y $\in$ *Dom*(f), f(x) = f(y) $\rightarrow$ x = y.

This means, by contraposition, that if x $\neq$ y, then f(x) $\neq$ f(y): Distinct arguments yield distinct values. The picture: No two strings emanating from *Dom*(f) are tied to the same member of *Rng*(f).

Function space operator. Some impressive insight is enabled by concentrating on certain sets of functions.

4B-14 DEFINITION. *(Function-space operator $\mapsto$)*

(X $\mapsto$ Y) is the set of all functions whose domain of definition is X, and whose range of values lies inside Y. That is, (X $\mapsto$ Y) is a set, and

f $\in$ (X $\mapsto$ Y) $\leftrightarrow$ f is a function & *Dom*(f) = X & *Rng*(f) $\subseteq$ Y. (Def. $\mapsto$)

In other words, for f to be a member of (X $\mapsto$ Y), f must be a function with domain X having the following further property:

$\forall$x(x $\in$ X $\rightarrow$ f(x) $\in$ Y).

f can be said to be from X *into* Y, in contrast to *onto* Y. In other equivalent jargon, one may call f an *injection*, whereas if f were onto Y, one would call f a *surjection*.

Since f-values for x $\in$ X are always found in Y, (X $\mapsto$ Y) satisfies the following kind of "modus ponens":

4B-15 COROLLARY. *(MP for $\mapsto$)*

1	f $\in$ (X $\mapsto$ Y)	
2	x $\in$ X	
	f(x) $\in$ Y	1, 2 MP for $\mapsto$

Also (see Axiom **4B-5**):

1	$f \in ((X \times Y) \mapsto Z)$	
2	$x \in X$	
3	$y \in Y$	
	$f(x, y) \in Z$	1, 2, 3 MP for $\mapsto$

You can use the principle of MP for $\mapsto$, Corollary **4B-15**, for what we sometimes call "type checking"—for making sure that what you write down makes sense.

4B-16 EXAMPLE. *(MP for $\mapsto$; examples)*

Let R be the set of real numbers and P the set of physical objects. So:

$$+ \in ((R \times R) \mapsto R)$$

$$\text{mass} \in (P \mapsto R)$$

Given that 3 and 5 are real numbers, we can check that it makes sense to write down "$3+5$" as follows:

1	$3 \in R$	Given
2	$5 \in R$	Given
3	$+ \in ((R \times R) \mapsto R)$	Given
4	$(3+5) \in R$	1, 2, 3, MP for $\mapsto$

So "$3+5$" makes sense by line 4. And we may continue in the same spirit:

5	$\text{mass} \in (P \mapsto R)$	Given
6	$\text{Frank} \in P$	Given (just now)
7	$\text{mass(Frank)} \in R$	5, 6, MP for $\mapsto$
8	$(\text{mass(Frank)} + 3) \in R$	1, 7, 3, MP for $\mapsto$

So "mass(Frank)" is o.k. by line 7, and line 8 yields an o.k. for "mass(Frank)$+$3."

In a more negative spirit, we can see that what we would need for "mass(3)" to make sense would be for 3 to belong to P (since P is the domain of definition of mass), and what we would need for "Frank$+$3" to make sense is for Frank to belong to R.

In the same spirit, we can see that it makes no sense to write down something like "mass(mass(Frank))"—the argument to the inner occurrence of "mass" is o.k., but the argument to the outer occurrence is thought to be a member of R rather than a member of P. What guides your thinking here is always MP for $\mapsto$: What *would* you need in order to apply it to the case at hand?

$X \mapsto Y$ also has the following conditional-proof-like property.

4B-17 COROLLARY. *(CP for ↦)*

1	f is a function	
2	*Dom*(f) = X	
3	| a ∈ X	hyp, flag a
.	| .	
.	| .	
.	| .	
k	| f(a) ∈ Y	
k+1	f ∈ (X ↦ Y)	1, 3–k, CP for ↦

Exercise 45 *(Exercise on function space)*

The "composition" of two functions is written $f \circ g$. Take the following as premisses: $f \in (Y \mapsto Z)$, $g \in (X \mapsto Y)$, $\forall x[(f \circ g)x = f(g(x))]$, and $Dom(f \circ g) = Dom(g)$. Now try to prove that $(f \circ g)$ is a member of $(X \mapsto Z)$—a kind of transitivity for the function space operator. (This proof is short and in a sense easy, but you may well become confused and need assistance.)

▷ . . . . ◁

Occasionally we need an "identity criterion" for functions; the following is ultimately a consequence of extensionality, Axiom **4A-7**.

4B-18 FACT. *(Identity of functions)*

Functions are identical if they have the same domains, and deliver the same argument for each value: For functions f and g, if $Dom(f) = Dom(g)$, and if $f(x) = g(x)$ for all $x \in Dom(f)$, then $f = g$.

PROOF. Tedious. □

Since we have an "identity criterion" for functions, we can count them. It turns out that the number of functions in $(X \mapsto Y)$ depends in an easy way on the number of elements in X, and in Y: Namely, let X have x elements and let Y have y elements; then (X↦Y) has y^x functions in it. For this reason, most set-theoreticians use the notation "Y^X" in place of "(X↦Y)"; but the notation is unmemorable and unhandy in applications such as ours in which we shall want to make frequent use of MP and CP for ↦, Corollary **4B-15** and Corollary **4B-17**.

Exercise 46 *(Exercises on relations and functions)*

1. Without being told what set X is, informally describe a function that is bound to belong to (X↦X).
2. a∈X; b∈Y; f ∈ (X↦(Y↦Z)); so (f(a))b∈? (Consult Convention **4B-10** for help disentangling parentheses; they are indeed casually placed.)
3. f∈(X↦Y); x∈W; g∈((X↦Y)↦(W↦Z)); so ?
4. f∈((X×Y)↦C); x∈X; y∈Y; so ?
5. For every x in Y, there are y and z such that y∈Y and z∈X and f(x)=⟨y, z⟩. So f∈? (You will need to make an obvious guess about *Dom*(f)).
6. Let f∈({0, 1, 2, 3, 4, 5} ↦ {a, b, c}) and have the following among its members: ⟨3, b⟩, ⟨4, b⟩, ⟨0, c⟩. Then:
   (a) f(4)=?
   (b) is "f(5)" meaningful?
   (c) Is "f(b)"?
   (d) As far as we know, could ⟨5, c⟩ be a member of f?
   (e) Could ⟨5, 3⟩?
   (f) Could ⟨4, c⟩?
   (g) Could ⟨4, b⟩?
   (h) Could both of the foregoing be simultaneously in f?

▷ . . . . ◁

Characteristic function. There is a standard move between sets and functions that makes one's technical life easier. You have to start with a set known to have exactly two members. Any such set will do, but for us it is convenient to use the set of truth values $\{T, F\}$, to which we give the mnemonic name **2**. (It is essential that we lay down as a fact that $T \neq F$.) We may use **2** in order to define the "characteristic function" of a given set Y relative to a domain X. The idea is that we will tag each member of $X \cap Y$ with T, and everything in $X - Y$ with F.

4B-19 DEFINITION. *(Characteristic function)*

---

f is *the characteristic function of* Y relative to a domain X $\leftrightarrow_{df}$

> $f \in (X \mapsto \mathbf{2})$, and for all y, if $y \in X$, then
>
> $f(y) = T \leftrightarrow y \in X \cap Y$, and
>
> $f(y) = F \leftrightarrow y \in (X - Y)$.

Since $X = (X \cap Y) \cup (X - Y)$, this recipe tags each member x of X with exactly one of T or F according as $x \in Y$ or $x \notin Y$.

Exercise 47 *(Characteristic function)*

1. Let X be $\{1, 2, 3, 4, 5, 6\}$. Let X_0 be $\{1, 3, 5\}$. Let f be the characteristic function of X_0 relative to its superset X. Give a six-row "input-output" table describing f.
2. Show by Easy Set Theory that the third clause of Definition **4B-19** is implied by the first two clauses.

▷ . . . . ◁

Powerset. The powerset operator takes us "up" a set-theoretical level. Here we merely explain it and say a word or two about it.

4B-20 AXIOM. *(Powerset operator $\mathcal{P}$)*

$\mathcal{P}(X)$ is a set, and

$\forall Y[X \in \mathcal{P}(Y) \leftrightarrow X \subseteq Y]$.

So $\mathcal{P}(X)$ is the set of all subsets of X. Cantor, in what has to count as a remarkable feat of intellect, came to see and then demonstrated that the powerset of X always has more members than X has. You can convince yourself of this for finite sets easily by jotting down all the members of $\mathcal{P}(X)$ when, for example, $X = \{1, 2, 3\}$. (If you are careful to include the empty set and $\{1, 2, 3\}$ itself, you should wind up with eight = 2^3 members in $\mathcal{P}(\{1, 2, 3\})$.) You have to internalize Cantor's famous thought that for sets, "same size" means "can be put into one-one correspondence," and then use Cantor's famous "diagonal argument" to see that the principle also holds for infinite sets.

It is also intuitive (and provably true) that the two sets $X \mapsto \mathbf{2}$ and $\mathcal{P}(X)$ are exactly the same "size."

While we are considering "more set theory," it is good to pair with the powerset operator, which takes us up a set-theoretical level, the generalized union operator, that takes us down a level.

4B-21 AXIOM. *(Generalized union operator $\bigcup$)*

$\bigcup Z$ is a set (Z is normally a set of sets, but that is not absolutely required), and for all y,

$\forall y[y \in \bigcup Z \leftrightarrow \exists Y[y \in Y \text{ and } Y \in Z]]$.

So $\bigcup Z$ is the set of members of members of Z, and thus descends a level. It should turn out that $\bigcup \mathcal{P}(X) = X$. Does it?

(For this "edition" of NAL, we omit exercises covering these ideas. Their definitions are included here so that we can refer back to them if we wish.)

Chapter 5

Symbolizing English quantifiers

Having mastered proofs involving quantifiers, we turn to symbolization in this chapter and semantics in the next.

5A Symbolizing English quantifiers: Four fast steps

We have studied the proof theory of the universal quantifier and the existential quantifier. Here we ask you to devote some effort to improving your ability to understand quantification as it occurs *in English* by means of symbolizing into our first order language. We approach symbolizing English quantifiers twice. We begin in this section with a brief summary of a procedure that has its roots in Russell's famous *Mind* article "On denoting" (1906) and Richard Montague's work to be found in Montague (1974). You should be able to pick up enough to do some interesting exercises, which we provide. We then treat the process of symbolizing quantifiers at (perhaps tedious) length in §10A.

Taken as words or phrases, English quantifiers are such as those on the following list: some, a, any, all, each, every, at least one, at most one, between three and five. Taken grammatically, a "quantifier" in English is a kind of functor (Definition **1A-1** on p. 5): The English quantifier gives us instructions for combining an English count-noun phrase (something that is not in "logical" grammar) with a predicate (a functor that takes singular terms into sentences—see Definition **1A-2**) in a certain way in order to make a sentence. So there are not only English connectives (they operate on one or more English sentences to make new sentences); there are also

English-quantifier functors. Among the logically complex English sentences that we can symbolize with profit, some have one of the truth-functional operators as major functor, and others have an English quantifier as major functor. In symbolizing such a sentence, the first thing to do is to locate the major functor, whether it is a truth-functional connective or an English quantifier. Upcoming §10A.2 may help a little in this task, but English grammar offers no sure-fire clues as to which functor is major, so that you need constantly to use your judgment. The rest of this section gives a procedure to use when you have decided that the major functor is an English quantifier.

This topic is inevitably confusing because of the mismatch between English grammar and symbolic Q-grammar. English grammar has no variables and no symbolic quantifiers, and symbolic Q-grammar has no count-noun phrases and no English quantifiers. We get around this confusion by means of the following underlying thought:

5A-1 THOUGHT. *(English quantifiers)*

Each English quantifier corresponds to a symbolic pattern for combining two semi-symbolic or "middle English" *open* sentences by means of a symbolic quantifier. Each of the two open sentences contains a *free* occurrence of the same variable. The result of the combination *binds* the free variable.

5A-2 EXAMPLE. *(Middle English)*

"Each" is an English quantifier that corresponds to the symbolic pattern

$$\forall x(\_\__1 \rightarrow \_\__2),$$

where each blank is filled with an open sentence using x, such as "x is a sheep" in the first blank, and "x is placid" in the second. The result, "∀x(x is a sheep →x is placid)," is evidently a part of neither English nor of our symbolic language; we say from time to time that such a figure is in "middle English," as are the open sentences "x is a sheep" and "x is placid." Such sentences are typically generated during logical analysis or symbolization/desymbolization.

We introduce jargon for the two open sentences: the *logical subject* (LS) and the *logical predicate* (LP).

Here is a list of the principal English quantifiers with which we shall at present be dealing, and the symbolic pattern-of-combination to which that English quantifier corresponds. You must learn this list—as well, eventually, as the longer list in §10A.1—if you are ever to understand English.

5A-3 TABLE. *(Patterns of symbolic combinations matching English quantifiers)*

English quantifiers	**Pattern of combination**
all, any, each, every	$\forall\mathbf{v}[$ LS $\rightarrow$ LP $]$
only	$\forall\mathbf{v}[$ LP $\rightarrow$ LS $]$
some, a, at least one	$\exists\mathbf{v}[$ LS & LP $]$
no, nary a	$\forall\mathbf{v}[$ LS $\rightarrow\sim($ LP $)]$

The English quantifier gives you the right pattern, as the table indicates. But you also need to know how to extract the *logical subject* (LS) and the *logical predicate* (LP) from the sentence that you wish to symbolize. We will come to that as part of the procedure that we are about to outline as four fast steps for symbolizing English quantifiers (that occur on our list). Suppose you are given the following to symbolize.

> **English:** Each horse in the stable is larger than some dog.

Step 1. Find the major English functor, whether connective or quantifier. We are now assuming that it is one of the English quantifiers on our list. In this step we tinker with the English and we write the skeleton of a symbolization.

Step 1-English Underline the English quantifier, for example as follows.

> **English:** <u>Each</u> horse in the stable is larger than some dog.

Step 1-Symbolization Use a *new* variable x (or y or z—it must be new) to write down the symbolic pattern that corresponds to this English quantifier, for example as follows. (Warning: Do *not* use the variable "**v**" that is used in the table. You will just wind up being confused. Use x, y, or z, maybe with subscripts.)

Symbolic 1: $\forall x[\quad \overset{\text{LS}}{} \quad \rightarrow \quad \overset{\text{LP}}{} \quad]$.

Write the entire pattern down, not omitting the final closing bracket. *Leave plenty of space* for later insertion of the logical subject and the logical predicate. *It is a good idea* to write "LS" and "LP" under or above the respective blanks to remind you which blank is to receive the logical subject and which is to receive the logical predicate.

Step 2. That quantifier will (according to the grammar of English) be followed by an *English count-noun phrase*.[1] Extend the underlining on the English quantifier so that it includes the following English count-noun phrase.[2] This is an essential step, as here indicated:

English: <u>Each horse in the stable</u> is larger than some dog.

Step 3. Now is the time to extract **the logical subject** from the English sentence that you are symbolizing. Obtain **the logical subject** from the phrase that you have underlined (that is, the English quantifier plus its following count-noun phrase) by replacing the English quantifier by "x is a." You must of course use the same (new) variable that you used in Step 1-Symbolization. For this example, the logical subject is

x is a horse in the stable.[3]

[1] If not, this method will not work.

[2] As jargon, we call the combination of English quantifier plus its following count-noun phrase the *English quantifier term*, as we did in §1A and will again. This is good jargon, since an English quantifier term can be used in the blank of any predicate, such as "_ is larger than some dog." You can put "Jack" in that blank, or you can put in "each horse in the stable." Also you can take any English quantifier term and put it into the blank of the predicate "Each horse in the stable is larger than _. (With plural quantifiers such as "all" and some uses of "some," you need to adjust to the singular.) Nevertheless, we avoid the jargon *English quantifier term* for this brief section, always saying instead "English quantifier plus its following count-noun phrase."

[3] If your sentence began with a plural quantifier such as occurs in "<u>All brown horses</u> are fast," you would need to adjust plural to singular so that your logical subject would be "x is a brown horse."

Now put **the logical subject** into the proper blank of the pattern that you wrote down in Step 1-Symbolization.

> **Symbolic 1:** ∀x[x is a horse in the stable → LP]

At this point you will be glad that you left plenty of room when you wrote down the pattern in Step 1-Symbolization.

Step 4. Obtain **the logical predicate** from the *entire* English sentence by replacing the underlined English quantifier plus its following count-noun-phrase by the variable x (still using the same (new) variable from Step 1-Symbolization). In other words, the logical predicate comes from the whole English sentence by *taking out* the underlined English quantifier plus its following count-noun phrase and *inserting* the variable. For this example, the logical predicate is

> x is larger than some dog.

Now put the logical predicate into its proper blank:

> **Symbolic 1:** ∀x[x is a horse in the stable → x is larger than some dog].

Observe that the method requires that you *mark up* the existing original English sentence and that you *write down* a new sentence in "middle English" that is partly English and partly symbolism. It is the new sentence that we called "**Symbolic 1**"; the number indicates that it is the first new sentence that you have been asked to write down anew.

Finish or recurse. To "recurse" is to go back and do it again. You are now finished with the major English quantifier of the example, but in this particular example there remains amid your partial symbolization yet another English quantifier, "some," which has now become major as a functor in your logical predicate. So here is how we apply the *same* steps. We begin by "marking up" the existing sentence **Symbolic 1**.

Step 0. Starting with your unfinished symbolization **Symbolic 1**, isolate the *part* on which you wish to work; let's call that part the "target." You probably do not need to write the target separately in order to keep track of what you are doing—just keep track in your head. The point is that you are working on only *part* of **Symbolic 1**, not on all of it. In this example, we are going to use bold face, just once, to indicate the target.

> **Symbolic 1:** ∀x[x is a horse in the stable → **x is larger than some dog**].

Step 1. Locate the major English quantifier of the target, underline it, and rewrite your *entire* partly-symbolized sentence, but with the symbolic pattern with a *new* variable in place of the part that we have called the "target." (In this method, you must write down *one new line* for each English quantifier—on pain of hopeless confusion and more generally suffering the consequences that follow upon the sin of arrogance.)

> **Symbolic 1:** ∀x[x is a horse in the stable → x is larger than <u>some</u> dog].
>
> **Symbolic 2:** ∀x[x is a horse in the stable →
> ∃y[LS & LP]].

We call this "**Symbolic 2**" because it is the second complete sentence that so far you have written down. This method asks you to write one complete sentence for each English quantifier. Experience teaches that utter confusion follows if you try to shortcut this requirement at any point at which you are still learning your way around the intricacies of English quantification.

Step 2. Extend the underlining in the existing target (which is part of **Symbolic 1**). Don't rewrite it, just mark it up.

> **Symbolic 1:** ∀x[x is a horse in the stable → x is larger than <u>some dog</u>].

Step 3. Locate **the logical subject** of the target by prefixing "y is a." (That y is a *new* variable is critical.) Put the logical subject into its proper blank in **Symbolic 2**.

> **Symbolic 2:** ∀x[x is a horse in the stable →
> ∃y[y is a dog & LP]]

Step 4. Locate **the logical predicate** by putting variable "y" in place of the underlined English quantifier together with its following count-noun phrase. Put the logical predicate into its proper blank. (Do this mindlessly; just follow directions.)

> **Symbolic 2:** ∀x[x is a horse in the stable →
> ∃y[y is a dog & x is larger than y]].

Finish or recurse. In some contexts "the" also counts as an English quantifier (as Russell discovered for us), but its treatment as such is more advanced. Here we might just as well think of "x is a horse in the stable" as a conjunction "x is a horse & x is in the stable," and be satisfied with going on to complete our symbolization by turning the following into predicative symbolic form: "x is a horse," "x is in the stable," "y is a dog," and "x is larger than y" going respectively to "Hx," "Sx," "Dy," and "Lxy." So except for this routine symbolizing, we are finished.

"That" and other connectives. When "that" or "who" fronts a relative clause, conjunction is just what is wanted. The same goes for "which." In the case of both "who" and "which" there is, however, the threat of a difficult ambiguity between their so-called restrictive and nonrestrictive uses, a topic that we here avoid.[4]

1. "Every bear that climbs a tree loves honey"

leads as first step to "∀x[(x is a bear and x climbs a tree) → x loves honey]." (The conjunction is part of the logical subject.) As an example of another connective,

2. "If any bear climbs a tree then he or she loves honey"

leads to "∀x[x is a bear → (if x climbs a tree then x loves honey)]." (The conditional is part of the logical predicate.) In each of these two cases, "x climbs a tree" is symbolized by ∃y[y is a tree & x climbs y]; note that the *scope* of the new quantifier "∃y" is as small as possible.

[4]The ambiguity crops up chiefly in connective with "the," whose quantifier pattern we list among those given in Definition **10A-1**. In good writing the ambiguity is resolved by commas. "The owner who is rich lives in luxury," where "who is rich" is so-called restrictive, puts "& x is rich" as part of the logical subject. In contrast, "The owner, who is rich, lives in luxury," where "who is rich" is so-called nonrestrictive, puts "x is rich &" as part of the logical predicate. The best writing, incidentally, always uses "that" *without* commas (and so as unambiguously restrictive), while always using "which" *with* commas (and so as unambiguously nonrestrictive), thus giving the reader the most possible help.

Observation. The proposed symbolizations of these two sentences are the only ones that correctly follow the English grammar. In particular, the English grammar of (2) forces "any" to have wide scope, and therefore forces the logical subject to be "x is a bear" and the logical predicate to be "if x climbs a tree then x loves honey." No choice. *After* these grammar-following symbolizations, you learn *from logic* that the two symbolizations are Q-logically equivalent. (Stop to check your understanding of this remark.) This serves as a partial explanation of an undoubted fact: Most people when faced with (2) leave its English grammar in the dust on their way to an "intuitive" symbolization that is identical to (and not just Q-logically equivalent to) their symbolization of (1). This is of course harmless—except when your task is to learn and use the four fast steps.

Note on "one" and "body" as in "someone" or "nobody," and "thing" as in "anything" or "something." It seldom hurts to treat "one" and "body" as equivalent to "person," as in "some person" or "no person." Thus, symbolize "Someone entered" as "$\exists$x(Px & Ex)." Also, given "anything," "something," or "nothing," it is all right to leave out the logical subject entirely, using just "$\forall$x," "$\exists$x," or "$\forall$x$\sim$" directly in front of the logical predicate, as in symbolizing "Something is out of joint" as "$\exists$xOx."

Pronouns that refer back to English quantifier terms. Some pronouns and some pronoun-like phrases refer back to English quantifier terms. Example: "Anything is expensive if *it* costs hard work." Here you need to use the same variable both for the logical subject, e.g. "x is expensive," and for the occurrence of "it," e.g. "x costs hard work." Some "the" phrases play this role: "If a (any) rat chases and catches a (any) dog, then *the rat* is faster than *the dog*." Here you want to use appropriate variables in place of "the rat" and "the dog."

Special note on passives. Suppose you are symbolizing "No one who loves Mary is loved by someone who loves Sally." Eventually the English passive "x is loved by y" will be symbolized in exactly the same way as the English active "y loves x." *It is essential, however, that this "passive-to-active" transformation be delayed until you are working on a target that has no quantifiers.* Do not assume, for example, that the above example means the same as "Someone who loves Sally loves no one who loves Mary." Maybe, maybe not. You can find that out accurately only if you delay the "passive-to-active" switch until you come to a sentence without quantifiers. The reason is this: As long as there are two English quantifiers, there is the question of which is major. English is ambiguous in this respect (whereas our symbolic language is not). Resolving this ambiguity in a particular case can make a great difference.

General disclaimer. These steps work for many, many cases of English, but there are also many cases for which they will not work. Such is life.

More details. See §10A for numerous additional details.

5A-4 METHOD. *(Four fast steps)*

Here are the "four fast steps" enumerated. The "zero" step is to locate the *major* English quantifier.

1. Underline the *English quantifier*, and write down its complete symbolic pattern, including all parentheses.

   Good idea: Make notes about where to put the logical subject and the logical predicate.

2. Extend the underlining to cover the entire *English quantifier **term*** (the quantifier plus its "following count-noun phrase").

3. The English quantifier term that you underlined has two parts: an English quantifier and a following common noun phrase. Get the *logical subject* by mechanically prefixing "x is a" to *the following count-noun phrase*. Put this into the proper blank.

4. Get the *logical predicate* by putting "x" in for the *entire English quantifier term* (and also cross-references if necessary). Put this into the proper blank.

In using the four fast steps, much can be done in your head, without writing. Not, however, everything. It is essential to write a separate line, usually semi-symbolic, for each quantifier. As an example, we process "No dog is larger than only cats."

<u>No dog</u> is larger than only cats.

∀x[x is a dog →∼x is larger than <u>only cats</u>]
  LS_1 LP_1

∀x[x is a dog →∼∀y[x is larger than y → y is a cat]]
  LP_2 LS_2

∀x[Dx →∼∀y[Lxy →Cy]]

Observe that logic tells us that the last line is equivalent to "$\forall$x[Dx$\rightarrow\exists$y[Lxy&$\sim$Cy]]," which results by the four fast steps from "Each dog is larger than some non-cat." Hence, the two English sentences should be equivalent. What do you think?

Exercise 48 *(Using four fast steps to symbolize English)*

Symbolize the following by using the four fast steps. Use initial letters of "animal" count-nouns for your one-place predicate constants. Use the following dictionary for the relational concepts: L$v_1v_2 \leftrightarrow v_1$ is larger than v_2; C$v_1v_2 \leftrightarrow v_1$ chases v_2. Two remarks: (1) If you find yourself at a loss, come back to this exercise after having worked through §10A. (2) Use the four fast steps mechanically, without trying to figure out what these sentences "really mean." If you do, you may very well lose your bearings. In other words, do not let your own intuitions compete with the genius of English grammar as encoded in the four fast steps!

1. Every dog is larger than some cat.
2. Not every cat is larger than every dog.
3. Only dogs are larger than no cats.
4. Every dog that is larger than some cat is larger than every dog.
5. Some dog is larger than some cat that is larger than every dog.
6. If no dog is larger than no dog then every dog is larger than some dog.
7. Optional: Only dogs who chase no rat and who are chased by only cats chase all dogs that are larger than no dogs. [Remember, one new line for each English quantifier.]

You do not need to rewrite the original English sentence, but you *must* write a separate symbolic or semi-symbolic line for each English quantifier. Just jotting down your "final" result does not count as using the four fast steps.

▷ . . . . ◁

Exercise 49 *(Using four fast steps to symbolize* TF *Semantics)*

Idiomatic English is not good at expressing complicated relational ideas with accuracy, but it can be done. The four fast steps may help, but may well not do the entire job. In symbolizing the following, use the following conventions governing restricted range for certain letters: A and B range over sentences, G ranges over sets of sentences, and $\mathbf{i}$ ranges over TF interpretations. (This means for instance that you can symbolize "any sentence" merely with "∀A" and "no sentence" with "∀A∼.") Also use the following dictionary.

> $A \in G \leftrightarrow$ A is a member of the set G;
> $\vDash_{TF} A \leftrightarrow$ A is a tautology;
> $G \vDash_{TF} A \leftrightarrow$ G tautologically implies A;
> $A \vDash_{TF} \leftrightarrow$ A is inconsistent;
> $G \vDash_{TF} \leftrightarrow$ G is inconsistent;
> $G \nvDash_{TF} \leftrightarrow$ G is consistent;
> $Val_{\mathbf{i}}(A) = T \leftrightarrow$ A has the value T on $\mathbf{i}$;
> $Val_{\mathbf{i}}(A) = T \leftrightarrow$ $\mathbf{i}$ gives A the value T (two pieces of English symbolized in the same way).

Hint: Each of the first five English sentences is a definition, and should be symbolized by means of a universally generalized biconditional. English quantifiers occurring to the *right* of "iff" need to be symbolized to the right of "↔." *If you find yourself at a loss, come back to this exercise after having worked through §10A.*

1. A is inconsistent iff A has the value T on no TF interpretation. Answer: $\forall A[A \vDash_{TF} \leftrightarrow \forall \mathbf{i} \sim (Val_{\mathbf{i}}(A) = T)]$.
2. A is a tautology iff A has the value T on every TF interpretation.
3. G tautologically implies A iff A has the value T on every TF interpretation that gives every member of G the value T.
4. G is consistent iff some TF interpretation gives the value T to every member of G.
5. G is inconsistent iff no TF interpretation gives the value T to every member of G.

Now *(a)* symbolize the following and *(b)* use your symbolizations 1–5 above as premisses from which to prove each of the following. Hint: These are all universally quantified conditionals. Bear in mind that A is a sentence and G is a set of sentences.

6. If A is a member of G, then G tautologically implies A.

7. If A is a tautology, then any set of sentences tautologically implies A.

8. If G is inconsistent, then G tautologically implies every sentence.

▷ . . . . ◁

Exercise 50 *(Proofs requiring symbolization)*

When symbolizing, restrict yourself to the method explained in this section. Dictionary: Rv ↔ v is a rat; Dv ↔ v is a dog; $Cv_1v_2 \leftrightarrow v_1$ chases v_2. *If you find yourself at a loss, come back to this exercise after having worked through* §10A. For this set of problems, in each case assume that the leftmost quantifier is "outermost" or "major," and thereby count the rightmost quantifier as within its scope.

1. Symbolize "No rats chase no rats." Symbolize "Every rat chases some rat."[5] Then prove your symbolizations logically equivalent. (You will be teaching yourself how careful you have to be with the suggestion that "two negatives make a positive.")

2. Symbolize "Only rats chase only dogs." Prove your symbolization logically equivalent to the following: $\forall x\exists y[(\sim Cxy \rightarrow Rx) \& (Dy \rightarrow Rx)]$.

3. Symbolize and prove: No rats are dogs. Therefore, no dog that chases only rats chases a dog.

▷ . . . . ◁

5B Symbolizations involving identity

Any branch of mathematics is full of explicit identities. And see §10A.1 for some *English* quantifier expressions that involve identity in their formal unpacking.

[5] These sentences are grammatically ambiguous; the ambiguity makes a difference, however, only in the second sentence. Resolve the ambiguity by supposing that the speaker intended the leftmost quantifier term to be symbolized first. (This makes "every rat" the "major" quantifier term, or, equivalently, makes "every rat have widest "scope." You must symbolize it first.)

Exercise 51 *(Identity symbolizations)*

Let the domain be all persons. Let $f(x) = $ the father of x, and let $m(x) = $ the mother of x. Let $j = $ John, $s = $ Sue and b = Bill. Let $Fxy \leftrightarrow$ x is faster than y, $Kx \leftrightarrow$ x is a present king of France, and $Bx \leftrightarrow$ x is bald. Express the following using *only* these non-logical symbols—and of course any logical symbols you like, including identity. (You should, however, avoid using *redundant* quantifiers or identities.)

1. No one is his or her own mother. [Answer: $\forall x(x \neq m(x))$.]
2. No one's mother is the same as that person's father.
3. In fact, no one's mother is the same as anyone's father.
4. John and Sue have the same father.
5. John and Sue are (full) siblings.
6. Bill is John's father's father and Sue's father.
7. Sue's father's father is not the same person as her mother's father.
8. Two persons are faster than John.
9. Sue is faster than at most one person.
10. No one is faster than himself.
11. John is the fastest person. That is, John is faster than anyone *except himself*.
12. There is a present king of France, and there is at most one present king of France, and every present king of France is bald.

▷ . . . . ◁

Chapter 6

Quantifier semantics—interpretation and counterexample

Part of the art of logic consists in a rough command of semantics. We first give an account of interpretations for quantifier logic (we'll call them "Q-interpretations" in order to distinguish them from TF interpretations as in Definition **2B-7**), and then put this account to use in some standard semantic concepts.

6A Q-interpretations

The *point* of an interpretation in any branch of logic is to give *enough information to determine semantic values of all categorematic expressions* (Definition **1A-4** on p. 7). Here "semantic values" means "truth values" for *sentences*, and "entities" for *terms*. It follows from this and our understanding of how our quantifier language works that a "Q-interpretation" must do much more than a TF interpretation: A Q-interpretation must state

1. The *domain* of quantification; what the quantifiers (or variables) range over. The domain must be nonempty. If we did not specify a domain of quantification, we should have no way of evaluating a sentence such as "$\exists x(5+x=3)$," whose truth value depends on whether or not negative numbers are part of the intended domain.

2. An "*extension*" for each "Q-atom" as defined in §3A. That the interpretation of the Q-atoms be "extensions" is required by our wish to confine ourselves to "extensional" logic.[1]

Keep in mind that by the jargon of Definition **1A-4**, a "Q-atom" is defined as either an individual variable or an individual constant or an operator constant or a predicate constant. Therefore, (2) breaks up as into four parts. A Q-interpretation must state the following.

2a. Individual variable. What it denotes or is assigned. It must be assigned something in the domain.

2b. Individual constant. What it denotes. It must denote something in the domain. One may think of an individual constant as "naming" a certain entity (in the domain). When the Q-atom is an individual variable, however, it is better to think of it as "being assigned" a certain entity (in the domain) as value. "Denotation" is intended to cover both of these modes of expression.

2c. Operator constant.

One-place. Since one-place operator constants are used to make up one-place operators, simple input-output analysis requires that the Q-interpretation of an operator constant f should be a function. The function should take its arguments (inputs) from the domain, and for each such argument, its value (output) must also be in the domain. That is, just as an operator takes terms as inputs and gives a term as output, so the Q-interpretation of an operator constant should take entities (in the domain) as inputs and give an entity (in the domain) as output. It figures.

Two-place. Recall that a two-place operator constant is used to make up a two-place operator, which has two terms as input and a term as output. The Q-interpretation of a two-place operator constant f should therefore be a two-argument function: Given any (ordered) pair of arguments (inputs) in the domain, the Q-interpretation of f must hand you back an entity (in the domain) as output.

And so on.

2d. Predicate constant.

[1]We don't offer a general account of "extension" or "extensional"; it suffices for us to say what we mean in a retail fashion by explaining what we mean by the extension of each kind of Q-atom and of each kind of categorematic expression.

One-place. A one-place predicate constant F is used to make up a one-place predicate, which takes a term as input and delivers a sentence as output. Therefore, the Q-interpretation of F should be a function that takes an entity (in the domain) as input and delivers a truth value as output.

Two-place. A two-place predicate constant F is used to make up a two-place predicate, which has two terms as input and a sentence as output. The Q-interpretation of a two-place predicate constant F should therefore be a two-argument function: Given any (ordered) pair of arguments (inputs) in the domain, the Q-interpretation of F must hand you back a truth value as output.

And so on.

The above constitutes a rough definition of what a Q-interpretation "really" is. If you explain the "extension" of individual variables and predicate, individual, and operator constants in this way, you have indeed supplied enough information to determine an entity (in the domain) for each term, and a truth value for each sentence. In order to upgrade the rough definition of "Q-interpretation" to an exact definition, we should need more in the way of set theory than the portion to which we are currently entitled. Instead of going that far towards exactness, however, we settle for the following, which is enough for us to work with.

6A-1 SEMI-ROUGH DEFINITION. *(Q-interpretation)*

- Anything **j** *is a* Q-*interpretation* iff **j** fixes the domain of quantification as a particular set, and also fixes an appropriate extension for each Q-atomic term (whether individual variable or individual constant), operator constant, and predicate constant, as indicated roughly above and somewhat more exactly below.
- We let "**j**" range over Q-interpretations.
- We use three notations involving **j**, as follows.

1. $Domain_{\mathbf{j}} \leftrightarrow_{df}$ the domain of quantification fixed by the Q-interpretation **j**.
2. We uniformly write **j**(E) whenever E is any Q-atom, whether individual variable, individual constant, operator constant, or predicate constant. We may read "**j**(E)" with some variant of "the interpretation of E in **j**." Because

there are three) kinds of Q-atoms, you should expect to see any of the following: **j**(x), **j**(a), **j**(f), and **j**(F). Clauses below become specific about the various cases of this usage.

(a) When a is a Q-atomic term, any Q-interpretation **j** must tell what entity in the domain is denoted by a. We may express this as follows: For each individual variable x, **j**(x) $\in Domain_{\mathbf{j}}$; and for each individual constant a, **j**(a) $\in Domain_{\mathbf{j}}$.

(b) When f is an n-place operator constant, **j**(f) must be a function defined on each array z_1, ..., z_n of n inputs chosen from the domain $Domain_{\mathbf{j}}$. For each of these arrays of inputs from $Domain_{\mathbf{j}}$, the output must also belong to $Domain_{\mathbf{j}}$. In that way, we can be sure that the Q-interpretation of f in **j** is a function defined on each array z_1, ..., z_n of entities in $Domain_{\mathbf{j}}$, and always outputting values in $Domain_{\mathbf{j}}$. We may express this as follows: Take any operator constant f and any sequence z_1, ..., z_n such that $z_i \in Domain_{\mathbf{j}}$ for every i with $1 \leqslant i \leqslant n$. Then also **j**(f)($z_1$, ..., z_n) $\in Domain_{\mathbf{j}}$.

(c) When F is an n-place predicate constant, **j**(F) must be a function defined on each array z_1, ..., z_n of n inputs chosen from the domain $Domain_{\mathbf{j}}$. For each of these arrays of inputs from $Domain_{\mathbf{j}}$, the output must be a truth value. In that way, we can be sure that the Q-interpretation of F in **j** is a function defined on each array z_1, ..., z_n of entities in $Domain_{\mathbf{j}}$, and always outputting truth values. We may express this as follows: Take any predicate constant F and any sequence z_1, ..., z_n such that $z_i \in Domain_{\mathbf{j}}$ for every i with $1 \leqslant i \leqslant n$. Then **j**(F)($z_1$, ..., z_n) $\in \{$T, F$\}$.

3. We uniformly write $Val_{\mathbf{j}}$(E) whenever E is a categorematic expression (Definition **1A-4**), whether term t or sentence A. We may read "$Val_{\mathbf{j}}$(E)" as "the value of E on the Q-interpretation **j**."

   (a) When t is a term, $Val_{\mathbf{j}}$(t) $\in Domain_{\mathbf{j}}$. We may read this as "the denotation of t on **j**."

   (b) When A is a sentence, $Val_{\mathbf{j}}$(A) $\in \{$T, F$\}$. We read this as "the truth value of A on **j**."

Observe the big difference between **j**(__), which gives the Q-interpretations of Q-atoms, and $Val_{\mathbf{j}}$(__), which gives the values of (usually complex) categorematic expressions (terms and sentences), values that depend on the Q-interpretation **j**.

The difference here for Q-logic repeats a difference that we noted for TF-logic in Rough idea **2B-6**. The correspondence indicated in the following table is Good.

TF **logic**	Q **logic**
TF-atom	Q-atom
TF interpretation $\mathbf{i}$(p)	Q-interpretation $Domain_{\mathbf{j}}$ $\mathbf{j}$(x), $\mathbf{j}$(a), $\mathbf{j}$(f), $\mathbf{j}$(F)
$Val_{\mathbf{i}}$(A)	$Val_{\mathbf{j}}$(t), $Val_{\mathbf{j}}$(A)

So Semi-rough definition **6A-1** tells us what a Q-interpretation $\mathbf{j}$ *is*: Every Q-interpretation gives us a domain of quantification and an appropriate extension to each of the four kinds of Q-atoms (individual variables, individual constants, operator constants, and predicate constants). Since there are infinitely many Q-atoms, that is a great deal of information to pack into $\mathbf{j}$. For practical purposes, however, we can ignore most of it. As should be obvious, the denotation of each term and the truth value of each sentence depend only on the Q-atoms that actually occur in the term or sentence. Furthermore, in the case of individual variables, only those that occur *free* are going to count. (See Fact **6D-1** for a definitive statement of this property of "local determination.") For this reason, in describing a Q-interpretation $\mathbf{j}$ relevant to determining the truth values of some particular set of sentences, you need only describe how $\mathbf{j}$ interprets those individual, operator, and predicate constants that actually occur in those sentences, and how $\mathbf{j}$ interprets any individual variables that are free in those sentences. We exploit this observation silently in what follows.

There are two standard methods of *presenting* Q-interpretations $\mathbf{j}$ informally. The first is via equivalences and identities, and the second is via tables.

6B Presenting Q-interpretations via biconditionals and identities

In order to present a Q-interpretation $\mathbf{j}$ by means of biconditionals and identities, do the following.

1. State the domain, $Domain_{\mathbf{j}}$.
2. State the extension that $\mathbf{j}$ gives to predicate constants by biconditionals, and state the extension that $\mathbf{j}$ gives to individual constants and operator constants by identities, as in Example **6B-1** just below.

This will be enough information to fix the critical ideas $Val_{\mathbf{j}}$(t) and $Val_{\mathbf{j}}$(A) for all *closed* terms t and *closed* sentences A, which is often all that we care about. Later we will deal with open expressions, i.e., with terms and sentences with free variables. Then the values that $\mathbf{j}$ assigns to the variables will become important. In the meantime, we can ignore $\mathbf{j}$(x), where x is a variable.

Suppose you are asked to provide a Q-interpretation sufficient to give a definite truth value to the following sentence:

$$\exists x[Mx \,\&\, \forall y[Fxy \to (Py \vee (f(x) = a)]] \qquad [1]$$

You must (1) describe a domain, and you must (2) describe the extension of each constant, using identities and biconditionals.

6B-1 EXAMPLE. *(Q-interpretations via biconditionals and identities)*

First the domain of Q-interpretation $\mathbf{j}_1$. You might write

$$Domain_{\mathbf{j}_1} = \{0, 1, 2, \ldots\}.$$

Your example describes an infinite domain, the set of non-negative integers. Now you must move to (2), the description by identities and biconditionals of the Q-interpretation of each constant. A look at [1] makes it plain that you must explain the interpretation of each of the following constants: M, F, P, f, and a. The left side of the biconditional or identity contains what you are interpreting, and the right side explains its Q-interpretation *in previously understood language*. So you must complete the following in order to define your Q-interpretation $\mathbf{j}_1$:

$Mv \leftrightarrow$
$Fv_1v_2 \leftrightarrow$
$Pv \leftrightarrow$
$fv =$
$a =$

You might complete step (2) by filling in the right sides as follows:

$Mv \leftrightarrow$ v is even
$Fv_1v_2 \leftrightarrow v_1 < v_2$
$Pv \leftrightarrow$ v is divisible by 4
$fv = (3 \times v) + 4$
$a = 14$

Your interpretation is now complete: With this Q-interpretation $\mathbf{j}_1$, [1] receives a definite truth value, either T or F. You could figure out which, but we'll postpone that exercise. For now it is enough that you know exactly what counts as a Q-interpretation.

Observe that to write "$a = .5$" would make no sense, since .5 is not in your domain. Similarly, "$fv = (1 - v)$" would be nonsense, since, for example, when v is 2, $1 - v$ is not in your domain. It is absolutely essential that each constant denote something in the domain, and that the interpretation that a Q-interpretation **j** gives to an operator be such that the output is in the domain $Domain_{\mathbf{j}}$; that is part of the very definition of a Q-interpretation.

Keep in mind that it is really the predicate constant M, for instance, that is being interpreted, and that the variable v is only used in order to present the Q-interpretation. It is not Mv but M that is being interpreted. For this reason it is not sensible to give both of the following in the same presentation of a Q-interpretation:

$Mv_1 \leftrightarrow v_1$ is a positive integer
$Mv_2 \leftrightarrow v_2$ is an even number

For only *one* Q-interpretation must be given to M. What variable is used to present it makes no difference at all. Well, not quite *no* difference. When you are giving a Q-interpretation, you always have certain quantified sentences in mind. You are interested in the truth value of these sentences. If you use the *same* variables in presenting your Q-interpretation as the variables that occur in those quantified sentences, you are extremely likely to become confused.

So in presenting Q*-interpretations via biconditionals and identities,* never *use the variables that occur in your sentences of interest.*

Let us give a second example. We will provide another interpretation for the same sentence, [1] on p. 182.

6B-2 EXAMPLE. *(*Q*-interpretations via biconditionals and identities)*

This time we will choose a small finite domain:

$Domain_{\mathbf{j}_2} = \{1, 2, 3\}$.

And here are our identities and biconditionals:

Mv $\leftrightarrow$ (0$<$ v and v$<$ 14)
$\text{Fv}_1\text{v}_2 \leftrightarrow \text{v}_1 < \text{v}_2$
Pv $\leftrightarrow$ v is divisible by 4
fv = 4$-$v
a = 2

Observe that our baroque explanation of M boils down to this: "M is true of everything in the domain." To write "Mv$\leftrightarrow$v$=$v" would have been more straightforward. Similarly, we have used a number of words just to say that P is false of everything in the domain, as if we had written "Pv$\leftrightarrow$v$\neq$v."

6B-3 ADVICE. *(Presenting* Q*-interpretations)*

You will do well to carry out Exercise 52 as follows:

1. State the domain.
2. Make an entire list of identities and equivalences, one for each constant, leaving the right side blank.
3. Fill in the right sides.

Exercise 52 *(*Q*-interpretations via biconditionals and identities)*

Make up and present, via biconditionals and identities, two Q-interpretations, $\mathbf{j}_1$ and $\mathbf{j}_2$, each of which confers a definite truth value (any truth value you like) on the following sentence.

$\forall x\exists y[(Mxy \rightarrow Px) \& P(f(a))]$.

Choose your domain, $Domain_{j_1}$, as the set of all nonnegative integers {0, 1, 2, 3, ...}. Choose your domain, $Domain_{j_2}$, to have a size about three. Note the list of Q-atoms that occur in the given sentence: You will have to explain how each of $\mathbf{j}_1$ and $\mathbf{j}_2$ interprets each of a, f, P, and M. (You do not need to worry about any variables, since none occur free in the given sentence.) Use identities to explain a and f, and biconditionals to explain P and M. Consult Example **6B-1**.

▷ . . . . ◁

6C Presenting Q-interpretations via tables

If the domain happens to be finite, and indeed rather small, the extension of predicate constants and operator constants can conveniently be given by means of tables. The following set of tables gives an exactly equivalent description of the Q-interpretation $\mathbf{j}_2$ of Example **6B-2**.

6C-1 EXAMPLE. *(Q-interpretations via tables)*

First, $Domain_{j_2}$ = {1, 2, 3}. Second, here are the explanations via tables of M, F, P, f, and a that are exactly equivalent to those of Example **6B-2**.

M1 = T	P1 = F	f1 = 3	a = 2
M2 = T	P2 = F	f2 = 2	
M3 = T	P3 = F	f3 = 1	
F11 = F	F21 = F	F31 = F	
F12 = T	F22 = F	F32 = F	
F13 = T	F23 = T	F33 = F	

You need to compare the two presentations of $\mathbf{j}_2$ (**6B-2** and **6C-1**) in order to verify that on this small, finite domain they come to the same thing. Observe that in these tables, *sentences* such as F11 are assigned *truth values*, while *terms* such as f1 are assigned *entities* in the domain. It figures.

Here is another example.

6C-2 EXAMPLE. *(Q-interpretations)*

Suppose you are thinking about a couple of sentences:

(a) $\forall$xSxx

(b) $\exists$y[Ra&Gya&R(fy)]

(a) and *(b)* are closed sentences. For them to have a truth value, you must interpret all of their constants: S, R, G, a, and f. Of course you must pay attention to which are predicate constants (and how many places), which are individual constants, and which are operator constants (and how many places). Here is how *one* Q-interpretation $\mathbf{j}_3$ might be presented via tables:

$Domain_{\mathbf{j}_3} = \{1, 2\}$

S11 = T	G11 = F	R1 = T	f1 = 2	a = 1
S12 = F	G12 = F	R2 = F	f2 = 1	
S21 = T	G21 = T			
S22 = T	G22 = F			

This fixes a truth value for the sentences *(a)* and *(b)*: *(a)* is true, since the open sentence Sxx is true for *each* value of x; and *(b)* is also true, since Ra&Gya& R(fy) is true for *some* value of y.

Note: This Q-interpretation is just made up. You can obtain *another* Q-interpretation, one you may prefer, by changing truth values, or changing what a denotes, or changing the value (an entity in the domain) of f1 or f2. Or you can change the domain, in which case your table would be more or perhaps less complicated.

Exercise 53 *(Presenting* Q*-interpretations via tables)*

1. Make up a Q-interpretation $\mathbf{j}_4$ different from that of Example **6C-2** that has the *same domain* ($Domain_{\mathbf{j}_2} = \{1, 2\}$), but that also gives definite truth values to both *(a)* $\forall$xSxx and *(b)* $\exists$y[Ra&Gya&R(fy)]. Present your Q-interpretation with tables.

2. Make up and present by tables a Q-interpretation that gives truth values to those same sentences *(a)* and *(b)* based on a domain of just one individual, $\{1\}$.

3. Optional. Make up and present by tables a Q-interpretation for *(a)* and *(b)* based on a domain of three individuals, $\{1, 2, 3\}$.

4. Make up a Q-interpretation for *(a)* and *(b)* based on an infinite domain, $\{0, 1, 2, 3, \ldots\}$. In this case, you obviously cannot present the Q-interpretation of the non-logical constants with a table; you will *have* to present your Q-interpretation via biconditionals and identities.

▷ . . . . ◁

6D Truth and denotation

Review §2B, where it is brought out that there are two concepts of truth each of which is useful in semantics: garden-variety truth, which is a one-place sentence-predicate, and truth relative to an interpretation, which is two place. Here we are primarily interested in a two-place concept, which we carry with the notation "$Val_{\mathbf{j}}$(A) = T," read as "the value of A on **j** is T," or perhaps "A is true on **j**" or (to change the English word order) "**j** makes A true." Particularly when A contains free variables, one also says "**j** satisfies A." The notation $Val_{\mathbf{j}}$(A) was introduced in Semi-rough definition **6A-1**(3). It was also indicated in that place that in quantifier logic we have, among our categorematic expressions, not only sentences, but also terms; and that we may read "$Val_{\mathbf{j}}$(t)" as "the denotation of t on **j**."

It is illuminating to note that the idea of a Q-interpretation **j** can be broken up into two parts.

1. The first part consists of the domain $Domain_{\mathbf{j}}$ together with what **j** does to each *constant* (individual constant, operator constant, predicate constant).

   This much of **j** is enough to fix a value for every categorematic expression that does not contain any *free* variables. Since our assertions, predictions, guesses, conjectures, premisses, conclusions, etc., never contain free variables, this portion of a Q-interpretation is of fundamental importance.

2. The second part of a Q-interpretation assigns a value to each *variable*: **j**(x) $\in Domain_{\mathbf{j}}$. This part is important because we use it in explaining how each sentence comes to have its truth value. The assignments of values to variables is, however, "auxiliary" in the sense that the truth values of our assertions and so forth, which have no free variables, are independent of these assignments. So to speak: We use them, and then throw them away.

The sense in which the truth values of closed expressions are independent of assignments to variables is worth stating with some care. While we are at it, we generalize to the feature called "local determination."

6D-1 FACT. *(Local determination)*

Let E be any categorematic expression and $\mathbf{j}_1$ any Q-interpretation. Let $\mathbf{j}_2$ be any other Q-interpretation that is exactly like $\mathbf{j}_1$ with respect to its domain and with respect to those constant Q-atoms a, f, or F that *actually occur* in E, and also exactly like $\mathbf{j}_1$ with respect to any variables x that occur *free* in E. Then $Val_{\mathbf{j}_1}$(E) = $Val_{\mathbf{j}_2}$(E).

In other words, the truth value of A on **j** depends entirely on how **j** interprets the constants that occur in A, and on how it assigns values to the variables that occur free in A (and analogously for the denotation of terms t).

Therefore, if A contains no free variables, the assignment that **j** gives to variables is entirely irrelevant to the truth value of A.

Why do we then care about the portion of the Q-interpretation that assigns values to variables? The answer is that this portion is essential when we come to explaining how the truth values of quantified sentences depends on the pattern of truth values of their parts. That is, we need to look at how **j** assigns values to variables in order to explain how the truth values of ∀xAx and ∃xAx depend on the pattern of values of the *open* sentence, Ax. Here is the explanation, couched as a definition.

6D-2 DEFINITION. *($Val_{\mathbf{j}}$(∀xAx) and $Val_{\mathbf{j}}$(∃xAx))*

Let Ax be any sentence, let x be any variable, and let $\mathbf{j}_1$ be any Q-interpretation.

- **Universal quantifier.** $Val_{\mathbf{j}_1}$(∀xAx) = T iff for every entity d and every Q-interpretation $\mathbf{j}_2$, if (d ∈ $Domain_{\mathbf{j}_1}$ and $\mathbf{j}_2$ is exactly like $\mathbf{j}_1$ except that $\mathbf{j}_2$(x) = d) then $Val_{\mathbf{j}_2}$(Ax) = T.

  In other words, a sentence ∀xAx is true on one Q-interpretation just in case the ingredient sentence Ax is true on *every* Q-interpretation that can be obtained from the given one by re-assigning x a value in the domain—and keeping fixed the domain and the interpretations of other variables (and constants as well).

- **Existential quantifier.** $Val_{\mathbf{j}_1}$(∃xAx) = T iff there is an entity d and there is a Q-interpretation $\mathbf{j}_2$ such that (d ∈ $Domain_{\mathbf{j}_1}$ and $\mathbf{j}_2$ is exactly like $\mathbf{j}_1$ except that $\mathbf{j}_2$(x) = d) and $Val_{\mathbf{j}_2}$(Ax) = T.

  In other words, a sentence ∃xAx is true on a certain Q-interpretation just in case the ingredient sentence Ax is true on *some* Q-interpretation that can be obtained from the given one by re-assigning x a value in the domain—and keeping fixed the domain and the interpretations of other variables (and constants as well).

These confusing explanations of the quantifiers should be compared with the much more easily understood explanations for the truth-functional connectives. See Definition **2B-11** for the TF versions; here, for the Q versions, we take negation and the conditional as examples.

- **Negation.** $Val_{\mathbf{j}}$(∼A) = T iff $Val_{\mathbf{j}}$(A)≠T.
- **Conditional.** $Val_{\mathbf{j}}$(A→B) = T iff either $Val_{\mathbf{j}}$(A)≠T or $Val_{\mathbf{j}}$(B) = T.

Compare these simple explanations with the complexities of our explanation **6D-2** of the meaning of the quantifiers.

In finite cases, these quantifier explanations can be laid out in a kind of "truth table," with one row for each assignment over the domain. Let's call such a table an *assignment table*, since there is one row for each possible assignment to the variables that are of interest. Suppose, for instance, that we are considering the closed sentence, ∀y∀x(Fxy→Fyx). Let the Q-interpretation be $\mathbf{j}_1$, the same Q-interpretation as at the beginning of Example **6C-1**, with $Domain_{\mathbf{j}_1} = \{1, 2\}$, and with the table for F given by F11=T, F12=T, F21=F, F22=F. We have not yet needed to say what $\mathbf{j}_1$ assigns to variables, but now we do; let us suppose (it won't make any difference) that $\mathbf{j}_1$ assigns each and every variable the number 1. The Q-interpretations $\mathbf{j}_2$, $\mathbf{j}_3$, and $\mathbf{j}_4$ come by varying the assignment to the variables x and y. Then we can do a routine assignment-table calculation as follows:

6D-3 PICTURE. *(Assignment table for two variables)*

Q-interp	x	y	Fxy	Fyx	(Fxy→Fyx)	∀x(Fxy→Fyx)	∀y∀x(Fxy →Fyx)
$\mathbf{j}_1$	1	1	T	T	T	T	F
$\mathbf{j}_2$	1	2	T	F	F	F	F
$\mathbf{j}_3$	2	1	F	T	T	T	F
$\mathbf{j}_4$	2	2	F	F	T	F	F

Key ideas:

- Although each of the Q-interpretations $\mathbf{j}_i$ interprets F in just one way, the patterns for the open sentences Fxy and Fyx are *not* the same.

- Suppose you are calculating some truth functional compound, for example, A→B. You are doing the calculation for a certain row. You need to look at the truth values of the parts, e.g. A and B, *only* in that *same* row.

- Suppose, however, you are calculating a quantifier compound, for example ∀xAx. You are doing the calculation for a certain assignment (row). You must look at the truth values of the part, e.g. Ax, not only in that same row. You must also look at the truth value of Ax in some *other* rows (assignments).

  You need the truth value of Ax *not only in the row you are calculating. You also need the truth value of* Ax *in each row whose assignments to variables other than* x *are exactly like the one you are calculating. In these rows, while every other assignment is held fast, the assignment to* x *will run over the entire domain.*

  You can see, for example, that in order to calculate that ∀x(Fxy→Fyx) has the value T in row $\mathbf{j}_1$, we needed to see that the value of Fxy→Fyx was T not only in row $\mathbf{j}_1$, but in row $\mathbf{j}_3$ as well.

  This idea is not clearly brought out in Picture **6D-3**; it is difficult to see the point if you are considering only two variables. For a better idea, consult Picture **11A-1** on p. 273.

  Since x and only x varies, you will consult just as many rows (assignments) as the size of the domain.

- A *closed* sentence, such as ∀y∀x(Fxy→Fyx), will always have the *same* truth value relative to every assignment (row) on this one Q-interpretation. That is precisely the idea of "local determination," Fact **6D-1**. The part of $\mathbf{j}_1$ that assigns values to variables does no work for *closed* sentences.

- If you have some operator constants and individual constants, then simple or complex *terms* can also head columns. But in each row (assignment), the value of a term will not be a truth value. It will be an entity in the domain. (Not illustrated. We'll do this on the blackboard.)

The appendix §11A goes more deeply into how assignment tables work. They are not, however, of much practical utility: If an interpretation has a large domain or if a sentence contains many free variables, assignment tables just get out of hand too quickly. And for very small problems, the *method of reduction to truth functions* makes easier work than does the method of assignment tables.

The idea of "reduction to truth tables" is this.

6D-4 METHOD. *(Reduction of quantifications to truth functions)*

On small domains in which we can give everything a name, one can easily evaluate a universal quantification $\forall$xAx by treating it as a conjunction $At_1 \& \ldots \& At_n$, where $t_1, \ldots, t_n$ run through the n members of the domain. Similarly, one can treat an existential quantification $\exists$xAx as a disjunction $At_1 \vee \ldots \vee At_n$, where $t_1, \ldots, t_n$ run through the n members of the domain.

For instance, suppose you are given $Domain_{\mathbf{j}} = \{1, 2\}$ (just to keep things small). Then the following.

> $Val_{\mathbf{j}}$($\forall$xFx) = T reduces to $Val_{\mathbf{j}}$((F1 & F2)) = T.
>
> $Val_{\mathbf{j}}$($\forall$x$\exists$yFxy) = T reduces to $Val_{\mathbf{j}}$(($\exists$yF1y & $\exists$yF2y)) = T; which in turn reduces to
> $Val_{\mathbf{j}}$((F11 $\vee$ F12) & (F21 $\vee$ F22)) = T.

Try out the method of reduction-to-truth-tables calculation on the following examples. We have worked out the first two. Observe our shortcut: Since $\mathbf{j}_1$ is fixed throughout, we write for example "A = T" instead of "$Val_{\mathbf{j}_1}$(A) = T," just to save writing.

6D-5 EXAMPLE. *(Truth values of quantified sentences on a* Q*-interpretation)*

Let the domain $Domain_{\mathbf{j}_1}$ be $\{1, 2\}$, and let $\mathbf{j}_1$ interpret M, F, and f exactly as in the Q-interpretation $\mathbf{j}_1$ in the first part of Example **6C-1** on p. 185.

1. $Val_{\mathbf{j}_1}$($\forall$xMx) = T, since the sentence reduces to (M1 & M2).
2. $Val_{\mathbf{j}_1}$($\exists$x$\forall$yFxy) = F, since the quantifier sentence reduces to ($\forall$yF1y $\vee$ $\forall$yF2y), which reduces to (F11 & F12) $\vee$ (F21 & F22).
3. $Val_{\mathbf{j}_1}$($\forall$x$\forall$yFxy) = F
4. $Val_{\mathbf{j}_1}$($\exists$xFxx) = F
5. $Val_{\mathbf{j}_1}$($\forall$x$\forall$y(Fxx $\rightarrow$ Fyx)) = T
6. $Val_{\mathbf{j}_1}$($\exists$x$\forall$yFyx) = F

7. $Val_{\mathbf{j}_1}$(∃xF(x, fx)) = T
8. $Val_{\mathbf{j}_1}$(∀x[Fxx↔F(fx, fx)]) = T

Exercise 54 *(Determining truth values)*

Consider the following Q-interpretation $\mathbf{j}_3$.

$Domain_{\mathbf{j}_3} = \{1, 2, 3\}$. a=2, b=2, c=3.

F1=T	G1=T	g1=1	H11=F	J11=F
F2=T	G2=F	g2=3	H12=F	J12=T
F3=T	G3=T	g3=3	H13=F	J13=F
			H21=F	J21=F
			H22=F	J22=T
			H23=F	J23=F
			H31=F	J31=F
			H32=F	J32=T
			H33=F	J33=F

Using this Q-interpretation $\mathbf{j}_3$, determine the truth values on $\mathbf{j}_3$ of each of the following. Take any shortcuts that you like.

1. Jbc
2. J(a, ga)
3. ∀xJxx
4. ∀xJ(gx, x)
5. ∀x(Fx→Gx)
6. ∀x(Gx→Fx)
7. ∀x((Jax∨Jbx)→Fx)
8. ∀x((Jxa∨Jxb)→Fx)
9. ∃x∀yJyx
10. ∀x(∼Hxx→Jxx)
11. ∀x(Hxx→Jxx)
12. ∀x(∃yHxy→(Gx∨Hxx))

▷ . . . . ◁

6E Validity, etc.

Now that we are clear on "$Val_{\mathbf{j}}$(A) = T" ("A is true on **j**," or "**j** makes A true"), we go on to define a family of logical concepts by "quantifying over Q-interpretations."

These definitions are *exactly* analogous to those of §2B.2, which you should review. In our present circumstances, it is customary to pretend (or believe) that quantifier logic exhausts all of logic. This explains why e.g. "tautology" is replaced by "logical truth." We prefer, however, to tone down the claim implied by such a choice of vocabulary by routinely inserting a "Q." This insertion reminds us of the modesty of our goals; for example, "is a tautology" means "is true in virtue of our understanding of our six truth-functional connectives," and "is Q-valid" (or "is Q-logically-valid") means "is true in virtue of our understanding of the workings of individual constants and variables, of (extensional) operators and predicates, of our six truth-functional connectives, and of the two quantifiers." Nothing more, nothing less.

6E-1 DEFINITION. *(Q-logical truth or Q-validity; falsifiability or invalidity)*

Let A and B be sentences and let G be a set of sentences.

1. $\vDash_Q A \leftrightarrow_{df} Val_{\mathbf{j}}(A) = T$ for *every* Q-interpretation $\mathbf{j}$.

   In English, A is a Q-*logical truth* (or is Q-*valid*) $\leftrightarrow_{df} \vDash_Q A$.

2. $A \nvDash_Q \leftrightarrow_{df} Val_{\mathbf{j}}(A) = T$ for *some* Q-interpretation $\mathbf{j}$.

   In English, A is Q-*consistent* $\leftrightarrow_{df} A \nvDash_Q$.

3. $A \vDash_Q \leftrightarrow_{df} Val_{\mathbf{j}}(A) \neq T$ for *each* Q-interpretation $\mathbf{j}$.

   In English, A is Q-*inconsistent* $\leftrightarrow_{df} A \vDash_Q$.[2]

4. $\nvDash_Q A \leftrightarrow_{df} Val_{\mathbf{j}}(A) \neq T$ for some Q-interpretation $\mathbf{j}$. In English, A is Q-*falsifiable* (or Q-*invalid*) $\leftrightarrow_{df} \nvDash_Q A$.

5. $G \nvDash_Q \leftrightarrow_{df}$ there is *some* Q-interpretation $\mathbf{j}$ such that $Val_{\mathbf{j}}(A) = T$ for *every* member A of G.

   In English, G is Q-*consistent* $\leftrightarrow_{df} G \nvDash_Q$.

6. $G \vDash_Q \leftrightarrow_{df}$ there is *no* Q-interpretation $\mathbf{j}$ such that $Val_{\mathbf{j}}(A) = T$ for *every* member A of G.

   In English, G is Q-*inconsistent* $\leftrightarrow_{df} G \vDash_Q$.

---

[2]Of course "$Val_{\mathbf{j}}(A) = F$" is just as good as (or better than) "$Val_{\mathbf{j}}(A) \neq T$" in the definition of inconsistency; but if you use it, then in proofs you need to supply an extra (obviously true) premiss, namely, $\forall \mathbf{j} \forall A[Val_{\mathbf{j}}(A) = F \leftrightarrow Val_{\mathbf{j}}(A) \neq T]$ (the principle of bivalence).

7. $G \vDash_Q A \leftrightarrow_{df}$ for every Q-interpretation $\mathbf{j}$, if (for all B, if $B \in G$ then $Val_{\mathbf{j}}(B)$ = T) then $Val_{\mathbf{j}}(A)$ = T.

   In English, G Q-*logically implies* A $\leftrightarrow_{df}$ $G \vDash_Q A$. An argument with premisses G and conclusion A is then said to be Q-*valid*.

8. $A \approx_Q B \leftrightarrow_{df} Val_{\mathbf{j}}(A) = Val_{\mathbf{j}}(B)$ for all Q-interpretations $\mathbf{j}$.

   In English, A and B are Q-*logically equivalent* $\leftrightarrow_{df}$ $A \approx_Q B$.

   logical

9. A set of sentences G is Q-*independent* $\leftrightarrow_{df}$ there is no member A of G such that $G - \{A\} \vDash_Q A$.[3]

   That is, G is Q-independent iff no member of G is Q-logically implied by the rest of G. For example, {A, B, C} is Q-logically independent if (1) {A, B} does not Q-logically imply C, and (2) {A, C} does not Q-logically imply B, and (3) {B, C} does not Q-logically imply A.

Exercise 55 *(Symbolizing definitions of logical ideas)*

Symbolize the above definitions Definition **6E-1**(1)–(9) as well as you can. Helping remarks:

1. Whenever you use a quantifier, pin down a "type" for it by using one of the specially restricted variables: A, B, C for sentences, G for sets of sentences, and $\mathbf{j}$ for Q-interpretations (with or without subscripts or primes or whatever).

2. See that your definition is a universally quantified biconditional; you may, however, omit *outermost* universal quantifiers if you wish. On the other hand, you may never omit internal quantifiers! Let us add that no harm comes if you omit the "df" subscript.

3. These definitions are written in "middle English." When you see "for every member A of G," This just means "for every member of G," with the extra information that you should use "A" as the variable with which you express the universal quantification. But there are no *rules* available; there is no substitute for trying to figure out what is meant.

[3]We forbore offering an account of "TF-independence" in §2B.2 because such a concept, though respectable, is not of much use or interest.

Example. Given the task of symbolizing "*any* sentence A is Q-falsifiable iff A is false on *some* Q-interpretation **j**" you could write "$\forall A[\nvDash_{Q} A \leftrightarrow \exists \mathbf{j}[Val_{\mathbf{j}}(A) \neq T]]$."

▷ . . . . ◁

6F Using Q-interpretations

An "interpretation problem" asks you to produce a Q-interpretation that demonstrates that some sentence (or group of sentences) has one of the just-defined properties. Note well: (a) The produced Q-interpretation must *be* a Q-interpretation and (b) the produced interpretation must have the required properties. We break (a) into parts (1) and (2) below, and we break (b) into parts (3) and (4): An answer to a problem requiring the presentation of a Q-interpretation is best seen as having four parts, as follows. (1) State the domain of your proposed Q-interpretation. (2) Present (using one of two methods) the Q-interpretation of each non-logical symbol. (3) Exhibit what the various sentences come to on your proposed Q-interpretation. (4) State the truth values of the various sentences on your Q-interpretation, defend your claim that they have those truth values, and be explicit as to how this information solves the problem you began with. Parts (1) and (2) present the Q-interpretation itself, and parts (3) and (4) establish that the Q-interpretation does the work required.

6F-1 EXAMPLE. *(Q-interpretations)*

Problem: Show that

$$\exists xRxx \qquad [2]$$

is Q-falsifiable (not Q-logically true). We present the Q-interpretation using a biconditional.

1. Domain $= \{1, 2\}$

2. $Rv_1v_2 \leftrightarrow v_1 < v_2$

3. [2] would be true on this Q-interpretation just in case some number in the domain, that is, either 1 or 2, is less than itself. In other words, [2], on this Q-interpretation, says that either $1 < 1$ or $2 < 2$.

4. [2] is false on this Q-interpretation, since, by common knowledge, neither 1 nor 2 is less than itself ($\sim(1 < 1)$ and $\sim(2 < 2)$). Therefore [2] is false on some Q-interpretation; that is, [2] is falsifiable—not a logical truth.

Note that you must consult *both* instances R11 and R22 to justify the Fs under ∃xRxx.

Problem: Show that the same sentence, [2], is Q-consistent.

Observe that you *cannot* show this with the *same* Q-interpretation as used above, since on any given Q-interpretation, [2]—or any other sentence—can have exactly *one* truth value! We will present this second Q-interpretation via a table.

1. Domain = {1, 2}

2. R11 = T; R12 = T; R21 = F; R22 = F.

3. [2] would be true on this Q-interpretation just in case either R11 = T or R22 = T.

4. [2] is true on this Q-interpretation, since R11 = T. Therefore, [2] is true on some Q-interpretation, that is; [2] is Q-consistent.

Again, you must in principle consult *both* instances R11 and R22 in order calculate a value for ∃xRxx. Of course a T for *any* instance is enough for giving the existentially quantified formula a T; but the *principle* of checking every row is just the same.

Exercise 56 *(Q-interpretations)*

Do the calculational work by "reduction to truth functions" (Method **6D-4** on p. 191) if you wish—keeping in mind that if your domain is infinite, this will be impossible.

1. Give a Q-interpretation showing that the sentence below is Q-consistent.

   ∀xRax

2. Give a Q-interpretation showing that the sentence below is Q-falsifiable.

   ∃xR(x, f(x))

3. Give Q-interpretations to show that the sentence below is both Q-falsifiable and Q-consistent. (How many Q-interpretations will you need?)

$\forall x((Px \vee Qx) \rightarrow Px)$

4. Give a Q-interpretation to show that the following set of sentences is Q-consistent.

   $\forall x((Px \& Qx) \rightarrow Rx)$; $\forall x(Px \rightarrow \sim Rx)$

5. Give Q-interpretations to show that the following set of sentences is Q-independent. (How many Q-interpretations will you need?)

   $\forall x(Px \& Qx)$; $\exists x(Px \& Rx)$; $\exists x(Qx \& \sim Rx)$

6. Construct Q-interpretations to show that each of the following sentences is not Q-logically true.

   (a) $(\forall xPx \rightarrow \forall xQx) \rightarrow \forall x(Px \rightarrow Qx)$
   (b) $\forall x\forall y\forall z((Pxy \& Pyz) \rightarrow Pxz) \rightarrow \forall xPxx$
   (c) $\exists x\forall y(Pxy \rightarrow Pyx)$
   (d) $(\exists xPx \& \exists xQx) \rightarrow \exists x(Px \& Qx)$

7. Construct Q-interpretations to show that each of the following arguments is Q-invalid.

   (a) $\forall x(Fx \rightarrow Gx)$, $\forall x(Gx \rightarrow Hx)$ $\therefore$ $\exists x(Fx \& Hx)$
   (b) $\exists xFx$, $\forall x(F(hx) \rightarrow Gx)$ $\therefore$ $\exists xGx$
   (c) $\forall xF(gx)$ $\therefore$ $\forall xFx$
   (d) $\forall x\forall y(Mxy \rightarrow Nxy)$ $\therefore$ $\forall x\forall y(Mxy \rightarrow (Nxy \& Nyx))$

▷ . . . . ◁

Exercise 57 *(More symbolization and Q-interpretation)*

This needs to be supplied. Contributions are welcome.

▷ . . . . ◁

6G Semantics of identity

We need to say exactly when an identity sentence $t = u$ is true (and when false) on a given Q-interpretation:

> $Val_j(t{=}u) = T$ iff $Val_j(t) = Val_j(u)$.
>
> In other words, t = u is *true* on a Q-interpretation just in case that Q-interpretation assigns exactly the same individual (in the domain) to both t and u.[4]

Identity is the *only* predicate that receives a fixed "meaning" on every Q-interpretation; it is in this sense, too, a "logical" predicate. Identity is *sheer* identity, not some unexplained or context-relative "equality." Leibniz' "definition" of identity, which uses quantifiers ∀F over predicate positions—the grammar of the branch of logic that we are currently studying does not happen to include quantification over predicate positions—is as follows:

$$a{=}b \leftrightarrow \forall F(Fa \leftrightarrow Fb)$$

This says that a and b are identical iff they have exactly the same properties; if, that is, every property of a is also a property of b, and every property of b is also a property of a. In pseudo-English: a and b are identical iff they are indiscernible. (Left-to-right is the principle of identity of indiscernibles, and right-to-left is the principle of indiscernibility of identicals; or, contrapositively, they are the principles of differentiability of distinguishables, and distinguishability of differentiables. If, however, you cannot discern the difference between distinctness and difference, then perhaps you will wish to identify them. In any event, we mean to be identifying distinctness or distinguishability with non-identity, and differentiability with discernibility (that is, with non-indiscernibility).

Exercise 58 *(Identity semantics)*

1. Consider the following Q-interpretation.

   D = {1, 2, 3}, a = 2, b = 3.

F1 = T	G1 = T	f1 = 1	g1 = 2
F2 = T	G2 = F	f2 = 3	g2 = 1
F3 = T	G3 = T	f3 = 2	g3 = 2

h11 = 2	h12 = 1	h13 = 2
h21 = 3	h22 = 2	h23 = 1
h31 = 1	h32 = 2	h33 = 2

   Using this Q-interpretation, determine the truth values of the following.

[4] This clause in effect redefines what we mean by a "Q-interpretation."

(a) gb = hab
(b) $\text{haa} \neq \text{f(g(b))}$
(c) $\exists x \forall y(\text{hxy} \neq x)$
(d) $\exists x\text{Fgx}$
(e) $\exists x \forall y(\text{Fx} \leftrightarrow \text{G(fy)})$
(f) $\exists x\text{Fhxx}$
(g) $\forall x(\text{F(fx)} \rightarrow \text{Gx})$
(h) $\exists x(\text{hxx} = \text{gx})$
(i) $\forall x \forall y(\text{Ghxy} \leftrightarrow \text{Fhxy})$
(j) Ghab
(k) $\forall x\text{Gh(x, fx)}$
(l) $\forall x(x \neq a \rightarrow \text{F(fx)})$

2. Construct a Q-interpretation to show that the following sentence is not Q-valid.

   $\forall x \forall y(\text{fx} = \text{fy} \rightarrow x = y)$

3. Construct a Q-interpretation to show that the following argument is invalid.

   $\forall x \forall y(\text{fxy} = \text{fyx}) \;\therefore\; \exists x(x = \text{fxx})$

4. Two of the following equivalences are Q-logically valid and two are not. For each of the Q-invalid ones, provide a Q-interpretation on which it is false.

   (a) $\text{Fa} \leftrightarrow \forall x(x = a \,\&\, \text{Fx})$
   (b) $\text{Fa} \leftrightarrow \exists x(x = a \,\&\, \text{Fx})$
   (c) $\text{Fa} \leftrightarrow \forall x(x = a \rightarrow \text{Fx})$
   (d) $\text{Fa} \leftrightarrow \exists x(x = a \rightarrow \text{Fx})$

▷ . . . . ◁

6H Proofs and interpretations

Existential claims are easy to defend, hard to attack. To defend them, one only needs a single example. To attack them, one must show that nothing counts as an example. Contrariwise, universal claims are hard to defend, easy to attack. To attack them, one has only to produce a single counterexample; to defend them, one must show that nothing counts as a counterexample.

We have four key ideas: Q-validity, Q-implication, Q-inconsistency, and Q-equivalence. All of these receive semantic definitions of universal form. Hence, they are hard claims to defend. They are, however, easy to attack—one must only provide a *single* Q-interpretation to show a formula *not* valid, or that an Q-logical implication does *not* hold, or that a set is consistent (*not* inconsistent), or that two formulas are *not* equivalent. So in showing any of these "*not*" things, *use* Q-*interpretations*. But how?

What to do with Q-interpretations

Q-invalidity. To show A invalid, produce a Q-interpretation on which A is *false*.

Q-nonimplication. To show that a set G of formulas does not Q-logically imply a formula B, produce a Q-interpretation on which all members of G are true, while B is false.

Q-consistency. To show that a set G of formulas is consistent (not inconsistent), produce a Q-interpretation on which every member of G is true.

Q-nonequivalence. To show that A is non-equivalent to B, produce a Q-interpretation on which A and B have different truth values.

But *proofs* must be used to establish Q-validity, Q-logical implication, Q-inconsistency, or Q-equivalence. Why? Because these are *universal* semantic claims, claims about *all* Q-interpretations; and you could never finish the work by treating each Q-interpretation separately; there are just too many. Exactly how should one use proofs?

What to do with proofs

Q-validity. Show that A is Q-valid by producing a categorical proof of A.

Q-logical implication. Show that a set G of formulas Q-logically implies a formula A by producing a proof with hypotheses G and conclusion A.

Q-inconsistency. Show that G is a Q-inconsistent set by producing a proof with hypotheses G and conclusion $\perp$ (or, if you prefer, any explicit contradiction $A \& \sim A$ instead of $\perp$).

Q-logical equivalence. Produce two proofs, one that A Q-logically implies B and one that B Q-logically implies A (as required for showing Q-logical implication). Or produce a proof that $A \leftrightarrow B$ is Q-valid (as above).

The techniques using Q-interpretations work by definition. But the techniques using proofs rely on the fact that our logical system, Fi, is sound and complete. Let us concentrate on validity so as to cut down on repetition (exactly the same remarks apply to the other three concepts).

We are supposed to know what an Fi-*proof* is, namely, a figure using only the rules hyp, reit, TI, TE, Taut, Conditional Proof, RAA$\perp$, Case Argument, Biconditional Proof, UI, UG, EG, EI.

6H-1 DEFINITION. *(Fi-provable)*

A is Fi-*provable* if and only if there is a categorical Fi-proof of A.

6H-2 FACT. *(Fi-provability and Q-validity)*

A is Fi-provable if and only if A is Q-valid.

Fact **6H-2** has been mathematically proven by paid logicians. *We* use Fact **6H-2** to sanction our use of proofs—and of *only* proofs—in order to establish Q-validity. Note that F-provability is an existential notion: A is Fi-provable iff *there is* an Fi-proof of A. *That* is why we want to use the technique of proof when showing something *is* valid—like all existential claims, provability claims are easy to support (just produce an example proof). But the method of proof *cannot* be used to show that something is *not* valid, for in terms of proofs, A is invalid just in case *no* proof yields it—and that, again, suggests an infinite task. So use Q-interpretations for Q-invalidity and use proofs for Q-validity. Let's go back to Fact **6H-2**, and restate it, unpacking the defined terms "Fi-provable" and "Q-valid":

6H-3 COROLLARY. *(Fi-provability and Q-validity, restated)*

There is an Fi-proof of A just in case A is true on *every* Q-interpretation.

This is equivalent (as you can prove) to the following, which negates both sides:

6H-4 COROLLARY. *(Fi-provability and validity once more)*

There is *no* Fi-proof of A if and only if *some* Q-interpretation makes A false.

Point: We *always* use the existential side of Corollary **6H-3** to prove Q-validity, and we *always* use the existential side of Corollary **6H-4** to prove Q-invalidity. Thus, we *always* make things easy on ourselves.

Penultimate note. It is interesting that the *same* property can be expressed *both* existentially *and* universally (as in Corollary **6H-3**). As you can imagine, it is non-trivial to *prove* that the existential fact of Fi-provability is equivalent to the universal fact of validity. Oddly enough, the most difficult part of the proof is the part showing that the universal form implies the existential form; when one considers the apparent strength of universals and of existentials, one might have thought it was the other way around. That is what Gödel's famous completeness theorem is all about; see §7C for a sketch of its proof.

Final note. Given a sentence and asked to decide whether or not it is Q-valid, what should you do? This is what to do. (1) Try for a while to prove it (being careful not to make any flagging mistakes). Then (2) try for a while to find a Q-interpretation on which it has the value F. And then continue by alternating between (1) and (2) until you become bored. Alas, no better advice is available since, as noted in §13A, there is no mechanical method for deciding between Q-validity and its absence. On a more practical note, however, sometimes you can "see" from a failed attempt at a proof of Q-validity just why the proof is not working out; and that insight *may*, if you are fortunate, lead you to a definite Q-interpretation on which your candidate is false.

Exercise 59 *(Using proofs and* Q-*interpretations)*

Explain how you would use proofs or Q-interpretations in the following cases. (Do not actually produce any proofs or Q-interpretations. The exercise is simply to explain, abstractly, how you would use them)

Example. Show that A & B →(A →C) is invalid. Answer: *You need to produce a* Q-*interpretation on which it is false.*

1. Show that the set {A →B, A & ∼B} (taken together) is Q-inconsistent.
2. Show that A →(B →A) and A →(B ↔A) are Q-nonequivalent.
3. Show that the set {A, B, C} (taken together) fails to Q-logically imply D.
4. Show that the set {A, B, C, D} (taken together) is Q-consistent.[5]
5. Show that A →B and ∼A ∨ B are Q-equivalent.
6. Show A →(B →A) is Q-valid.

[5]If someone asks you to believe that free will, determinism, optimism, and the existence of evil are Q-consistent, that person is not asking you to believe merely that *each* is Q-consistent. You are instead being asked to believe in the consistency of the set of those four doctrines *taken all together*. At this point in your education, you need *already* to have learned enough about quantifiers to distinguish between "For each sentence in the set, there is a Q-interpretation on which it is true" and "There is a Q-interpretation on which every sentence in the set is true."

7. Show that A and B (taken together) Q-logically imply A & B.

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Chapter 7

Theories

This chapter is somewhat of a miscellaneous collection of theories based on Q-logic. Its section headings speak for themselves, if only in muted tones.

7A More symbolization of logical theory

The semantic theory of TF was developed roughly in §2B. Parts of the development can now be made a little more revealing with the use of set-theoretical concepts from §4A.

Also, we are using A and the like as variables ranging over sentences, G and its cousins to range over sets of sentence, and **i** as a variable ranging over TF interpretations. We use "$Val_{\mathbf{i}}$(A) = T" to mean that A is true on TF-interpretation **i**.

We rely on the previously given definition of tautological implication, which we here repeat, changing bound variables (CBV, **3E-4** on p. 129) as we see fit.

7A-1 DEFINITION. *($\vDash_{\text{TF}}$)*

$$G_1 \vDash_{\text{TF}} A_1 \leftrightarrow \forall \mathbf{i}[\forall B(B \in G_1 \to Val_{\mathbf{i}}(B) = T) \to Val_{\mathbf{i}}(A_1) = T].$$

That is, $G_1 \vDash_{\text{TF}} A_1$ just in case A_1 is true on every TF interpretation that makes every member of G_1 true—which is precisely the idea of implication. From this definition and earlier postulates, we can prove several facts about the semantics of truth-functional logic.. These are important, but here we are using them only as exercises.

7A-2 FACT. *(Structural properties of tautological implication)*

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1. *Identity.* $A \in \text{G} \rightarrow \text{G} \vDash_{\text{TF}} A$.
2. *Weakening.* $\text{G} \subseteq \text{G}' \rightarrow (\text{G} \vDash_{\text{TF}} A \rightarrow \text{G}' \vDash_{\text{TF}} A)$.
3. *Cut.* $[\text{G} \cup \{A\} \vDash_{\text{TF}} \text{C}\ \&\ \text{G} \vDash_{\text{TF}} A] \rightarrow \text{G} \vDash_{\text{TF}} \text{C}$.
4. *Finiteness.* $\text{G} \vDash_{\text{TF}} A \rightarrow \exists \text{G}_0[\text{G}_0$ is finite & $\text{G}_0 \subseteq \text{G}$ & $\text{G}_0 \vDash_{\text{TF}} A]$.

The following definition and fact are here just to offer an exercise—although they are in another context important ideas. (G* is just another variable over sets of sentences, and A* over sentences.)

7A-3 DEFINITION. *(Maximal* A**-free)*

G* is maximal A^*-free $\leftrightarrow_{df}$ $\text{G}^* \nvDash_{\text{TF}} A^*$ and $\forall \text{G}'[\text{G}^* \subset \text{G}' \rightarrow \text{G}' \vDash_{\text{TF}} A^*]$.

7A-4 FACT. *(Maximal* A**-free)*

G* is maximal A^*-free $\rightarrow \forall A(\text{G}^* \vDash_{\text{TF}} A \leftrightarrow A \in \text{G}^*)$

Exercise 60 *(Theory of* $\vDash_{\text{TF}}$ *formulated set-theoretically)*

1. Prove Fact **7A-2**(1)–Fact **7A-2**(3), using instances of Definition **7A-1** as many times as needed. You should also expect to use some set theory and the logic of identity, since these concepts occur in the statements of what you are to prove. (It is perhaps a little surprising that these facts do not rely on any properties of the "truth-value-of" operator $Val_{\mathbf{i}}$(A).)
2. Prove Fact **7A-4**. There is a proof of this fact that does not require additional appeal to the definition of $\vDash_{\text{TF}}$; in fact, you can rely on just the property Cut (provided you have succeeded in proving it).
3. Refrain from proving Fact **7A-2**(4). This is not only Hard, but does depend on properties of the truth-value-of operator.)

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7B Peano arithmetic

A fundamental theory finding a natural reconstruction amid quantifiers and identity is the arithmetic you learned in grade school, and whose theory you began to study under the heading "algebra" in somewhat later years. This reconstruction, due in essentials to Dedekind, was elaborated by Peano, from whom Whitehead and Russell learned it, from whom we all learned it, and is always named after Peano.

Grammar. There are four primitives: one constant, 0, and three operators that make terms looking like x', $x+y$, and $x\times y$.

Semantics. The operator "+" represents ordinary addition, while "$\times$" represents multiplication. The operator "$'$" represents the successor operation, so that x' is the successor of x, i.e., $x'=x+1$. The constant "0" represents 0. The intended domain of quantification is constituted by the non-negative integers $\{0, 1, 2, \ldots\}$.

Proof theory. First the axioms PA1–PA7 of Peano arithmetic (PA = Peano axiom). PA3—the induction axiom—is an "axiom schema," representing not one but infinitely many axioms—one for each choice of A(x).[1]

7B-1 Axioms. *(Peano axioms for arithmetic)*

PA1. $\forall x\forall y(x'=y' \rightarrow x=y)$

PA2. $\forall x{\sim}(x'=0)$

PA3. $[A(0) \,\&\, \forall x(A(x) \rightarrow A(x'))] \rightarrow \forall xA(x)$

PA4. $\forall x(x+0=x)$

PA5. $\forall x\forall y(x+y'=(x+y)')$

PA6. $\forall x(x\times 0=0)$

PA7. $\forall x\forall y(x\times y'=(x\times y)+x)$

[1] Here we write "A(x)" instead of "Ax" because of improved readability when terms with operators are substituted for "x," as for example in PA3; but the intention remains to rely on Convention **3A-2**.

"Peano's Axioms" usually include: N(0) and $\forall x(N(x) \rightarrow N(x'))$, and also "N" figures in P3; this is essential when "N" is thought of as representing only part of the domain of quantification, rather than (as here) all of it.

In carrying out proofs, it is convenient to replace the induction *axiom* by an induction *rule*, as follows:

	.		
	.		
	.		
1	A(0)	**?**	[0/x]
2	| A(a)	hyp (of induction), flag a for Ind.	[a/x]
.	| .		
.	| .		
.	| .		
k	| A(a′)	**?**	[a′/x]
n	$\forall$xA(x)	1, 2–k, Induction 0/x, a/x, a′/x	

That is, $\forall$xA(x) follows from A(0) and a proof flagged with (say) a, the hypothesis of which is A(a) and the last step of which is A(a′). So the first premiss for Induction says that 0 has the property A, and the second says that there is a way to show that a′ has the property A if a does—so every number must have A, which is what the conclusion says. The induction rule should be treated as a new kind of universal-quantifier-introduction rule, applicable only in the context of Peano arithmetic. It gives one a new strategy, apart from UG, for proving universal generalizations. Observe that its annotation involves *three* separate instantiations for x; this is the most confusing part of using the induction rule.

Just to help visualize what an inductive proof might look like, we repeat the above layout, except using triple dots instead of A.

	.		
	.		
	.		
1	… 0 … 0 …	**?**	[0/x]
2	| … a … a …	hyp (of induction), flag a for Ind.	[a/x]
.	| .		
.	| .		
.	| .		
k	| … a′ … a′ …	**?**	[a′/x]
n	$\forall$x(… x … x …)	1, 2–k, Induction 0/x, a/x, a′/x	

Exercise 61 *(Peano arithmetic)*

What follows are a few theorems of Peano arithmetic (PT = Peano theorem). Take these as exercises, and try to prove them in the order indicated, using old ones in the proof of new ones if wanted. Each is as usual to be construed as a universal generalization, as indicated explicitly in the first.

PT1. $\forall x\forall y\forall z[(x+y)+z=x+(y+z)]$

PT2. $0+x=x$

PT3. $x'+y=(x+y)'$

PT4. $x+y=y+x$

PT5. $x\times(y+z)=(x\times y)+(x\times z)$

PT6. $(x\times y)\times z=x\times(y\times z)$

PT7. $x'\times y=(x\times y)+y$

PT8. $x\times y=y\times x$

▷ . . . . ◁

Here is a proof of the first one. Note how two of the quantifiers are introduced by UG, while one is introduced by induction. "PA" refers to the Peano axioms above.

1	flag a, b	for UG	
2	$(a+b)+0=a+b$	PA4, a+b/x	
3	$(a+b)+0=a+(b+0)$	2, PA4, b/x, =	[0/z]
4	$(a+b)+c=a+(b+c)$	hyp of ind., flag c	[c/z]
5	$((a+b)+c)'=(a+(b+c))'$	4=(add. context: **3D-4**)	
6	$((a+b)+c)'=a+(b+c)'$	5, PA5 a/x, b+c/y, =	
7	$((a+b)+c)'=a+(b+c')$	6, PA5 b/x, c/y, =	
8	$(a+b)+c'=a+(b+c')$	7, PA5 a+b/x, c/y	[c′/z]
9	$\forall z[(a+b)+z=a+(b+z)]$	3, 4–8, Ind. 0/z, c/z, c′/z	
10	$\forall x\forall y\forall z[(x+y)+z=x+(y+z)]$	1–9, UG a/x b/y	

We have flagged here with constants a, b, c (and not with variables) because any time the induction rule comes into play, there is an extra threat of confusion. Using constants helps to avoid it.

One caution: When you attack a problem by "setting up for induction," be sure put a question mark on the "zero step" and *leave room in which to prove it*.

A hint: Unless we are mistaken (which is possible), you only need to use induction on one of the variables—the alphabetically last such. The alphabetically earlier quantifications can be proven by way of plain UG.

Another hint: In order to prove PT8 above, you are probably going to have to use induction to prove something that stands to PT8 as PT2 stands to PT4.

Exercise 62 *(Definitions in arithmetic)*

Many standard concepts can be defined in Peano arithmetic. What follows is a mixture of examples and exercises; the ones marked "exercise" call for definitions (not proofs).

1. Less-than: $\forall x \forall y[x < y \leftrightarrow \exists z(z \neq 0 \;\&\; x+z=y)$.
2. Less-than-or-equal ($\leqslant$): *Exercise*.
3. "Funny" subtraction: $\forall x \forall y \forall z[x - y = z \leftrightarrow (y \leqslant x \;\&\; x = z+y) \vee (\sim(y \leqslant x) \;\&\; z = 0)]$. (Recall that the domain for Peano arithmetic does not include negative numbers.)
4. Divisibility: x divides y (evenly): *Exercise*.
5. Primeness: x is prime: *Exercise*.
6. Even: $\forall x[x \text{ is even} \leftrightarrow \exists y[0'' \times y = x]]$
7. True or false? Every even number is the sum of two primes. *Exercise*: Symbolize this statement.

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7C Gödel's completeness theorem for the system Fi of intelim proofs

We have used the *double* turnstiles $\vDash_{TF}$ and $\vDash_{Q}$ for speaking of semantic relations. It is convenient to introduce turnstiles also on the side of proofs; for these we use *single* turnstiles $G \vdash^{Fi}_{TF} A$ and $G \vdash^{Fi}_{Q} A$. We want to use them in connection with *intelim Fitch proofs* as follows:

7C-1 DEFINITION. *(Fitch-intelim-provable:* $G \vdash^{Fi}_{TF} A$ *and* $G \vdash^{Fi}_{Q} A$*)*

- $G \vdash^{Fi}_{TF} A$, read "A is intelim TF Fitch-provable from G," $\leftrightarrow_{df}$ there is a hypothetical truth-functional *intelim Fitch proof* (only truth-functional intelim rules permitted) all of whose hypotheses belong to G and whose conclusion is A. For a list of permitted rules, see Definition **2C-23**.
- $G \vdash^{Fi}_{Q} A$, read "A is intelim Q Fitch-provable from G," $\leftrightarrow_{df}$ there is a hypothetical *intelim Fitch proof* (only truth-functional or quantifier intelim rules permitted) all of whose hypotheses belong to G and whose conclusion is A. For the list of permitted rules, see Definition **3E-14**.

Note that G can include "junk": Not all the members of G need to be listed among the hypotheses, whereas all the hypotheses must belong to G.

7C-2 THEOREM. *(Soundness and completeness)*

The systems of Fitch intelim proofs for truth-functional logic and for quantifier logic are sound and complete:

- $G \vdash^{Fi}_{TF} A$ iff $G \vDash_{TF} A$, and
- $G \vdash^{Fi}_{Q} A$ iff $G \vDash_{Q} A$.

From left to right is called *soundness* (or sometimes *semantic consistency*) of the proof-system: Every argument that the system of intelim proofs certifies as "OK" is in fact semantically valid (in the appropriate sense). Soundness means that the system of intelim proofs does not "prove too much," inadvertently letting in some semantically bad arguments.

The argument for soundness is an "induction" rather like induction in Peano arithmetic. You show that intelim proofs of length 1 make no mistakes, and then show (more or less) that if all intelim proofs of length n are all right, then so are all intelim proofs of length $n+1$. Hence you may conclude by induction that all intelim proofs, no matter their length, are OK. Working out the details is at this point inappropriate.

From right to left is called *completeness* of the proof system: If a certain argument is semantically valid (in the appropriate sense), then the proof system certifies it as "OK." Completeness means that no semantically valid argument is omitted by the proof system.

The argument for completeness is entirely different. We restrict attention to the simpler truth-functional case, and even then we lay out only a bare outline, suppressing almost all details.

To begin with, we need some lemmas. They contain all the hard work of proving that the system of intelim proofs is complete. You should be able to understand these lemmas, but their proofs are well beyond our scope. All of these are to be interpreted as having universal quantifiers on the outside governing G, G*, A, and A*, as well as A_1 and A_2.

In Fact **7A-2** we considered identity, weakening, etc., as properties of the double turnstile, $G \vDash_{TF} A$, of semantics. The following lemma claims that strictly analogous properties hold for the single turnstile, $G \vdash^{Fi}_{TF} A$, of proof theory.

7C-3 LEMMA. *(Structural properties of* $\text{G} \vdash^{\text{Fi}}_{\text{TF}} \text{A}$*)*

The properties identity, weakening, and cut all hold for $\text{G} \vdash^{\text{Fi}}_{\text{TF}} \text{A}$, and so also does finiteness.

1. *Identity.* $A \in \text{G} \to \text{G} \vdash^{\text{Fi}}_{\text{TF}} A$.
2. *Weakening.* $\text{G} \subseteq \text{G}' \to (\text{G} \vdash^{\text{Fi}}_{\text{TF}} A \to \text{G}' \vdash^{\text{Fi}}_{\text{TF}} A)$.
3. *Cut.* $[\text{G} \cup \{A_1\} \vdash^{\text{Fi}}_{\text{TF}} A_2 \;\&\; \text{G} \vdash^{\text{Fi}}_{\text{TF}} A_1] \to \text{G} \vdash^{\text{Fi}}_{\text{TF}} A_2$.
4. If $\text{G*} \nvdash^{\text{Fi}}_{\text{TF}} \text{A*}$ and $\forall \text{G}'[\text{G*} \subset \text{G}' \to \text{G}' \vdash^{\text{Fi}}_{\text{TF}} \text{A*}]]$ (that is, if G* is maximal A*-free in the single-turnstile sense), then $\forall \text{A}[\text{G*} \vdash^{\text{Fi}}_{\text{TF}} \text{A} \leftrightarrow \text{A} \in \text{G*}]$.
5. *Finiteness.* $\text{G} \vdash^{\text{Fi}}_{\text{TF}} A \to \exists \text{G}_0[\text{G}_0$ is finite & $\text{G}_0 \subseteq \text{G}$ & $\text{G}_0 \vdash^{\text{Fi}}_{\text{TF}} A]$.

In the following we make *explicit* reference only to Lemma **7C-3**(1); the remainder are needed only for the proofs of upcoming lemmas, whose explicit proofs we omit. You proved the double-turnstile analogs of all except Lemma **7C-3**(5) as Fact **7A-2**. Those proofs, however, are worthless here, since they relied on Definition **7A-1** of the *double* turnstile, whereas Definition **7C-1** of the *single* turnstile is entirely different in form: For instance, the former is a *universal* quantification over TF-interpretations, whereas the latter is an *existential* quantification over intelim proofs. The only exception is Lemma **7C-3**(4), which follows from the single-turnstile cut (Lemma **7C-3**(3)) in *exactly* the same way as in your Exercise 60. All proofs omitted—but it is easy enough to convince yourself informally that they must be true.

7C-4 LEMMA. *(Lindenbaum's lemma)*

If $\text{G} \nvdash^{\text{Fi}}_{\text{TF}} \text{A*}$ (which is to say, if G is A*-free in the single-turnstile sense) then G can be extended to a G* that (in the single-turnstile sense) is maximal A*-free. That is,

> if $\text{G} \nvdash^{\text{Fi}}_{\text{TF}} \text{A*}$ then there is a G* such that $[\text{G} \subseteq \text{G*}$ and $\text{G*} \nvdash^{\text{Fi}}_{\text{TF}} \text{A*}$ and $\forall \text{G}'[\text{G*} \subset \text{G}' \to \text{G}' \vdash^{\text{Fi}}_{\text{TF}} \text{A*}]]$.

You will see below how we use Lindenbaum's lemma (as it is called). We omit its proof because the proof of Lindenbaum's lemma uses ideas and principles not present in this book.[2]

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[2]The proof can, however, be arm-waved on a suitably large blackboard.

The next (and last) lemma delivers a critical property of maximal A*-free sets.

7C-5 LEMMA. *(Truth like)*

If G* $\nvdash^{\text{Fi}}_{\text{TF}}$ A* and $\forall$G′[G* $\subset$ G′ $\rightarrow$ G′ $\vdash^{\text{Fi}}_{\text{TF}}$ A*] (that is, if G* is a maximal A*-free set), then there is a TF interpretation **i** such that membership in G* exactly agrees with truth on **i**:

$$\exists \mathbf{i} \forall A[Val_{\mathbf{i}}(A) = T \leftrightarrow A \in G^*].$$

We sometimes say that G* is *truth like* if its membership agrees with truth on some one TF interpretation. So this lemma says the following: *Every maximal* A*-*free set is truth like.* As above, we omit the proof.

Having stuffed all the "hard" parts into lemmas, we are now ready to give you an outline of a proof of the completeness of the system of intelim proofs. The easiest proof to follow proceeds via contraposition. Here is how it goes, in bare outline, for the truth-functional case (the argument for completeness in the case of quantifiers is substantially more complicated). Assume for arbitrary G and A* that *(a)* G $\nvdash^{\text{Fi}}_{\text{TF}}$ A*; we show that under this hypothesis, *(z)* G $\nvDash_{\text{TF}}$ A*. (The standard statement of completeness—the right-to-left part of Theorem **7C-2**—then follows by contraposition.) Our strategy is to show *(z)* by showing its definitional equivalent: *(y)* there is a TF-interpretation **i** such that *(y_1)* $\forall$A[A $\in$ G $\rightarrow$ $Val_{\mathbf{i}}$(A) = T] and *(y_2)* $Val_{\mathbf{i}}$(A*) $\neq$ T (all members of G are true on **i**, but A* is not).

First put Lemma **7C-4** together with *(a)*, and then use EI on the result (flagging G*) to obtain *(c)* G $\subseteq$ G*, *(d)* G* $\nvdash^{\text{Fi}}_{\text{TF}}$ A*, and *(e)* $\forall$G′[G* $\subset$ G′ $\rightarrow$ G′ $\vdash^{\text{Fi}}_{\text{TF}}$ A*]. Now *(d)* and *(e)* give us the antecedent of Lemma **7C-5**, so that we may use EI on the result (flagging **i**) to give *(f)* $\forall$A[$Val_{\mathbf{i}}$(A) = T $\leftrightarrow$ A $\in$ G*]. It is easy to see that *(f)* and *(c)* imply *(y_1)* to the effect that all members of G are true on **i**.

We also need to show *(y_2)*; we do so by a *reductio*, starting with the assumption to be reduced to absurdity, namely *(h)* $Val_{\mathbf{i}}$(A*) = T. Note that *(h)* and *(f)* together imply that *(i)* A* $\in$ G*. Now this in turn implies that *(j)* G* $\vdash^{\text{Fi}}_{\text{TF}}$ A* by Identity for the single turnstile, Lemma **7C-3**(1). But this contradicts *(d)* and completes the *reductio*, thereby establishing *(y_2)*. Since we have shown *(y_1)* and *(y_2)*, we may existentially generalize to *(y)*, which by definition implies *(z)*, and completes the proof that (given the lemmas whose proofs were omitted) the system of intelim proofs for truth-functional logic is complete.

Exercise 63 *(Completeness of the system of intelim proofs for truth-functional logic)*

Set out the foregoing proof as a strategic proof in the sense of Definition **3E-14** on p. 138, using of course the quantifier rules, and omitting only the wholesale rules TI, TE, and Taut. You should present the proof as hypothetical, with the lemmas Lemma **7C-3**(1), Lemma **7C-4**, and Lemma **7C-5** serving as its hypotheses and with conclusion $\forall G \forall A[G \vDash_{TF} A \to G \vdash^{Fi}_{TF} A]$. For accuracy, when you enter the lemmas, include all needed outermost universal quantifications.

▷ . . . . ◁

7D Gödel incompleteness

A certain form of Gödel's argument shows that each formal arithmetic that is semantically consistent must be semantically incomplete. This conclusion is less strong and less pure then Gödel's own proof-theoretical result (as modified by Rosser) to the effect that each formal arithmetic that is negation consistent must be negation incomplete. But it does reveal at least some of the central ideas.

7D.1 Preliminaries

Words are used as follows.

7D-1 DEFINITION. *(Arithmetic)*

Arithmetic. The *grammar* of truth functions, quantifiers, identity, zero, one, addition, and multiplication (as in §7B). The obvious and usual *semantic* accounts of "semantic value": denotation (of arithmetic terms) and truth (of arithmetic sentences), where the domain of quantification is understood as the set of non-negative integers $\{0, 1, \dots\}$.

***Formal* arithmetic.** A system of arithmetic is *formal* iff its *proof theory* is based on a family of axioms and rules such that it is effectively decidable to say of an appropriate candidate whether or not it is an axiom, and whether or not it is an instance of one of the rules. In other words, theoremhood in the arithmetic is "formalizable" (§13A).

Let S be a particular formal system of arithmetic,

Semantically consistent. S is *semantically consistent* iff all theorems of S are true.

Semantically complete. S is *semantically complete* iff all truths are theorems of S.

Negation-consistent. S is *negation consistent* iff for no A are both A and ∼A theorems of S.

Negation-complete. S is *negation complete* iff for each A, at least one of A and ∼A are theorems of S.

Now we may state

7D-2 THEOREM. *(Gödel's incompleteness theorem, semantic version)*

There is no formal arithmetic S such that theoremhood in S coincides with arithmetic truth; or, equivalently, there is no semantically consistent formal arithmetic S that is semantically complete. That is, the concept of arithmetic truth is not formalizable.

To show how it goes, we pick on a group of islanders far away who have a formal system S that they think of as arithmetic. We will show that the notion "theorem of S" does not coincide with truth, so that they have failed fully to "formalize arithmetic."

When we speak of the islander's arithmetic, we use the following.

Our variables	**What they stand for**
n	Numbers (if we ever need to speak of them).
K, M, N	Islanders' number terms (actually, these are the only terms they have). The islanders use their number terms to stand for numbers. We use the letters, "K," "M," and "N" to range over their number terms.

A, T(v), G	Islanders' sentences. They use their sentences to make arithmetic statements. We use "A," etc., as variables ranging over their sentences.
v, v′	Islanders' variables (over numbers 0, 1, ...). We are using "v" and "v′" as names of two of these variables; it turns out that we will not need to talk of any of the others

You understand that what is written down here is in our own language. No part of the islanders' language will ever be displayed. In this heavy-handed way we entirely avoid the threat posed by Confusion **1B-16** on p. 19.

There are three pieces of "hard work" that we will just assume so as to get on with it. The first work is "Gödel-numbering." Or better, for our immediate purposes, "Gödel-number-terming." Gödel needed to show that our language contains an (islander number term)-to-(islander sentence) operator, which we are writing for transparency as "the N-sentence":

> the N-sentence is an islander sentence whenever N is an islander number term.

Exactly how to find the N-sentence when you are given the number term N is important for a deeper understanding of Gödel's work, but not for what we are doing. All we need is a guarantee that each islander sentence is in fact correlated with a number term:

7D-3 FACT. *(Gödel numbering)*

Every islander sentence has a number term: For each islander sentence A (whether open or closed), there is an islander number-term N such that A = the N-sentence. In symbols: ∀A[A is an islander sentence → ∃N[N is an islander number term & (A = the N-sentence)]].

In other words, the function expressed by "the N-sentence" is from the islander number terms onto the islander sentences.

The next work is to show that the islanders themselves can find an open sentence (only v will be free in it) that will stand in for our concept of theoremhood (of their system S of arithmetic). This is Really Hard Work. The idea is that this theoremhood-of-S-expressing open sentence shall be true when completed by an islander number term just in case the islander sentence associated with that number term is a theorem of the islanders' system S of arithmetic. As notation, we'll assume that the open sentence in question is T(v), so that what we are promising to locate is an islander open sentence T(v) that expresses "theoremhood in S" in the following sense.

7D-4 FACT. *(Expressibility of theoremhood)*

There is an islander sentence T(v), with v its only free variable, such that for every islander number term K, T(K) is true $\leftrightarrow$ the K-sentence is a theorem (of the islanders' arithmetic). So for an islander to say "T(K)" is tantamount to our saying "the K-sentence is a theorem of S."[3]

The final work is to show that substitution is expressible in the islanders' language. We'll say this technically first, because that is what is important. The assumption is that there is in the islanders' arithmetic an open term Sub(v, v′) such that for every pair of islander number terms N and M, Sub(N, M) is correlated with the result of putting the number term M for every free occurrence of the variable v in the N-sentence.

7D-5 FACT. *(Expressibility of substitution)*

There is in the islander language an open term Sub(v, v′), with v and v′ its only free variables, such that for any islander open sentence A(v), and any islander number term N, if A(v) = the N-sentence, then for every islander number term M, A(M) = the Sub(N, M)-sentence.[4]

In other words, if the number term N is correlated with A(v), then the number term Sub(N, M) is correlated with the result of putting the number term M for every free occurrence of v in A(v). A picture should be helpful in engendering a feel for Sub(v, v′).

[3]The existential Fact **7D-4** is squarely dependent on the islanders' arithmetic being formal. All bets are off if they appeal to Delphi. Maybe there is a Delphic concept of theoremhood, but it is only formalizable notions of theoremhood that we promise to locate inside of arithmetic.

[4]In fact there is no such term as Sub(v, v′): Addition and multiplication and 0 and 1 are not enough to express substitution. But what we have said is nearly true, since one can express the corresponding three-term relation with an open sentence of the islanders' arithmetic. The nice thing about open sentences is that they can contain quantifiers, whereas terms cannot, so that their expressive power is far greater. But pretending that we express substitution with a term is harmless, and it makes our presentation easier to follow.

Number Term	Sentence	Ponderously
N	A(v)	A(v) = the N-sentence
Sub(N, M)	A(M)	A(M) = the Sub (N, M)-sentence
Sub (N, K)	A(K)	A(K) = the Sub (N, K)-sentence

Hence, if you have a number term for the open sentence A(v), you automatically have a number term for each of its instances. Good!

7D.2 The argument

We now proceed as follows.[5] Fact **7D-4** guarantees the expressibility of theoremhood in the islander's language, so consider the critical open sentence T(v) mentioned in Fact **7D-4**. We know by Fact **7D-3** that T(v) has to have a number term:

T(v) = the K-sentence (some number term K).

As a special case of Fact **7D-5**, we note that we could, if we wished, calculate a number term for any substitution instance of T(v): For any term M, T(M) must be the Sub(K, M)-sentence. We do not, however, care about this. We are really more interested in unprovability or non-theoremhood than we are in theoremhood. That is, we want to focus for a moment on the open sentence ∼T(v), which expresses non-theoremhood in S. The open sentence ∼T(v), too, will have a number term, by Fact **7D-3**, and so by Fact **7D-5** we would be able to calculate a number term for any of its substitution instances. But even the open sentence expressing non-theoremhood is not what we can finally use as a basis for Gödel's argument. What we are really interested in is the more complex open sentence, ∼T(Sub(v, v)) that is obtained by putting the open term Sub(v, v) in place of v in the open sentence ∼T(v) that expresses non-theoremhood. The observation that this more complex open sentence has a number term is the first step in the argument:

1. ∼T(Sub(v, v)) = the N-sentence (some number term N, which we are flagging). Reason: Fact **7D-3**.

As a special case of Fact **7D-5**, we note that we can now calculate a number term for any substitution instance of ∼T(Sub(v, v)): For any term M, ∼T(Sub(M, M)) must be the Sub(N, M)-sentence. We do not, however, care to calculate all these number terms. We are interested in only the particular instance of this generalization for

[5] The argument itself is given in Fact **7D-3**, Fact **7D-4**, Fact **7D-5** and the steps numbered 1–12, which are self-contained and can be read by themselves.

the number term N promised in step 1, and this gives us the critical second step of the argument.

2. ∼T(Sub(N, N)) = the Sub(N, N)-sentence. Reason: 1, Fact **7D-5**.

The trick is that the number term N is playing a double role here, both as the number term of the open sentence and as the number term to be substituted for the variable. The outcome is a sentence of the islander's language that "says of itself" that it is unprovable. All we need to do is verify that (1) this not quite viciously self-referential claim is true, and that (2) it leads to the incompleteness of the islander's formal system.

The next step, 3, just gives us a short way of writing; it is redundant but helpful. Step 4 is keeping track.

3. G = ∼T(Sub(N, N)) (local definition).

4. G = the Sub(N, N)-sentence. Reason: 2, 3, identity.

So G is the sentence that "says of itself" that it is unprovable. The rest of the argument falls into two parts. In Part I we conclude that G does indeed have the self-referential property we just informally ascribed to it. We do this by establishing that

> (I) G is true ↔ G is not a theorem of S.

This is step 9 below. In Part II we rely on (I) in order to conclude that G itself is a witness to either the semantic inconsistency or the semantic incompleteness of S:

> (II) Either G is true and not a theorem of S or else it is a theorem of S but not true.

This is step 10 below. Then we immediately obtain the conclusion of the whole argument as step 12.

All of this will be easy. The hard work lies behind us in Fact **7D-4** and Fact **7D-5**.

Part I.

5. [Extra number.]

6. G is true ↔ ∼T(Sub(N, N)) is true. Reason: 3, = (Added context)

7. G is true ↔ T(Sub(N, N)) is not true. Reason: 6, semantics for ∼.

8. G is true ↔ the Sub(N, N)-sentence is not a theorem of S. Reason: 7, Fact **7D-4**, logic.

9. G is true ↔ G is not a theorem of S. Reason: 8, 4, identity.

Part II

10. Either G is true and not a theorem of S or else it is a theorem of S but not true. Reason: 9, truth tables.

11. Either some sentence is true and not a theorem of S or else some sentence is a theorem of S but not true. Reason: 10, logic.

12. Either the islander's formal arithmetic is not semantically consistent or else it is not semantically complete. In either event, the notion of "theoremhood in S" does not agree with the notion of "truth." Reason: 11, Definition **7D-1** of key terms.

Since step 12 is what we wanted, we are done. We can add a little more information if we want. We all think that the islanders' arithmetic is in fact consistent (after all, if it weren't, surely someone would have found a contradiction by now). Let us then use this additional hypothesis: All theorems of S are true. Now we can show that the Gödel sentence, G, is itself true and not a theorem of S (exercise). Of course the clever islanders might decide (Delphically?) to add G as an axiom to their formal arithmetic; why not? Let $S' = S + G$. But they will not approach completeness more closely, for we can find a new Gödel sentence, say G', with which we can pull off the same trick, but now relative to their new formal arithmetic S' instead of relative to their old formal arithmetic.

This argument should be compared with Tarski's "Paradox of the Liar." Suppose arithmetic can express its own truth:

Tr(K) is true ↔ the K-sentence is true.

Then run the same argument with "Tr" in place of "T." You will find that the analog of (I) above, which is Step 9 above, is now not just strange, but an outright contradiction. As so often, Gödel skirts the very edge of paradox.

Chapter 8

Definitions

Definitions are crucial for every serious discipline.[1] Here we consider them only in the sense of explanations of the meanings of words or other bits of language. (We use "explanation" as a word from common speech, with no philosophical encumbrances.) Prominent on the agenda will be the two standard "criteria"—eliminability and conservativeness—and the standard "rules."

8A Some purposes of definitions

There are two especially clear social circumstances that call for a meaning-explaining definition, and then many that are not so clear. The clear ones call either for (1) "dictionary" definitions or for (2) "stipulative" definitions; some of the less clear circumstances call for (3) "analyses."

8A.1 "Dictionary" or "lexical" definitions

One might need to explain the existing meaning of a word already in use in the community, but unfamiliar to the person wanting the explanation.

[1] This chapter draws on Belnap (1993), which in turn found its basis in earlier versions of this chapter. The cited article contains material not present here, including an exceedingly small history of the theory of definitions. Even though much is omitted, however, you may well find this chapter more "talky" (more "philosophical"?) than some others.

8A-1 EXAMPLE. *(Lexical definitions)*

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1. What is a *sibling*? A sibling is a brother or a sister. In notation: $\forall$x[x is a sibling $\leftrightarrow$ (x is a brother $\vee$ x is a sister)].
2. What does it mean to *square* a number? One obtains the square of a number by multiplying it by itself. In notation: $\forall x[x^2 = (x \times x)]$.
3. What do you mean by *zero*? Zero is that number such that when it is added to anything, you get the same thing back. In notation: $\forall x[0 = x \leftrightarrow \forall y((y + x) = y)$.
4. What is a *brother*? A brother is a male sibling. In notation: $\forall$x[x is a brother $\leftrightarrow$ (x is male & x is a sibling)].

We postpone discussion of the question whether these interchanges should be sprinkled with quotation marks and whether or not the circularity threatened by combining the sibling/brother examples (1) and (4) is vicious.

8A.2 "Stipulative" definitions

One might wish to explain a proposed meaning for a *fresh* word. The purpose might be to enrich the language by making clear to the community of users that one intends that the new word be used in accord with the proposed meaning. This case squarely includes the putting to work of a previously-used word with a new technical meaning.

8A-2 EXAMPLE. *(Stipulative definitions)*

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1. Let ω be the first ordinal after all the finite ordinals.
2. Let a *group* be a set closed under a binary operation satisfying the following principles . . . .
3. By a *terminological realist* I mean a philosopher who subscribes to the following doctrines . . . .

The person who introduces a new word might have any one of various purposes. Here is a pair:

1. The person might want a mere abbreviation, to avoid lengthy repetition: "By NAL I refer to *Notes on the art of logic*."
2. Or the person might take it that he or she is cutting at a conceptual joint: An *elementary functor* is a functor such that either all of its inputs are terms or all of its inputs are sentences; and whose output is either a term or a sentence (Definition **1A-2**).

Note that in both cases the defined expression has no meaning whatsoever prior to its definition—or in any event no meaning that is relevant in any way to its newly proposed use.

8A.3 "Analyses" or "explications"

There are many cases not exhibiting either of these clear purposes, including perhaps most distinctively philosophical acts of definition. In these cases (Carnap calls some of them "explications") one wants both to rely on an existing meaning and also to attach a *new*, proposed meaning; it seems that one's philosophical purposes would not be served if one let go of either pole.

8A-3 EXAMPLE. *(Explicative definitions)*

1. Let *knowledge* (in the present technical sense) be justified true belief. In notation: $\forall x \forall p$[x knows p $\leftrightarrow$ x believes p & x is justified in believing p & p is true].
2. A choice is *right* if it produces the greatest good for the greatest number.
3. We say that a set G *tautologically implies* a sentence A iff A is true on every TF interpretation in which every member of G is true. In notation (see Definition **2B-14**(4)): $G \vDash_{TF} A \leftrightarrow_{df}$ for every TF interpretation $\mathbf{i}$ [if (for every sentence B, if $B \in G$ then $Val_{\mathbf{i}}(B) = T$) then $Val_{\mathbf{i}}(A) = T$].

Observe that in these cases the philosopher neither intends simply to be reporting the existing usage of the community, nor would his or her purposes be satisfied by substituting some brand new word. In some of these cases it would seem that the philosopher's effort to explain the meaning of a word amounts to a proposal for "a good thing to mean by" the word. We learned this phrase from Alan Ross Anderson. Part of the implication is that judging philosophical analyses is like judging eggs: There are no shortcuts; each has to be held to the candle.

8A.4 Invariance of standard theory across purposes: criteria and rules

The extraordinary thing is this: The applicability of the standard theory of definition remains invariant across these purposes—just so long as the purpose is to "explain the meaning of a word." An early manifestation of this invariance is that in any case, the word or phrase being defined is called the *definiendum* (the defined), and the word or phrase that is intended to count as the explanation is called the *definiens* (the defining).

This standard theory has two parts. In the first place, it offers two *criteria* for good definitions: the criterion of eliminability and the criterion of conservativeness. Later, in §12A.1 and §12A.2 we give full explanations of these criteria. For now, the following extremely rough account will do.

8A-4 CRITERIA. *(Criteria of eliminability and conservativeness, extremely roughly put)*

- **The criterion of eliminability** requires that the definition permit the elimination of the defined term in favor of previously understood terms.
- **The criterion of conservativeness** requires that the definition not only not lead to inconsistency, but not lead to anything—not involving the defined term—that was not obtainable before.

In the second place, the standard theory offers some *rules* for good definitions, rules which if followed will guarantee that the two criteria are satisfied.

The *criteria* are like the logician's account of the Q-validity of an argument (Definition **6E-1**(7)): Q-validity is proposed as an account of "good inference," and satisfaction of the criteria as an account of "good definition." And the standard *rules* are like the logician's rules defining derivability in a particular system such as Fitch intelim proofs (Definition **7C-1**): As indicated in Theorem **7C-2**, if you follow the Fitch intelim rules for constructing derivations, you will derive all and only correct semantic consequences, i.e., you will make all and only "good inferences." In a similar way, if you follow the logician's rules for constructing definitions, you will offer all and only definitions that satisfy the criteria of eliminability and conservativeness. You will, that is, offer all and only "good definitions."[2] (There is, however, a different and richer account of "good definition." This we postpone to the end of §12A.3.)

[2]Thanks to M. Kremer for the analysis leading to this fact, and for communicating its (unpublished) proof.

8A.5 Limitations

For simplicity we severely limit our discussion of the theory of definitions as follows. First, the discussion applies only to a community of language users whose language can profitably be described by labeling it an "applied first-order logic" such as the one that we have been learning.[3] You should imagine that this language may contain a fragment of English, a fragment that the users think of as structured in terms of predicate, function, individual, and perhaps sentence constants; truth-functional connectives; individual variables; quantifiers; and identity. But you should *also* imagine that this language is *really used* and therefore has lots of English in it. Think of the language that some mathematicians sometimes employ, or indeed of the language that we employ in these notes: a mixture of English and notation, but with the "logic" of the matter decided by first-order logic. For example, the language will use English common nouns such as "set"; but only in locutions such as "Some sets are nonempty" that the users think of as symbolizable in the usual first-order way.

We need to say more about what this first limitation means. So far we have said that the users take the *grammar* of their language in the standard first-order way. They also think of their *proof theoretical* notions as given in the standard way: We are thinking of "axioms," "rules," "theoremhood," "derivability from premisses," "theory," "equivalent relative to a theory," and so forth. They have learned their Fitch, or something equally as good, and they freely use the notation $G \vdash_{Q}^{Fi} A$ as in Definition **7C-1** for "A is derivable by Fitch intelim rules from premisses G." And lastly, they think of their *semantic* concepts in the standard way: "Q-logical truth," "Q-logical implication by some premisses," "Q-logical equivalence relative to a theory," and so forth, as encoded in the notations $G \vDash_{Q} A$, etc. of Definition **6E-1**. They know that there is agreement between the appropriate proof-theoretical and semantic ideas (§7C).

In the second place, we consider only definitions (explanations of meaning) of predicate constants and function constants and individual constants. The defined constant is traditionally called the *definiendum*, and we will follow this usage. To keep our discussion suitably general, we let **c** stand for an arbitrary definiendum, whether predicate constant, function constant, or individual constant.

If **c** is a predicate constant, we expect **c** to have a definite "n-arity" and therefore to be associated with a unique predicate (a functor, with blanks; see Definition **1A-1**). For example, if the predicate constant is R of n-arity 2, we expect that it uniquely

[3]This limitation, made to keep our discussion within bounds, is severe; we have by no means been studying all of logic!

determines a predicate (functor) such as "_R_." To avoid circumlocution, we will sometimes speak of the predicate instead of the predicate constant as the definiendum, not being careful to distinguish which of these we are considering. We treat operator constants vs. operators in the same way: They are different, but we feel free to call either a definiendum. In the third place, we are going to follow the standard account in considering only definitions that are themselves sentential. In imposing this limitation we do not intend to be deciding whether the act of defining is "really" imperatival instead of assertional, or "really" metalinguistic rather than not. We do intend to assert, however, that the technical theory of definition goes very much more easily if in giving it one can assume that the only "logic" involved is a logic that applies to declarative sentences.

We can therefore see that the policy, exhibited in Example **8A-1**, of giving example definitions above as plain sentences correctly forecast this decision. It is simpler to take the definition of "sibling" to be "Anything is a sibling if and only if it is a brother or a sister" instead of "Replace 'sibling' by 'brother or sister' wherever found!."

Fourth, we consider only one definition and one definiendum at a time. Most theories require many, many definitions in order to be interesting, but you should picture that these are added in a series, one by one, and that the discussions to follow apply to the very point at which a single new definition is added to the underlying theory and all previous definitions.

8A.6 Jargon for definitions

The entire discussion goes more smoothly if we introduce some jargon. This will be confusing on first reading, but if you master this jargon, you will find the discussion to follow much more transparent. As indicated in §8A.5, we are imagining a single act of definition.

8A-5 CONVENTION. *(Definition terminology)*

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1. Let **c** be the constant that is the definiendum.
2. Let **D** be the sentence that counts as a definition of the definiendum. In every case the definiendum **c** occurs as part of the definition **D**.
3. We have to talk about two vocabularies, one vocabulary without the definiendum **c**, and one with it. A "vocabulary" here is just a set of constants.

(a) Let **C** be the set of constants *before* the definition **D** is entered. We sometimes call **C** *the Old vocabulary*, and we say that sentences confined to this vocabulary are in *the Old language*. You are supposed to understand every sentence in the Old language.

(b) Automatically, $\mathbf{C} \cup \{\mathbf{c}\}$ is the set of constants *after* the definition **D** is entered. We sometimes call $\mathbf{C} \cup \{\mathbf{c}\}$ *the New vocabulary*. The New vocabulary is obtained by adding the definiendum **c** to the Old vocabulary. Sentences confined to the New vocabulary may be said to be in *the New language*. The definition **D** is certain to be in the New language.

4. In addition to two vocabularies **C** and $\mathbf{C} \cup \{\mathbf{c}\}$, we need to keep track of two theories (two sets of sentences), one before and one after the definition **D** has been entered.

   (a) We let **G** be the theory in hand *before* the entering of the definition **D**. For example, **G** may be the theory of kinship before entering Example **8A-1**(1) as a definition of "sibling" (in notation, S); or **G** could be the theory of arithmetic before entering Example **8A-1**(2) as a definition of the squaring operator (in notation, ^{2}). We sometimes call **G** *the Old theory*. You are supposed to be willing to assume the entire Old theory.

   (b) Automatically, $\mathbf{G} \cup \{\mathbf{D}\}$ is the theory in hand *after* the definition **D** is entered. We sometimes call $\mathbf{G} \cup \{\mathbf{D}\}$ *the New theory*.

Keeping §8A.5 and Convention **8A-5** in mind, we discuss the rules first, and then the two criteria. From a "logical" point of view this is somewhat backwards; we choose this order because for most students the rules are easier to understand than are the criteria.

8B Rules of definition

Here, briefly, are the standard rules for producing definitions guaranteed to satisfy the criteria of eliminability and conservativeness. They are easy. And by Beth's definability theorem, they are complete for the first-order logic of truth functions with quantifiers. Why is it, then, that so much philosophy otherwise faithful to first-orderism is carried out contrary to the policies they enjoin? Answer: Logic books, excepting Suppes (1957), do not give these matters proper discussion. The consequence is that students of philosophy, even those who are thoroughly taught to manipulate quantifiers, are not taught the difference between acceptable and unacceptable definitions. Since philosophers spend vastly more time proposing and using definitions than they do manipulating quantifiers, this is sad.

8B.1 Rule for defining predicates

We will give a rule for defining predicates, and then, in §8B.3, a rule for defining operators. As promised, the rules are presented as definitions. (Surely it won't occur to anyone to think in this regard of circularity.) These rules have so many clauses that you may find them intimidating. A closer look, however, will make it clear that each clause is extremely simple, obvious, and boring.

8B-1 DEFINITION. *(Standard rule for defining a predicate by an equivalence)*

D *is a standard definition of an n-ary predicate constant* R *relative to* **C** *and* **G** iff for some variables v_1, …, v_n, and some sentence A (the *definiens*), the following all hold.

1. The definiendum R is in fact an n-ary predicate constant, v_1, …, v_n are in fact variables, **C** is a set of constants (the Old vocabulary), and **G** is a set of sentences (the Old theory) all of whose constants belong to **C**.
2. **D** is the n-times universally quantified biconditional

   $$\forall v_1 \ldots \forall v_n [R v_1 \ldots v_n \leftrightarrow A].$$

3. The definiendum R does not belong to the Old vocabulary **C**.
4. The constants occurring in the definiens A all belong to the Old vocabulary **C**.
5. The variables v_1, …, v_n are distinct.
6. The definiens A has no "dangling" or "floating" variables, that is, no *free* variables other than v_1, …, v_n.

Definition **8B-1** has something fishy about it: None of the requirements for a standard definition of a predicate constant mention the Old theory **G**, so that had we chosen we could have omitted any mention of **G**. Though fishy, this does not amount to a problem; it simply testifies to the fact that a standard definition of an n-ary predicate constant can be reliably added to *any* theory whatsoever. The point is that you will be able to show satisfaction of the criteria of eliminability and conservativeness (Criteria **8A-4**) regardless of the contents of the Old theory **G**. In contrast, the Old theory really does make a difference when it comes to standard definitions of operators, as we see later in §8B.3.

8B.2 Defining predicates: examples and exercises

Suppose we take the Old language as Peano arithmetic (§7B), with Old vocabulary 0, ′, +, and ×; and also suppose the Old theory is the theory given by Axioms **7B-1**.

8B-2 EXAMPLE. *(Less)*

$x < y$ if and only if $\exists v(v \neq 0 \;\&\; x+v=y)$.

Example **8B-2** is a proper definition, provided two conventions are understood. First, we have omitted *outermost* universal quantifiers in accord with Convention **1B-20**. Second, we have used English "if and only if" instead of symbolic ↔. Since it is obvious that we intend to be understood as if we had written

$$\forall x \forall y[x < y \leftrightarrow \exists z(z \neq 0 \;\&\; x+z=y)], \qquad [1]$$

the definition should be counted as proper according to Definition **8B-1**.

The following is a key point about Example **8B-2**: The variable z that occurs in the definiens (right side) but not in the definiendum (left side)[4] is *bound*. If it had been free, it would have violated Definition **8B-1**(6). The next example illustrates violation of this clause.

8B-3 EXAMPLE. *(Evenly divisible, violates* Definition **8B-1**(6)*)*

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y *is even* if and only if $w \times (0'') = y$.

If we treat this candidate exactly as before, we obtain

[4]This is a third use of "definiendum," to be added to the two explained on p. 224: A definiendum can be (1) a predicate or operator constant, or (2) a predicate or operator, or (2) the left side of a definition, in which case it will be a sentence with free variables. Only the most compulsive reader should feel the need to be in command of such subtleties.

$$\forall y\forall w[y \textit{ is even} \leftrightarrow w\times(0'')=y]. \qquad [2]$$

This is not of the correct form according to Definition **8B-1**(2) (too many universal quantifiers), but more profoundly, there is a violation of Definition **8B-1**(6). This faulty definition has as easy consequences (by UI) both "0 is even $\leftrightarrow$ $(0\times 0'')=0$" and "0 is even $\leftrightarrow$ $(0'\times 0'')=0$." It is now but an exercise in Peano arithmetic to prove "$(0\times 0'')=0$" (so that you can prove "0 is even") and also "$\sim((0'\times 0'')=0)$" (so that you can prove "$\sim$(0 is even)"). Whoops, a contradiction! This is a dreadful violation of the criterion of conservativeness in either its rough form (Criteria **8A-4**) or its more accurate form (Definition **12A-5** below). And all because of a dangling free variable in violation of Definition **8B-1**(6).

It seems clear that what the incautious definer *intended* as a definition of "_ is even" was the following.

$$y \textit{ is even} \text{ if and only if } \exists w(w\times 0'') = y), \qquad [3]$$

which when conventions are applied becomes

$$\forall y[y \textit{ is even} \leftrightarrow \exists w(w\times 0'') = y)], \qquad [4]$$

which satisfies all the rules for defining a predicate.

The given definition [4] is not only formally correct, but, in Tarski's phrase, "materially adequate." That is, it works as a faithful lexical definition to describe our use of "_ is even," which makes the definition interesting. The rules, however, by no means guarantee that a definition will have interest. Here is an example of a definition that is formally correct by the rules:

8B-4 EXAMPLE. *(Always false)*

$\forall x[Rx \leftrightarrow 0=0']$.

Definition **8B-1**(6) requires that each variable free in the definiens must occur also in the definiendum; but there is no rule insisting every variable in the definiendum must also occur in the definiens. In fact Example **8B-4** is o.k. Is it a silly definition? Perhaps; no one ever said that every defined predicate must be interesting. On the other hand, perhaps someone will find it useful to have in their language a predicate such as R that is false of everything. It certainly isn't going to hurt anything.

For this next attempt at definition, which is faulty, assume that the Old language already contains $<$ as defined by Example **8B-2**.

8B-5 EXAMPLE. *(Circular definition, violates* Definition **8B-1**(3)*)*

$\forall x[Gx \leftrightarrow (Gx \,\&\, (x < 0'))]$.

Here the circularity is obvious; this definition cannot count as an explanation of the meaning of G. It violates the criterion of eliminability (see Criteria **8A-4** or Definition **12A-2**): When you try to use Example **8B-5** in order to eliminate G in favor of previously understood language, you just run around in circles. The particular rule violated is Definition **8B-1**(3), which see.

Consider again Example **8A-1**(1) and Example **8A-1**(4). If you think of these as both definitions, you have to put them into some order. If you put the first definition of "sibling,"

$$\forall x[\text{x is a sibling} \leftrightarrow (\text{x is a brother} \vee \text{x is a sister})], \qquad [5]$$

then you must be treating "brother" as part of the previously understood Old language in order to satisfy Definition **8B-1**(4). But then if you later try to enter the definition of brother,

$$\forall x[\text{x is a brother} \leftrightarrow (\text{x is male} \,\&\, \text{x is a sibling})], \qquad [6]$$

you are trying to define a constant that is already in the Old language, in violation of Definition **8B-1**(3). And if you invert the order in which the definitions of "sibling" and "brother" are entered, you run into the same trouble, just inverted. You may not care which you take as a definition and which as an assertion, but you can't have it both ways: At most one of these can be taken as a formally correct standard definition that explains the meaning of its definiendum in language that is previously understood.[5]

The following faulty definition illustrates the need for Definition **8B-1**(5), which requires that all the variables $v_1, \ldots, v_n$ be distinct. Suppose that the Old language is Peano arithmetic enriched via the definition Example **8B-2** of $<$.

8B-6 EXAMPLE. *($\leqslant$, violates* Definition **8B-1**(5)*)*

$\forall x[(x \leqslant x) \leftrightarrow (x < x \vee x = x)]$.

[5] See §12A.5 on p. 289 for reference to a *non*-standard formal theory of *circular* definitions.

Why is this faulty? It seems to *look* all right. What difference does the violation of "distinctness" make? This: You can use this definition to eliminate $\leqslant$ *only* when it is flanked by the same term on each side, as in $(0'' \leqslant 0'')$. The definition gives no help at all in understanding the meaning of $\leqslant$ in a context such as $(0 \leqslant 0')$. Therefore, the definition does not satisfy the criterion of eliminability (Criteria **8A-4** or Definition **12A-2**). What the definer doubtless "had in mind" was the following absolutely correct definition:

$$\forall x \forall y[(x \leqslant y) \leftrightarrow (x < y \lor x = y)]. \qquad [7]$$

Here, since the variables are distinct, the definition explains $\leqslant$ in *every* context.

Exercise 64 *(Kinship terminology)*

Take the following as previously understood constants in the Old language: parent-of (use P_ _), ancestor-of (A_ _), male (M_), female (F_). Restrict the domain to people, past, present, or future. Create a *sequence* of definitions, feeling free to use constants defined earlier in the sequence when it comes to giving later definitions. Your defined constants should include all of the following. You *may* choose to define other helpful constants on the way, and you should *not* expect that the given order is a good order for your purposes—part of the exercise is to figure out the right order. You *may* choose to abbreviate the defined constants symbolically, or you *may* prefer to leave them in English. If you choose to use symbolic abbreviations and you run out of capital letters, mark some letters with subscripts or superscripts in order to avoid ambiguity.

Two-place predicates: sister, aunt, mother, child, half-sister, grandmother, grandparent, great-grandparent, blood-relative, father, sibling, first cousin, second cousin, first cousin once removed.

Optional: Think about the possibility of defining "parent" in terms of "ancestor," and the possibility of defining "ancestor" in terms of "parent."

One-place predicates: father, grandfather.

▷ . . . . ◁

8B.3 Rule for defining operators

The rule for defining an operator by an equivalence[6] is *very* much like the rule for defining predicates, **8B-1**, and you should begin by comparing the two rules clause-for-clause.

8B-7 DEFINITION. *(Standard rule for defining an operator by an equivalence)*

D *is a standard definition of an n-ary operator constant* f *relative to* **C** *and* **G** iff for some variables $v_1, \ldots, v_n$, and w, and some sentence A (the *definiens*), the following all hold.

1. The definiendum f is in fact an n-ary operator constant, $v_1, \ldots, v_n$ and w are in fact variables, **C** is a set of constants (the Old vocabulary), and **G** is a set of sentences the constants of which all belong to **C**.
2. **D** is the $(n+1)$-times universally quantified biconditional

   $$\forall v_1 \ldots \forall v_n \forall w[(fv_1 \ldots v_n = w) \leftrightarrow A].$$

3. The definiendum f does not belong to the Old vocabulary **C**.
4. The constants occurring in the definiens A all belong to the Old vocabulary **C**.
5. The variables $v_1, \ldots, v_n$, and w are distinct.
6. The definiens A has no "dangling" or "floating" variables, that is, no *free* variables other than $v_1, \ldots, v_n$, and w.
7. $\mathbf{G} \vDash_Q \forall v_1 \ldots \forall v_n \exists y \forall w[(y = w) \leftrightarrow A]$. That is, it is a Q-consequence of the Old theory (without the definition!) that, for each n-tuple of arguments $v_1, \ldots, v_n$, there is in fact exactly one w such that A; so A is "functional."

If you have seen the point of all the clauses for defining predicates, then you automatically see the point of all the rules for defining operators, except for the one critical difference: There is no clause in the rule for predicates that corresponds to (7) in the rule for operators. To illustrate the need for the earlier clauses would

[6] In simple cases it is perfectly all right to define an operator by means of an identity, an alternative standard form of definition that we do not illustrate.

accordingly be repetitious, and we illustrate only the need for (7). Take the special case of a one-place operator f, and permit us to display the variables v and w that you would expect to occur in A:

$$\forall v \forall w[(fv = w) \leftrightarrow Avw]. \qquad [8]$$

What does the definiens Avw of [8] tell you about f? What you *want* it to tell you is, for each input v to f, the exact identity of the output fv. It can do this, however, only if Avw is "functional": For each input value of v, there is *exactly one* output value of w such that Avw. And that is exactly what Definition **8B-7**(7) says. (See Exercise 32(4) to review how the quantifier-form "$\exists y \forall w[y = w \leftrightarrow Avw]$" works, and to convince yourself that it does say what it is supposed to say.) If you cannot prove in the Old theory that Avw is "functional," then [8] has to be a faulty explanation of the meaning of the operator f.

There is a sharp way of putting this: Unless Definition **8B-7**(7) is satisfied, your definition will violate the criterion of conservativeness: You will be able to prove something in the Old language using the definition **D**, namely, $\forall v_1 \ldots \forall v_n \exists y \forall w[(y = w) \leftrightarrow A]$, that you could not prove in the Old theory, without the definition. It is a reasonable exercise for you to prove a simple case of this.

Exercise 65 *(Defining operators and clause (7))*

Take as a candidate definition **D** the following, where we have chose A as "Rvw."

$$\forall v \forall w[(fv = w) \leftrightarrow Rvw]. \qquad (\mathbf{D})$$

Using (**D**) as a hypothesis, prove the existence and uniqueness clauses for Rvw:

1. $\forall v \exists w Rvw$ (for every input there is an output), and
2. $\forall v \forall w_1 \forall w_2[(Rvw_1 \,\&\, Rvw_2) \rightarrow w_1 = w_2]$ (for every input there is at most one output).

▷ . . . . ◁

This is an exercise in the logic of identity. It is a matter now of the further use of that logic to see the equivalence of the conjunction of Exercise 65(1) and Exercise 65(2) with the fancier $\forall v \exists y \forall w[w = y \leftrightarrow Rvw]$.

Exercise 66 *(Defining operators)*

Assume that you have already entered definitions for all the kinship predicates listed in Exercise 64. Consider the problem of defining the following two operators:

1. father of __, and
2. son of __.

First write down a pair of candidate definitions, and check to see that each satisfies all of the conditions of Definition **8B-7** except perhaps (7). For each of your two candidate definitions, what must follow from the theory of kinship in order to satisfy clause (7) of rule Definition **8B-7** for defining operators? If you think about it, you will see that the clause stands in the way of giving a formally correct definition of one of the operators, but gives no trouble in defining the other. Explain.

▷ . . . . ◁

8B.4 Adjusting the rules

Even within the purview of the standard account, there are other forms of definition with an equal claim to propriety. In the context of the set-theoretical axioms Axiom **4A-17** of separation and Axiom **4A-7** of extensionality, for instance, one may properly define set-to-set operators such as $X \cap Y$ by way of membership conditions, for example taking Axiom **4A-13** as a *definition* instead of as a mere *axiom*. The axiom of separation will imply "existence," whereas the axiom of extensionality will imply "uniqueness," both needed for Definition **8B-7**(7).

In the context of higher-order axioms establishing an inductively generated structure (such as a higher-order version of Peano's axioms as listed in Axioms **7B-1** on p. 206), one may properly define operators by rehearsing the mode of inductive generation. Sometimes, in some contexts, some of these are called "implicit definitions." In these cases, "properly" means "in such a way as to satisfy the criteria of eliminability and conservativeness." So much is a built-in generality of the standard account.

Part II

Appendices to
Notes on the Art of Logic
2008

Chapter 9

Appendix: Parentheses

9A Parentheses, omitting and restoring

This section is in the nature of an appendix. Every exercise in this section is optional, and may be skipped.

Sometimes showing all proper grouping via parentheses leads to a thicket or forest of parentheses, which interferes with easy visual processing. Since logic is supposed to make this kind of processing easier, not harder, there is point in introducing conventions that reduce the severity of this problem. On the other hand, we have to be sure that we do not cut out anything valuable when we cut out some parentheses. That is, we must be sure that all our expressions remain *unambiguous*. The following conventions are of help in this regard.

(You do not have to learn to *write* these; it is good, however, to learn how to *read* them.)

1. Outermost parentheses are generally omitted.

2. Competition between either of $\rightarrow$ and $\leftrightarrow$ on the one hand, and either of & and $\vee$ on the other. In such competition, always take $\rightarrow$ or $\leftrightarrow$ as the *main* connective. Examples.

   $A\rightarrow B\vee C$ .... is $A\rightarrow(B\vee C)$
   $A\vee C\leftrightarrow A\,\&\,D$ .... is $(A\vee C)\leftrightarrow(A\,\&\,D)$
   $A\,\&\,(B\vee A\leftrightarrow C)\rightarrow D\,\&\,E$ .... is $(A\,\&\,((B\vee A)\leftrightarrow C))\rightarrow(D\,\&\,E)$

The above example correctly indicates that "competition" only occurs within a parenthetical part.

Exercise 67 *(Parentheses—skippable exercise)*

Cover the right hand sides of the examples of item (2) above, and then write them out, looking only at the left hand sides. Check this yourself (do not hand in your work). Extra credit: Do (and check) the reverse.

▷ . . . . ◁

3. Because & and ∨ are associative, so that parentheses for them don't make any difference anyhow, we can omit parentheses between a series of &, or a series of ∨ (but not, of course, a mixture of the two!):

   (A&B&C&D)
   ((A→B)∨∼∼C∨(∼E↔(F&G))

   If you require a definite rule for restoration in these cases, think of replacing the parentheses by grouping everything to the left:
   (((A&B)&C)&D).

4. Other groupers. In complicated cases, we can substitute "[" or "{" for "("—and similarly for the right mates—so that the eye can more easily pick up left-right matches.

5. Dots. The useful convention of Church (1956) is this: A dot is a left parenthesis, whose mate is as far away as possible; i.e., either at the end of the whole expression, or if the dot is inside a parenthetical part, at the end of that parenthetical part. (Church's dots are "heavy dots," whereas here our dots are mere periods. For this reason, you may have to look closely to be able to follow.) Example.

   A→B→.B→C→.A→C is A→B→(B→C→(A→C)).

6. But this, so far, is ambiguous. The second part of Church's convention is that any missing parentheses (when all the above conventions have already been employed) are to be restored by grouping to the *left*, so that the above example becomes

   (A→B)→((B→C)→(A→C)).

The left-grouping convention was used twice: once in the right-hand parenthetical part, once in the main expression. A few more examples:

$A \to . B \to C$ is . . . . $A \to (B \to C)$.
$(A \to . B \to C) \to . A \to B \to . A \to C$ is a tautology, namely, ...
. . . . $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.
$A \to B \to A \to A$ is Peirce's Law, ... $(((A \to B) \to A) \to A)$.

Exercise 68 *(Left grouping—skippable exercise)*

Use the examples above as an exercise: Cover the right hand formulas, and then write them out while looking at only the left formulas. Extra credit: Try the reverse. In both cases, check your own work (do not hand it in).

▷ . . . . ◁

7. Our conventions really do resolve all ambiguities, because (6) settles anything that hasn't already been settled, by grouping everything to the left. But in fact we mostly avoid using the full power of this part of the convention, because we find it confusing—and the point of all of these conventions is to be useful. For example, we never write $A \vee B \to . \sim A \to B$, but instead always $(A \vee B) \to . \sim A \to B$ or $(A \vee B) \to (\sim A \to B)$. For another perhaps even more potent example, we never write $A \vee B \,\&\, C$, which by the left-group convention is $(A \vee B) \,\&\, C$. We just write the latter instead. Also, we usually write $A \vee (B \,\&\, C)$ instead of $A \vee . B \,\&\, C$. In fact, we mostly use the dots only after arrows.

Exercise 69 *(Dots—skippable exercise)*

Restore parentheses. Optional (involving material not yet studied): Indicate which are tautologies in the sense of Definition **2B-20**.

1. $A \,\&\, A \to B \to B$
2. $A \,\&\, . A \to B \to B$
3. $(A \,\&\, A \to B) \to B$
4. $A \,\&\, (A \to B) \to B$
5. $(A \,\&\, . A \to B) \to B$
6. $A \,\&\, . A \to . B \to B$
7. $A \,\&\, A \to . B \to B$

▷ . . . . ◁

The only ones of these examples which we would ever write, incidentally, are (4) and (6). Or maybe also (1).

Chapter 10

Appendix: Symbolization in more detail

The following sections elaborate on symbolization, relying on an adaptation of "Montague grammar" (Montague, 1974).

10A Symbolizing English quantifiers: A longer story

This section expands on §5A. We begin by repeating the basic idea with emphasis, and then take up details of the technique.

> *Every English quantifier is an instruction to put together two open sentences according to a certain pattern.*

This is true not only for "all" and "some," but also for other English quantifiers like "only," "at most one," and even "the." It follows that to use this technique, you must learn (a) how to find the two open sentences, and (b) for each English quantifier, what pattern should be used in combining the two sentences. Neither part is quite easy (nothing worthwhile is), but you will find this particular study of English grammar most rewarding if what you care about is understanding what is said by persons who have something worthwhile to say, or saying something worthwhile to be understood by those very same persons. By way of referring to the most important examples, we shall call the two open sentences to be combined (1) the *logical subject* and (2) the *logical predicate*—though beware of false associations until you have learned our definitions of these two kinds of open sentences.

10A.1 English quantifier patterns

We begin by outlining for each English quantifier how the logical subject is to be combined with the logical predicate; this should be familiar territory. The examples are chosen so that it is easy to see what is meant in these cases by "logical subject" and "logical predicate":

> Sv symbolizes the logical subject "v is a sheep."
>
> Pv symbolizes the logical predicate "v is placid."

We will use special variables here—variables you will never use in your own symbolizations—so that you won't get mixed up later when consulting this table.

10A-1 DEFINITION. *(English quantifier patterns)*

The following list amounts, for each of the listed English quantifiers, to a definition of the symbolic quantifier pattern corresponding to that quantifier.

All sheep are placid. $\forall v(Sv \rightarrow Pv)$

***Each*, *every*.** Ditto.

Only sheep are placid. $\forall v(Pv \rightarrow Sv)$

No sheep are placid. $\forall v(Sv \rightarrow \sim Pv)$

The sheep is placid. $\exists v_1[(Sv_1 \;\&\; \forall v_2(Sv_2 \rightarrow v_2 = v_1)) \;\&\; Pv_1]$

Some sheep are placid. $\exists v(Sv \& Pv)$

At least one sheep is placid. Ditto.

A sheep is placid. Ditto. But " a " is sometimes used for "every."

At least two sheep are placid. $\exists v_1 \exists v_2[v_1 \neq v_2 \;\&\; (Sv_1 \& Pv_1) \;\&\; (Sv_2 \& Pv_2)]$

At least three sheep are placid. $\exists v_1 \exists v_2 \exists v_3[v_1 \neq v_2 \& v_1 \neq v_3 \& v_2 \neq v_3 \;\&\; (Sv_1 \& Pv_1) \;\&\; (Sv_2 \& Pv_2) \;\&\; (Sv_3 \& Pv_3)]$

At least four (five; etc.) sheep are placid. Similar

At most one sheep is placid. $\forall v_1 \forall v_2[((Sv_1 \& Pv_1) \& (Sv_2 \& Pv_2)) \rightarrow v_1 = v_2]$

At most two sheep are placid. $\forall v_1 \forall v_2 \forall v_3[((Sv_1 \& Pv_1) \& (Sv_2 \& Pv_2) \& (Sv_3 \& Pv_3)) \rightarrow (v_1 = v_2 \vee v_1 = v_3 \vee v_2 = v_3)]$

At most three (four;, etc.) sheep are placid. Similar.

Exactly one sheep is placid. Conjoin "at least one" with "at most one," or use the following mystery, due to Bertrand Russell: $\exists v_1 \forall v_2((Sv_2 \& Pv_2) \leftrightarrow v_1 = v_2)$

You have seen all or most of these forms before; the next step is to see "the pattern of the patterns": Every one of these symbolizations arises by combining the two open sentences (the logical subject and the logical predicate) according to a certain pattern—and the English quantifier tells you exactly what pattern to use in combining them. One thing that might blind you to this insight is the fact that in certain of the patterns, one or more of the open sentences is repeated, with a different variable. But in fact, as you know, the variable of an open sentence is irrelevant: Since it is just a place-holder, telling us where something is to be substituted, it doesn't matter if the variable is x, y, or z, whatever—as long as the same variable is used consistently. For example, "the" should be seen by you as giving directions for combining the two open sentences Sx and Px in a pattern that requires using Sx twice (once with v_1 for x, once with v_2 for x) and using Px once (with v_1 for x). Once again: English quantifiers tell you how to put together two open sentences. Though some of the patterns may not be memorable, they are memorizable, and to use this method you must learn them.

Exercise 70 *(English quantifier patterns)*

1. Which English quantifiers go with the following symbolic quantifier patterns?
   (a) $\exists v(Sv \& Pv)$
   (b) $\exists v_1 \exists v_2[v_1 \neq v_2 \ \& \ (Sv_1 \& Pv_1) \ \& \ (Sv_2 \& Pv_2)]$
   (c) $\exists v[(Sv \ \& \ \forall w(Sw \rightarrow w = v)) \ \& \ Pv]$
   (d) $\forall v(Sv \rightarrow \sim Pv)$

2. Which symbolic quantifier patterns go with the following English quantifiers?

   (a) At most two

   (b) A

   (c) The

   (d) No

   (e) Only

3. Now be sure to *memorize* the entire list of quantifier patterns, in both directions. Do not claim to be able to speak or understand English unless you have completed this straightforward task. Do not proceed until you are prepared to certify that you have done so.

▷ . . . . ◁

But knowing the pattern for each English quantifier is not enough; you must also know how to find the open sentences that are to be combined. This is a somewhat longer story; it relies on the work of R. Montague (Montague (1974)), who had the vision that perhaps the entire process of symbolizing English could be made as routine as truth-tables. (His ideas have of course been adapted to the present context.) To understand the procedure you must learn about the following concepts.

1. Major English quantifier.

2. English quantifier term.

3. Major English quantifier term.

4. Logical subject.

5. Logical predicate.

You will discover that the open sentences to be combined are precisely the logical subject and the logical predicate.

10A.2 Major English quantifier

So much for English quantifiers. Which English quantifier terms are "major"? Like the major connective, the major English quantifier is the one that must be symbolized first as we proceed from the outside to the inside. The reason we need the concept of major English quantifier is analogous to the situation in truth-functional logic: Just as a sentence may have more than one connective, so a sentence may have more than one English quantifier. We must decide which one to symbolize as outermost, in order to give the most plausible reading to the English sentence. The major English quantifier is the one that gives rise to the logical subject—the one that tells us what the sentence is "about."[1]

Consider "Each president consulted at least two elderly advisors." This sentence has two English quantifiers (underlined). It is therefore like a sentence with two connectives, say "and" and "or," and we must decide in the former case as in the latter which to make major. In this case it is perhaps not plausible that "at least two" be the major English quantifier. For if it were, the sentence would say that there were at least two elderly advisors such that each president consulted them, which is absurd. Instead, it is more plausible to suppose that the sentence has "each" as major English quantifier; for then we are analyzing the sentence as saying that for each president, he or she consulted at least two elderly advisors—which sounds right. (We don't know, of course, whether it is true, but at least it isn't silly.) You should be warned that there is no automatic way to pick out the major English quantifier (or indeed the major connective). Usually the leftmost quantifier is major in English—but, alas, not always. Surely "At least two elderly advisors were consulted by each president" has essentially the same content as the above sentence, which indicates that "each" is the major English quantifier, even though it is not leftmost. And there are cases of genuine ambiguity: Does "Some woman is loved by every man" have "Some woman" as its major English quantifier, so that the sentence says that there is some one woman (Helen) such that she is loved by every man, or is "every" major, so that the sentence says that every man is such that some woman (his mother) is loved by him? Alas, there is nothing else to say except that English is ambiguous.

Exercise 71 *(Major English quantifier or connective)*

[1]The notion of aboutness is useful in obtaining your bearings here, but worthless for serious conceptual analysis. That is one reason our discussion is based on *grammatical* distinctions.

Decide for each of the following sentences whether the major functor is an English quantifier or a connective, naming the English quantifier or connective that is major. No ambiguity is intended, but if you think there is an ambiguity (and it is difficult to make up unambiguous examples), explain.

1. No pitcher will win every ball game.
2. If Jake wins every ball game, he will gain the prize.
3. If someone helps with the dishes, Jake will be content.
4. Some numbers are both odd and even.
5. Some numbers are odd and some numbers are even.
6. The boy who kissed a girl who won every prize was wearing a tie.
7. He did not give her some cake. Answer: "not."
8. He did not give her any cake. Answer: "any."
9. Delilah wore a ring on every finger and a photograph on every toe.

▷ . . . . ◁

10A.3 English quantifier terms

What is an "English quantifier term"? The easiest way to identify one is just this: An English quantifier term (1) starts with an English quantifier and (2) can meaningfully be put anywhere a name can be put (though you may have to change singular to plural, or vice-versa). But where can names be put? They can be substituted for the variables of open sentences (of middle English). We are therefore led to the following (rough) definition: An English quantifier term is an expression of English that begins with an English quantifier and that can meaningfully be substituted for the variables in an open sentence (of middle English).

What would be some examples of English quantifier terms? Begin by considering the following sentence: "x disagrees with y." Now note that either "x" or "y" or both could sensibly be replaced by any of the following: "everyone," "every knowledgeable woman in Boston," "nobody in his right mind," "at least one Albanian," "a man from Georgia who raises peanuts," "some professor of logic," "the tallest midget in the world." All the following are then good English sentences or open middle-English sentences: (The term substituted for the variable has been underlined):

- Every knowledgeable woman in Boston disagrees with y;
- x disagrees with nobody in his right mind;
- at least one Albanian disagrees with some professor of logic.

The underlined expressions are English quantifier terms, because substituting them for variables makes sense. You can use the same procedure to verify that the other expressions above are also English quantifier terms. **Stop.** Do the following before proceeding: Read aloud each result of substituting one of the above terms for one of the variables in "x disagrees with y."

An English quantifier term can be recognized in another way: Each English quantifier term is constructed from an English quantifier and a "count-noun phrase." We go into this a little in §10E, which serves as a kind of appendix going more deeply into the idea of "English quantifier term"; at this stage, however, a circular definition is quite enough: A count-noun phrase of English is a phrase that nicely follows an English quantifier to make up an English quantifier term (which can be recognized because it goes in the blank of a predicate). A splendid test is this: Can you turn the phrase into a predicate by prefixing "_ is a"? If so, you have a count-noun phrase, and if the phrase is a single word, you have a count noun (see p. 269).

Exercise 72 *(English quantifier terms)*

Parenthesize all the English quantifier terms in the following sentences. Note that sometimes an English quantifier term can occur inside another English quantifier term.

1. No number is larger than every number.
2. The fastest runner was late.
3. Every dog has a day worth remembering.[2]
4. The pilot who flew a biplane during the recent war was not as big as any bread box.[3]

[2]The phrase "a day" *can* be an English quantifier term, but not in *this* sentence. Reason: "Every dog has x worth remembering" is ungrammatical. You have to take the entire "a day worth remembering" as the inner English quantifier term.

[3]The phrase "a biplane" is an English quantifier term here. Reason: "The pilot who flew x during the recent war was not as big as any bread box" *is* grammatical.

5. If only scientists seek a wise solution to the problem of pollution, at least two troubles will arise.

▷ . . . . ◁

10A.4 Major English quantifier term

The "major English quantifier term" is identified simply as the English quantifier term that begins with the major English quantifier. (Considering that a connective might be major, we need to add "if there is one.") It is the English quantifier term that starts with the major English quantifier and is completed by its following count-noun phrase.

Exercise 73 *(Identifying major English quantifier terms)*

Underline all English quantifier terms. Indicate the major English quantifier somehow, perhaps by circling it. (If the given sentence seems to you ambiguous as to major English quantifier, explain.)

1. A philosopher wrote "Moby Dick."
2. Sam wrote a brilliant novel while he was in Chicago.
3. Every philosopher writes a book.
4. Every book is written by a philosopher.
5. No philosopher that writes a book writes at most one book.
6. Every philosopher writes a good book or a bad article.
7. Someone who lives in every state receives fifty tax bills.
8. Someone who lives in each city specializes in fire safety.
9. A woman who kissed a man that owned two Mercedes that stood in the garage on Main Street stood up.
10. A book written by a philosopher is bound to be read by someone.

▷ . . . . ◁

10A.5 The logical subject and the logical predicate

Now that you have located the major English quantifier term of a sentence, regardless of where it is, it is easy to define the two open sentences that are to be combined according to the pattern of the major English quantifier, the two open sentences that we call the "logical subject" and the "logical predicate."

Logical subject. The logical subject comes from the major English quantifier term.

> *To obtain the logical subject from the major English quantifier term: Just strip off the English quantifier and replace it by "x is a."*

Thus, if

> two black-eyed unicorns wearing trousers

is the major English quantifier term, then

> x is a black-eyed unicorn wearing trousers

is the logical subject. Sometimes you need to adjust plural to singular: Given "all male lawyers," the logical subject is "x is a male lawyer." Sometimes if there is a pronoun in the major English quantifier term referring back to the English quantifier, you need to put in an extra variable; given "two barbers who shave themselves," the logical subject is "x is a barber who shaves x." One point: You can pick any variable as long as it is new to the problem. Given the term "some disease that crippled x," for instance, it should be obvious that the logical subject is "y is a disease that crippled x," rather than "x is a disease that crippled x." Once again, you obtain the logical subject from the English quantifier term by replacing the English quantifier by "x is a" (choosing a new variable in place of "x" if necessary).

Exercise 74 *(Logical subject)*

Write out the *logical subject* of each of the following sentences.

1. All humans are mortal.

2. Some mortals are not human.
3. Each owner of each cat in the yard gave Horace a present.
4. Two persons thanked Horace.
5. Horace thanked two persons.
6. Horace gave two presents to Sarah.
7. The boy who kissed a girl went to y.

▷ . . . . ◁

Logical predicate. Finding the other open sentence, the "logical predicate," is also easy.

> *To obtain the logical predicate, take the original sentence, underline the major English quantifier term, and then replace the major English quantifier term by a variable.*

That's all. So given

> each president consulted at least two elderly advisors,

the logical predicate is

> x consulted at least two elderly advisors

and given

> at least two elderly advisors were consulted by each president,

the logical predicate is

> at least two elderly advisors were consulted by x.

Here again you may need to introduce a new variable; given

> x gave nobody a book,

the logical predicate is

x gave y a book

(with “y” new). But sometimes you do want to repeat the (new) variable—if there is a pronoun referring to the English quantifier term. Thus, given

Every barber living in x shaves himself,

the logical predicate is

y shaves y

rather than “y shaves himself.”[4]

Exercise 75 *(Logical predicate)*

For the following sentences, write down the logical predicate.

1. All humans are mortal.
2. Some mortals are not human.
3. Horace thanked two persons.
4. Every male cat loves himself.
5. x went to three cities.

▷ . . . . ◁

Exercise 76 *(Logical subject and predicate)*

In the following sentences, the major English quantifier term is underlined. For each one, write down the logical subject and the logical predicate. Be sure to resolve all pronouns, which is perhaps the trickiest part of this exercise.

[4]In general so-called “reflexive” pronouns in English correspond to a repeated term (either variable or constant) in the symbolism.

1. Everyone older than thirty is suspect.
2. Jack gave at most two raisins to his brother.
3. If any man is meek, he will inherit the earth.
4. Mary is loved by only friends of her brother.

▷ . . . . ◁

10A.6 Putting it together

So now you know what you need to do to symbolize a sentence with an English quantifier that is major.

1. First find the major English quantifier (if there is one!). Underline it by all means. Write down the symbolic pattern from §10A.1 that corresponds to that English quantifier, leaving large blanks for logical subject and logical predicate. Remind yourself which blank is which by lightly writing a small "LS" or "LP" in the respective blanks.
2. Extend the underlining to cover the entire major English quantifier term, including its following count noun phrase.
3. Then obtain the logical subject by replacing the English quantifier (in the major English quantifier term) by "x is a" (choosing a new variable in place of "x" if necessary). Put the logical subject into its proper blank.
4. And obtain the logical predicate by replacing the entire English quantifier term (in the original sentence) by the variable "x" (or whatever variable you used for the logical subject). Put the logical predicate into its proper blank.

Let us first do an easy (but not trivial) example. With abbreviation scheme Px ↔ x is a president, Cxy ↔ x consulted y, Ex ↔ x is elderly, Ax ↔ x is an advisor, we might proceed as follows.

10A-2 EXAMPLE. *(Presidents and advisors)*

- Each president consulted at least two elderly advisors.
- ∀x(x is a president → x consulted at least two elderly advisors)

- $\forall$x(Px $\to$ x consulted at least two elderly advisors)
- $\forall$x(Px $\to$ $\exists$y$\exists$z[y$\neq$z & (y is an eld. adv. & x consulted y) & (z is an eld. adv. & x consulted z)])
- $\forall$x(Px $\to$ $\exists$y$\exists$z[y$\neq$z & ((Ey&Ay)&Cxy) & ((Ez&Az)&Cxz)])

The three most important things to see about this example are all features of our work on "x consulted at least two elderly advisers." In the first place, we didn't just dream up "x consulted y" or "x consulted z"; we obtained these forms of the logical predicate in a mechanical fashion by replacing the underlined major English quantifier term by "y," and then by "z." Nor, secondly, did we just fantasize the logical subject "y is an elderly advisor"; we found it by following the rule to prefix "y is a" to the part of the major English quantifier term following the English quantifiers (and changing to the singular). Ditto for "z is an elderly advisor." Third, we didn't just make up out of whole cloth the pattern according to which these two open sentences are combined: We routinely consulted the table for "at least two" given above in §10A.1, using the logical subject here for the logical subject there, the logical predicate here for the logical predicate there, and the variables "y" and "z" here for "v_1" and "v_2" there. All of which is harder to say than to do.

Before going on to the next, more complex, example, it should be emphasized that the chief thing "new" in the technique you are learning can be summed up in the slogan, "*Find the major English quantifier term!*"

One thing you should keep in mind in locating the major English quantifier term is that you will not have done it right unless you wind up with a grammatical logical predicate after replacing your candidate major English quantifier term with a variable. For instance, which of the following underlinings is correct?

1. Every auto made in the U.S. before the 20th century began is now an antique.
2. Every auto made in the U.S. before the 20th century began is now an antique.
3. Every auto made in the U.S. before the 20th century began is now an antique.
4. Every auto made in the U.S. before the 20th century began is now an antique.

Each of the underlined bits is in fact a potential English quantifier term since it could replace the variable in *some* open sentence; but only (4) is correct, because only in that case does the result of replacing the English quantifier term by a variable in *this* sentence yield a grammatical logical predicate.

Here, for example, is how the logical predicate would look if (3) were correct: "x began is now an antique." But that just doesn't make sense. You try putting "x" for the underlined bits of (1) and (2) and (4), noting that only (4) comes out right.

Let us now attempt a more complex symbolization. The scheme of abbreviations is as follows: Fxy ↔ x is a friend of y, Cx ↔ x is a carouser, Wx ↔ x is wise, Lx ↔ x is loaded, Px ↔ x is a pistol, Dx ↔ x is a drunk, Gxyz ↔ x gives y to z.

10A-3 EXAMPLE. *(Friends of carousers)*

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1. If any friend of only carousers is wise, he gives no loaded pistol to a drunk.
2. ∀x[x is a friend of only carousers → if x is wise, x gives no loaded pistol to a drunk]
3. ∀x[∀y(x is a friend of y → y is a carouser) → if x is wise, x gives no loaded pistol to a drunk]
4. ∀x[∀y(Fxy→Cy) → (x is wise → x gives no loaded pistol to a drunk)]
5. ∀x[∀y(Fxy→Cy) → (Wx → ∀z(z is a loaded pistol → ∼ x gives z to a drunk))]
6. ∀x[∀y(Fxy→Cy) → (Wx → ∀z((Lz&Pz) → ∼∃w(w is a drunk & x gives z to w)))]
7. ∀x[∀y(Fxy→Cy) → (Wx → ∀z((Lz&Pz) → ∼∃w(Dw&Gxzw)))]

Note that in Step 1, "he" had to be identified as "x" to make sense. You should verify that what follows "→" in step 1 is in fact the logical predicate (the original sentence with "x" in place of the major English quantifier term and in place of "he"). It is typical of "any," incidentally, to tend to be major over "if" when it is in the antecedent, and also major over negations. But the chief point to keep in mind is that you should not choose a connective as major if the choice will force you to leave some pronoun without an antecedent. "Only if a man believes will he be saved" requires taking "a man" (in the sense of "any man") as major; for to take "only if" as major would leave "he" without an antecedent. (There are difficulties with this doctrine. A favorite puzzle is typified by the following so-called "donkey sentence": If a man has a donkey, he beats it. See Exercise 79(3).)

Exercise 77 *(Some symbolizations)*

Symbolize some of the following. Do *not* rely on luck or intuition: The exercise is to *use the method!* You do not need to write each logical subject and predicate; that would be too tedious. You do, however, need to write a separate line for each quantifier term. That is the only way to keep things straight. You may wish to use Example **10A-2** and Example **10A-3** as models. Better yet, review §5A and use that example as a model. In any case, we repeat: *Write a separate line for each English quantifier term.*

1. Use r = Rex, Dx ↔ x is a dog, and other equivalences provided by you.

   (a) Rex chases only dogs.

   (b) Rex chases all dogs.

   (c) Rex chases no dogs.

   (d) Rex chases dogs.

   (e) Some dogs chase some rats.

   (f) Only dogs chase only rats.

   (g) No dog chases no rat.

   (h) No rat is chased by no dog.

   (i) No one who writes poetry is a professor of the student who is reading a comic book. (Take "the ..." as major—it usually is.)

   (j) At most one student is reading at least two books.

   (k) At least two books are being read by at most one student. (Understand the leftmost English quantifier as major.)

2. Use the method on the following, working carefully from the outside in, and writing down your intermediate results.

   (a) A person who succeeds at anything will be envied by everyone. Px ↔ x is a person, Sxy ↔ x succeeds at y, Exy ↔ x envies y.

   (b) Fred wore a rose on every toe, and hung a toe on every surfboard. f = Fred, Wxyz ↔ x wore y on z, Rx ↔ x is a rose, Txy ↔ x is a toe of y, Hxyz ↔ x hung y on z, Sx ↔ x is a surfboard.

   (c) Some woman is married to at least two men. Wx ↔ x is a woman, Mxy ↔ x is married to y, Mx ↔ x is a man.

3. We discussed the ambiguity of "some woman is loved by all men (or each man)." Explain the ambiguity by building two different symbolizations, the difference lying in the choice of major English quantifier.

4. Consider “all sheep are not placid.” Explain its ambiguity by building two different symbolizations, the difference being that “not” is taken as major connective in one, and “all sheep” as major English quantifier term in the other.

5. Optional. The following bits of English, which come out of the mouths of philosophers or their translators, do not come with a logician-given dictionary. In order therefore to exhibit some part of their structure that may be relevant to the logician’s task of separating the good arguments from the bad, you will need to consider the “logical grammar” of both English and our symbolic language. With this in mind, symbolize as many of the following sentences as you can. Be sure to provide a “dictionary” correlating each English term or functor with a symbolic term or functor (e.g., “Bx ↔ x is a brute”). If you think that one of them is ambiguous, try to expose its ambiguity by means of two different symbolizations. If you find one of the problems particularly difficult, or if you think that it cannot be adequately symbolized, explain why.

   (a) Of desires, all that do not lead to a sense of pain, if they are not satisfied, are not necessary, but involve a craving which is easily dispelled (Epicurus).
   (b) Some intelligent being exists by whom all natural things are directed to their end (Aquinas).
   (c) Whatever I clearly and distinctly perceive is true (Descartes).
   (d) There are no moral actions whatsoever (Nietzsche).
   (e) One substance cannot be produced by another substance (Spinoza).
   (f) All things that follow from the absolute nature of any attribute of God must have existed always and as infinite (Spinoza).
   (g) All ideas come from sensation or reflection (Locke).
   (h) There are some ideas which have admittance only through one sense which is peculiarly adapted to receive them (Locke).
   (i) Brutes abstract not, yet are not bare machines (Locke).
   (j) Nothing is demonstrable unless the contrary implies a contradiction (Hume).
   (k) All imperatives command either hypothetically or categorically (Kant).
   (l) There is one end which can be presupposed as actual for all rational beings (Kant).

(m) There are synthetic a posteriori judgments of empirical origin; but there are also others which are certain a priori, and which spring from pure understanding and reason (Kant).

6. Thoroughly optional: Try the following difficult and time-consuming exercise, taken from the l980 general instructions for income tax Form 1040A. (So far as we know, the set of students since 1980 who have tried this exercise is altogether empty.) You should probably treat "even if" clauses as rhetorical flourishes that are not symbolized ("John may go even if he doesn't finish his orange juice" just means that John may go). Watch out for the difference between "may be permitted (or required)" and "is permitted (or required)." The wording and punctuation is theirs; you must make the best of it. Though one cannot be sure, the following scheme of abbreviations appears to be best.

   u = you, t = your total income, i = your total interest income, d = your total dividend income, a = Form 1040A, b = Form 1040, \$400 = \$400, etc., $(x > y) \leftrightarrow$ x is a greater amount of money than y, Ixy $\leftrightarrow$ x is some of y's income, Pxy $\leftrightarrow$ you are able (can; may; are PERMITTED) to use y; P′xy $\leftrightarrow$ x may be able (PERMITTED) to use y; Rxy $\leftrightarrow$ x must (has to, is REQUIRED to) use y; R′xy $\leftrightarrow$ x may have to (be REQUIRED to) use y, Bxy $\leftrightarrow$ x should (it's BETTER for him if he does) use y, Exy $\leftrightarrow$ x is EASIER to complete than y, Lxy $\leftrightarrow$ x lets you pay LESS tax than y, Cx $\leftrightarrow$ x is filing only to get a refund of the earned income tax CREDIT, Wx $\leftrightarrow$ x is WAGES, Sx $\leftrightarrow$ x is SALARY, Tx $\leftrightarrow$ x is TIPS, Ux $\leftrightarrow$ x is UNEMPLOYMENT compensation, Mx $\leftrightarrow$ x is MARRIED, Jx $\leftrightarrow$ x is filing a JOINT return.

   You MAY Be Able to Use Form 1040A if: You had *only* wages, salaries, tips, unemployment compensation, and not more than \$400 in interest or \$400 in dividends. (You may file Form 1040A even if your interest or dividend income was more that \$400 if you are filing only to get a refund of the earned income credit), AND Your total income is \$20,000 or less (\$40,000 or less if you are married and filing a joint return). Since Form 1040A is easier to complete than form 1040, you should use it if you can unless Form 1040 lets you pay less tax. However, even if you meet the above tests, you may still have to file Form 1040.

▷ . . . . ◁

10B Extending the range of applicability

The following remarks will somewhat increase the range of English sentences that you can relate to symbolic notation.

Missing English quantifiers. You have to supply them, using your best judgment. "Whales are mammals" clearly intends "all whales," while "whales entered the harbor last night" clearly means "some whales." (The purist might want to say that it means "at least two whales.")

"A," "any." For these, sometimes you want to use the pattern for "all," sometimes for "some." The thing to do is try each and see which sounds best. "A whale is a mammal" means "every whale is a mammal," but "if any whale entered the harbor, there will be trouble" doesn't mean "if every whale entered the harbor, there will be trouble" (with "if" a major connective).

Special note. "x is a drunk" might be thought to involve the term "a drunk," read as "some drunk," and perhaps it does (Montague thought so). But for our elementary purposes it is best to ignore this possibility, and symbolize "x is a drunk" directly as "Dx." And similarly whenever you have "x is a" followed by a single count noun.

Exercise 78 *(Missing English quantifiers, and "any")*

Symbolize the following. (Symbolize h = the Hope Diamond.)

1. Philosophers are self-righteous.
2. Philosophers are to be found in Boston.
3. If everything is expensive, the Hope Diamond is so (trivially).
4. If the Hope Diamond is expensive, anything is expensive (doubtless false).
5. If anything is expensive, the Hope Diamond is expensive. (This statement is intended as nontrivially true. Be sure that your symbolization has the same feature.)
6. A philosopher is supposed to think.

▷ . . . . ◁

"Each" and "all." "Each" tends to yield a major English quantifier, while "all" tends to let other quantifiers or connectives be major. Can you hear the difference between (1) "*each*" in "Some woman is loved by <u>*each* man</u>" (his mother), and (2) "*all*" in "<u>Some woman</u> is loved by *all* men" (Helen)? The underlining indicates what appears to be the major English quantifier term of each sentence above, and the parenthetical phrases are just to help you hear it that way; but people do differ about this.

Possessives, etc. There are a number of terms that aren't names (because complex) and not English quantifier terms either. Possessives (Bill's dog, Bill's weight, Bill's father) are good examples. Possibly they should be paraphrased into definite descriptions before symbolization: the dog Bill has, the weight Bill has, the father Bill has. But you can see that this paraphrase might not help much, since you are still left with the problem of symbolizing the dreadful word "has"—Bill "has" his dog in the sense of owning it; but he doesn't own his weight or his father; he "has" his weight as a characteristic, but he doesn't "have" his dog or his father in that sense. He "has" his father in the sense that his father stands in the fatherhood relation to him; but his dog doesn't stand in any doghood relation, nor his weight in a weighthood relation. Point: No single technique, such as this one, will solve all problems. (Problem: Symbolize "no technique will solve all problems" by this technique.)

Exceptives. These are a bit of a bother because there is trouble regardless of whether you include the "except" clause as part of the major English quantifier term, or include it as part of the logical predicate. Starting with "all ticket holders except drunks will be admitted," we seem to have two choices for the major English quantifier term: either <u>all ticket holders</u>, or <u>all ticket holders except drunks</u>. The first choice gives us a sensible logical subject, "x is a ticket holder," but leaves us with "x except drunk(s) will be admitted" as an ungrammatical logical predicate. The second choice leaves us with a sensible logical predicate, "x will be admitted," but leaves us with "x is a ticket holder except drunk" as an ungrammatical logical subject.

There are some nice theoretical questions here, but we will bypass them and just give you one approach, which depends on thinking of "except" as part of the English quantifier phrase itself (just as "if-then" is a single connective). (1) Let the major English quantifier term include the exceptive; if in the English the exceptive is at a distance as in "all ticket holders will be admitted except drunks,"first "collect" it by paraphrase: "All ticket holders except drunks will be admitted." This automatically gives us the logical predicate, as before, by putting a variable for the major English quantifier term. (2) Derive two open sentences from the major English quantifier term, not one. Let the one obtained from the words following “all” or “no” be called the “logical subject” as before, and let the one obtained from the words following “except” be the “logical exceptive.” (3) Treat the pattern for exceptives as combining *three* open sentences instead of two, according to one of the following patterns

10B-1 DEFINITION. *(Exceptive patterns)*

This list defines the patterns associated with the English quantifier-exceptive phrases displayed.

All sheep ***except*** East Fresians are placid. $\forall x(Sx \rightarrow (\sim Ex \leftrightarrow Px))$

No sheep ***except*** East Fresians are placid. $\forall x(Sx \rightarrow \sim(\sim Ex \leftrightarrow Px))$

The double negative can of course be canceled.

Observe that “except” does not combine well with English quantifiers other than “all” and “no”: “Some sheep except East Fresians are placid” is just not English. Thus only these two patterns are needed.[5]

Quantifiers without terms. Quantifiers can enter English other than by way of terms: “There are placid sheep” is an example, and a number of others are given in §10D. Just take things as they come, if you can, and rest assured that there are countless constructions of English such that no one has a good theory of them (can you symbolize that?). But still, Montague’s idea of the English quantifier term has supplied us with a technique that at the very least is widely applicable.

Exercise 79 *(More symbolizations)*

Symbolize the following. Use the method as far as possible.

1. Oranges and apples, except the rotten ones, are expensive.
2. No oranges or apples, except the golden ones, are expensive.
3. Any farmer who owns a donkey beats it [this is a donkey sentence].

▷ . . . . ◁

[5]“Unless” has a different grammar and accordingly requires a different treatment.

10C Further notes on symbolization

The following are further notes on symbolization, in no particular order.

Definite descriptions. The definite article "the," like the indefinite article "a," can sometimes be used to convey a universal claim. Thus "The whale is a mammal" means "All whales are mammals." The instructions provided in §10A.1 for "the" cover only those cases where "the" is used to form a *definite description*, that is, a phrase which purports to denote a single object by describing it rather than naming it—for example "the smallest prime number," or "the president of the United States." Such phrases do not always succeed in describing a unique object, for which reason we don't want to symbolize them as singular terms. The method suggested in §10A.1 is Bertrand Russell's famous "theory of descriptions."

Restricting the domain. It is sometimes possible to simplify symbolizations by restricting the universe of discourse in an appropriate manner. In a context in which the only things being discussed are natural numbers, for example, it is reasonable to take the universe of discourse to be the set of natural numbers. This not only permits the use of operators such as +, but also permits another simplification in that one need not symbolize the predicate "x is a number." For example, the sentence "All numbers are even or odd" could be symbolized as

$$\forall x(Ex \lor Ox)$$

rather than as

$$\forall x(Nx \to (Ex \lor Ox)).$$

"Nobody," "someone," etc. These are best thought of as quantifiers over people—"no person," "some person" and so on. Certain adverbs can also be thought of as phrases of quantity: "Always," "sometimes," "never," and so on seem to involve quantification over *moments of time*, while "somewhere," "nowhere," and their ilk seem to involve quantification over *places*. In symbolizing sentences involving such phrases one must give a more complicated representation of the predicates involved. For example, to symbolize "Sometimes I feel like a motherless child" one might do this:

> Domain = the set of people and moments of time; a = me; Tx ↔ x is a time; Fxy ↔ x feels like a motherless child at time y; Symbolization: ∃x(Tx & Fax).

Here it would not be enough to set Fx ↔ x feels like a motherless child.

“Only” again. “Only” has a use which is not covered by the methods of §10B, namely, where it combines a *singular term* and an open sentence (rather than two open sentences), as in “Only Calpurnia knows what happened in the Forum.” Here the pattern to be followed is illustrated in the symbolization:

> Domain = the set of people; a = Calpurnia; Kx ↔ x knows what happened in the Forum; Symbolization: ∀x(Kx ↔ x = a).

“Except” again. “Except” also has a use in which the “exception” phrase involves a singular term—“All ticket holders except Bob will be admitted.” Here one can adopt the paraphrase “All ticket holders except those identical to Bob will be admitted,” illustrated by:

> Domain = the set of people; Tx ↔ x is a ticket-holder; Wx ↔ x will be admitted; b = Bob; Symbolization:∀x(Tx → (x≠b ↔ Wx)).

Relative clauses. Relative clauses beginning with “who,” “that,” “which,” and so on are symbolized by using *conjunction* (&). For example, the open sentence “x is a woman whom I love” should be understood as “x is a woman & I love x.” Thus “The woman whom I love has dark hair” comes out as:

> Domain = the set of people; a = me; Wx ↔ x is a woman; Hx ↔ x has dark hair; Lxy ↔ x loves y; Symbolization: ∃x((Wx & Lax) & ∀y((Wy & Lay) → y = x) & Hx).

Adjectives. Adjectives can often be treated in the same way as relative clauses, as involving conjunction. Thus, we can treat “x is a green frog” as “x is green & x is a frog.” Caution must be used here, however. Two broad classes of adjectives cannot be treated in this way. First, there are *relative* adjectives such as “large”—“x is a large Chihuahua” cannot be thought of as “x is large & x is a Chihuahua.” If we thought of things this way, we would end up validating the incorrect inference from "All Chihuahuas are dogs" to "All large Chihuahuas are large dogs." A more reasonable ploy is to think of "x is a large Chihuahua" as "x is a Chihuahua & x is large *for a Chihuahua*"—this isn't perfect, though.

Another problem class of adjectives includes words like "fake." While a large Chihuahua is after all a Chihuahua, a fake Chihuahua isn't really a Chihuahua at all. In this case the whole phrase "x is a fake Chihuahua" should perhaps be thought of as a simple one-place predicate, at least for the purposes of deductive logic.

Adverbs. Most adverbs cannot be represented. (Exceptions are "sometimes," "anywhere," and so on, discussed above.) Thus we have no choice but to symbolize "x walks quickly" as a simple one-place predicate. One disadvantage of this approach is that we fail to represent the inferential connection between "x walks quickly" and "x walks." This might lead one to try to modify one's logical thought to include functors taking predicates as both inputs and outputs (which is what adverbs are from the logical grammarian's point of view). We shall not pursue this, however. (D. Davidson suggests an alternative strategy involving the view that sentences which contain adverbs implicitly make use of first-order quantification over events. As you might expect, this works nicely for some examples and badly for others. See Belnap et al. (2001) for some observations and an alternative strategy using non-truth-functional connectives.)

Superlatives. Superlatives such as "the tallest mountain" can best be handled as follows. Treat "x is taller than y" as a two-place predicate, and treat "the tallest mountain" as "the mountain that is taller than all *other* mountains." Symbolize "x is other than y" as "x≠y." So for "x is a tallest mountain" we shall have "x is a mountain & ∀y[(y is a mountain & x≠y) → x is taller than y]." (That of course only gives us the logical *subject* of "the tallest mountain is such and such." The complete recipe for "the" can only be followed once you have also a particular logical *predicate* in place of "such and such.")

Most elementary logic books have lengthy discussions of these issues. A particularly useful discussion, especially of the expressive *limitations* of first-order logic, is found in chapter VIII of Thomason (1970).

Exercise 80 *(Contributions)*

As an optional exercise, we hereby solicit you to submit some exercises concerning the preceding section, §10C. Here are a few contributions from from former students, with our comments. Think about them all, and try symbolizing a couple.

Student 1 This batch, you will noticed, has been "pre-processed" so that the language is exactly the same as that of the text, hence ready for symbolization. (a) All dogs except Jack chase themselves. (b) Nobody always feels lonely. (c) Any dog who has fleas chases only rats. (d) Only Jack loves someone. (e) Jack is the fastest dog.

Student 2 This batch needs some "pre-processing" before symbolization. We have indicated our comments or suggestions in parentheses. (f) The Red Bull is a delicious energy drink. (Take "The Red Bull" as a proper name, not a definite description. Observe that the example is *not* like "The Red Bull is a large energy drink.") (g) This is a delicious pear. (This "this" needs to be accompanied by pointing, which is not a part of *our* logical grammar; the journals, however, have been full of formal observations to be made concerning such uses. Look under "indexicals" in, say, Wikipedia.) (h) He was the greatest magician to have lived in Pittsburgh. (Our logical grammar doesn't do tenses (but there is a rich literature on "tense logic," so that this needs pre-processing: "He is the greatest magician living in Pittsburgh on July 4, 1986." The pre-processing needs to include presupposing some fixed date to avoid problems with the *present* tense. Take " _ lives (or did live or will live) in _ on _" as a three-place predicate.

Student 3 (h) Sam wants to go to the movies. The movies are closed on weekday mornings. (This would need a bucket of pre-processing. Two examples: "wants" does not fit into (our) logical grammar; and "the movies" clearly begins a universal quantification.) (i) Nobody ever achieved anything with that attitude. (Tough. What shall we do with "ever," with "with," and with "that"?) (j) Only Jack is able to answer that. (You would have to mash together "is able" and the following infinitive phrase to form a two place predicate: "_ is able to answer _." There are, incidentally, logics that treat questions and answers. For one, see Belnap and Steel (1976). (k) The old boat pulls into the foggy harbor. (Grammatically this is easy, but semantically, not. There is the problem of existence-and-uniqueness, given that the two "the" phrases are to be definite descriptions; and then in symbolizing *two* definite descriptions, the question of scope has to be faced. We like this example: "The boy who loved her kissed the girl that didn't love him." Awful problem.) (l) The most amazing prize will go to the smartest contestant. (We don't see any pre-processing problems; what do you think?)

▷ . . . . ◁

10D Additional ways of expressing quantification

English quantifier terms are just one of the many ways that English has found to represent quantifications in the straitjacket of English grammar. These are best thought of as exercises, some to be worked on jointly in class.

Exercise 81 *(Additional quantification expressions)*

Group A. Here we illustrate a few more that are entirely idiomatic. As you read these, think about them in terms of "quantifier pattern," "logical subject," and "logical predicate."

1. There are (or exist) numbers between 0 and 1.
2. No matter which pair of numbers you take, adding the first to the second is the same as adding the second to the first.
3. For every pair of numbers you consider, there are numbers that lie between them.
4. There are (or exist) no codes that cannot be broken (by someone).
5. Sooner or later everyone dies.
6. A solution for the equation "$a+x=c$" always exists. (In the jargon of mathematics, this is a pure number-quantifier statement. The statement is not really quantifying over "solutions.")

Group B. This group involves quantification in middle English (Remark **1B-3** on p. 13). Their style fits much of our own prose, and so we choose some of the illustrations from these notes.

1. $G \vDash_{TF} A$ iff there is no TF interpretation in which all of the members of G are true and A is not. (Definition **2B-14**(5) on p. 39).

2. When G is truth-functionally inconsistent, one has G $\vDash_{TF}$ A quite regardless of the nature of A (taken from just after Examples **2B-24** on p. 44).

3. The lowest common multiple n of a pair of numbers a and b is the smallest number such that each of the pair divides it (evenly).

▷ . . . . ◁

Exercise 82 *(Further symbolizations)*

Give these a go. Assume the following domain: the natural numbers {1, 2, ... }. Use only the following notation (in addition to usual names for the numbers): Ex ↔ x is even; Ox ↔ x is odd; x+y=the sum of x and y; x < y ↔ x is less than y; Px ↔ x is a prime number. On some of them it makes sense to use the method, working always from the outside in. Others, however, including item (1), employ methods of expressing quantification in English that do not follow the simple pattern "quantifier + common noun phrase" that the method expects. You are on your own.

1. If a number is less than another, then the second is certainly not less than the first.
2. Every number is smaller than some number.
3. No number is smaller than every number.
4. No number is larger than every number.
5. Every number is larger than some number.
6. The sum of two odd numbers is an even number.
7. Every even number is the sum of two odd numbers.
8. Every even number is the sum of two primes.
9. Only 1 is smaller than every even number.
10. Every number except 1 is larger than some number.
11. The number that is less than 2 is odd.
12. Discuss: 4 is a small number.

13. 6 is the largest number.

14. If an operation $\circ$ satisfies the well-known group axioms, then there is an identity element for $\circ$; namely, there is some e such that no matter what element x of the group you choose, you will find that $(x\circ e)=x$, and also $(e\circ x)=x$.

▷ . . . . ◁

10E English quantifier terms

This section is in the nature of an appendix to our discussion of symbolizing English quantification.

We can do better than we did with respect to the concept of an *English quantifier term*. You must not, however, expect rigor, since we are speaking of English grammar, not an idealized grammar. We will frequently pretend to the absence of hard problems.

Altogether we shall be using the following English-grammar jargon: term, predicate, sentence, singular term, English quantifier, English quantifier term, count-noun phrase, count noun, common noun, mass noun. If you think this is complicated, please put the blame on English.

We begin with the term, and ask a pair of very good grammatical questions:

10E-1 QUESTION. *(English terms: Whence and whither?)*

Where do they come from? Where do they go?

Terms: Where do they come from? There are in English at least two kinds of *terms*:

1. *(Singular) terms* (which we introduced in §1A.1), and

2. *English quantifier terms* (which we have been discussing for numerous pages). We repeat: There are no "quantifier terms" in logical grammar; only in English. And we are leaving out plural terms of English such as "Mary and Tom."

We discuss English quantifier terms at length below. But first we turn to the other question, which is of critical importance.

Terms: Where do they go? Singular terms and English quantifier terms have something decisive in common: They can go in the blanks of English predicates. So given an English predicate, there are *two* ways of making a sentence:

1. You can fill it with a *singular term*,
2. or you can fill it with an *English quantifier term*.

This is a feature of English grammar of the highest importance. It is for this deep reason that we introduce the generic term "term," covering both cases.[6] (If there are other ways of filling predicate blanks, we are hereby ignoring them.)

We are *altogether* ignoring the plural in the case of name-like terms. So we are ignoring plural terms such as "Mary and John." Furthermore, we are ignoring the *difference* between the singular and the plural with respect to quantifier terms. So we will just continue to be fuzzy concerning for example "a horse" vs. "some horses" or "at most one rider" vs. "three riders."

So what is an "English quantifier term"?

English quantifier terms. The first half of our explanation of "English quantifier terms" is based on where they can go: They can go in the blanks of predicates to make a sentence. Anything that cannot do that is not an English quantifier term.

The second half is more complicated and may be put into a formula:

English quantifier term = English quantifier + count-noun phrase.

That is, a typical English quantifier term consists of two parts. The left part is an *English quantifier*, discussed below. The right part is a *count-noun phrase*, also discussed below. Here are some examples.

10E-2 EXAMPLES. *(English quantifier terms)*

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[6]Did you happen to notice that "term" is not a term in our jargon sense? Instead, the word "term" is a common noun (a count noun in the sense explained below), and therefore "term" is a word that (unlike terms) cannot be put into the blank of an English predicate.

English quantifier	count-noun phrase
a	horse
the	raid on the city
every	one (person) who admires Mira
some	dog with fleas
at most one	World War I pilot known for bravery
between three and five	success stories
a few	brave souls
many	daughters of the American Revolution

We have said that an "English quantifier term" is made up of two parts, (1) an "English quantifier" and (2) a "count-noun phrase." To complete our grammatical story, we need to explain each of these ideas.

English quantifiers. A dictionary of the English language should (but won't) tell you which words and phrases are English quantifiers; we shall pretend, however, that somewhere there is a definite list of them. Some are simple such as "three," whereas some are more complex, consisting of several words such as "at least seven." Most of the words and phrases have in common that they are all answers to "how many?" questions.[7] Here is a list of some English quantifiers.

10E-3 EXAMPLES. *(English quantifiers)*

> a, the, an, any, every, all, some, at most three, less than five, less than five but more than two, exactly one, at least seven, a finite number of, an infinite number of

English quantifiers have the following critical feature: They make sense when prefixed to a count-noun phrase (adjusting for singular vs. plural). "Make sense" is a longish story that in the end is helpfully circular: It means that the result of prefixing an English quantifier to a count-noun phrase gives you something that you can put in the blank of a predicate (and so the result is some kind of term).

[7]A deeper commonality is that in order to bear the sense that we give one of them when we symbolize its use with quantifiers and identity, the phrase following it must express a kind, and must permit making sense out of "individuating" what falls under that kind. So much is a necessary condition of enabling our location of the logical subject by replacing the English quantifier with "x is a." This is part of the reason that "the" belongs on the list even though it cannot be used to say "how many."

Take each one of the English quantifiers listed in Examples **10E-3** and try to put it in front of some count-noun phrases such as those listed in Examples **10E-2**. The idea is to see that they make good grammar (adjusting for singular vs. plural); for example, ”exactly one raid on the city.”

We have said that an English quantifier term is made up of two parts, (1) an English quantifier, which we have just discussed, and (2) a “count-noun phrase,” to which we now turn.

Count-noun phrases. We know that count-noun phrases can be prefixed with English quantifiers to make English quantifier terms (which go in the blanks of predicates to make sentences). That’s essential. But how are they made up? Certainly count-noun phrases can be exceedingly complex, and we do not know the rules for building them. We do know this: Each count-noun phrase has a “count noun” (discussed below) as its base. You start with a count noun and you can add adjectives and you can add subordinate clauses and doubtless much else. But you have to start with a count noun. Example. The “base” count noun (sometimes called the “head” noun) of the following is just: cow.

> vastly admired pregnant cow who rapidly walked through the Wisconsin State Fair grounds in 1962 while her owner whistled Dixie.

Exercise. Pick out the “base” or “head” count nouns of the various count-noun phrases given in Examples **10E-2**. You should come up with the following list of count nouns: horse, raid, one, dog, pilot, story, soul, daughter.

Some count-noun phrases may be built in more complex ways. We suppose that “brown horse or black cow” can be a count-noun phrase. We repeat: We just don’t know the rules for this.

Since every count-noun phrase starts with a “count noun,” we next ask after count nouns.

Count nouns. We know that count nouns serve as a basis from which we can make count-noun phrases by combining adjectives, relative clauses, and so on. (Also each count noun itself counts as a “count noun phrase.”)

We add the following. Each count noun is a single word that is a “common noun,” which is a more general category. A dictionary should (but won’t) tell you which single words are common nouns, and among those, which are count nouns.

Common nouns. You need first to distinguish a "common" noun from a "proper" noun.

- A common noun cannot be put in the blank of a predicate.
- A common noun can sensibly be prefixed by a "determiner" such as "a," "some," or "all." (These are all quantifier words, too; "determiner" is a wider class, e.g. possessives such as "Mary's," and (of most importance for us) "the.")

That's not so very helpful, but we have little more to say. Common nouns express "kinds." The most important point for our purposes is that common nouns are of two quite different species, which some folks mark by dividing common nouns into the "count nouns" and the "mass nouns."

Count nouns. Count nouns such as "horse," "person," and "box" can be prefixed by a determiner that does some counting, which is to say, a count noun can be prefixed by an English quantifier: a (single) <u>horse</u>, a <u>person</u>, three <u>boxes</u>. An even better test for our particular purpose is this: You can transform any English count noun into an English predicate by prefixing "_ is a": _ is a horse, _ is a person, _ is a box.

Mass nouns. Mass nouns such as "water," "air," "snow," and "mercury" cannot be prefixed by a determiner that does some counting. The following are o.k.: some water, all air. The following are bad grammar: three waters, at least one air. In parallel, you cannot make an English predicate from an English mass noun by prefixing "_ is a." For example, "_ is a water" is bad grammar. [8]

Even though quantifier terms are made from count nouns, much of what we say can be adapted to mass nouns, partly because associated with every mass noun there is a count noun phrase "a portion (or parcel) of _." We can count portions of snow. If we say "all snow is white," we could well mean that all portions of snow are white." In symbolizing, we could use "Sx for "x is a portion of snow," or even (in English) "x is some snow."

[8] English lets just about any word be used in just about any grammatical category. Not al1 languages are like that, but English is. In the face of this, we just need to count a word as "a different word" when it changes grammatical category. So go ahead and use "horse" as a verb (Jack horsed around) and use "water" as a count noun (Minnesota is the land of many waters). But don't lose sight of the advances we are describing.

Exercise. Give a number of examples of count nouns, choosing them as diverse as you can (e.g., give only one animal count-noun).

The grammatical difference between count nouns and mass nouns can be felt to have a “metaphysical” basis, and perhaps sometimes it does. But not always. For a striking example, consider that “chair” is grammatically a count noun, whereas “furniture” is grammatically a mass noun, as you can see from the following table.

a chair (ok)	a furniture (bad)
three chairs (ok)	three furnitures (bad)
all chair (bad)	all furniture (ok)
all chairs (ok)	all furnitures (bad)
every chair (ok)	every furniture (bad)
some chair (ok)	some furniture (also ok)
some chairs (ok)	some furnitures (bad)
a chair (ok)	a furniture (bad)
the chair (ok)	the furniture (also ok)

We don’t know the theory (metaphysics?) of this; but you can see that the *grammar* of “chair” is different from the *grammar* of “furniture.” “Chair,” like “mystery,” is a count noun, whereas “furniture,” like “water,” “snow,” and “intelligence,” is a mass noun.

Chapter 11

Appendix: Assignment tables

This chapter is in the nature of an appendix to our discussion of Q-interpretations.

11A Universals in assignment tables

Assignment tables provide a paradigm of the workings of connectives that are *not* truth functional; for someone who has had some exposure to such connectives, this section may prove of some use in putting into a common perspective the semantic explanations of quantifiers and those other non-truth-functional connectives. The section is, however, too complicated to help with the practical job of finding Q-interpretations in order to show Q-invalidity or the like, and should be passed over by all but the most compulsive.

The essential point emphasized by assignment tables is that you need to look in more than one row when you evaluate a quantifier statement.

Suppose $Domain_{\mathbf{j}} = \{1, 2, 3\}$, and that just the four variables w, x, y, and z are of interest. The relevant assignment table has $3^4 = 243$ rows. Let us suppose that we *already know* the truth value of some (it could be any) sentence Awxyz for each of the 243 rows, and that we are interested in the truth value of the four possible universal quantifications in the particular row, labeled "$\mathbf{j}$," in which $\mathbf{j}$(w) = 2, $\mathbf{j}$(x) = 1, $\mathbf{j}$(y) = 3, and $\mathbf{j}$(z) = 2. We will call this row the "target row."[1]

[1]The Q-interpretation $\mathbf{j}$ of course gives an extension to *all* Q-atoms. Apart from the assignments to the four relevant variables, however, this information is not represented in the pictured assignment table.

You cannot determine the truth value at **j** of a quantification of Awxyz—for example, $Val_{\mathbf{j}}$(∀xAwxyz)—merely by consulting $Val_{\mathbf{j}}$(Awxyz). You need to know the truth value of Awxyz at *other* rows (at other Q-interpretations $\mathbf{j}_1$, etc.). Which other rows? The recipe (p. 156) says that

> $Val_{\mathbf{j}_1}$(∀xAx) = T iff for every entity d and every Q-interpretation $\mathbf{j}_2$, if (d∈ $Domain_{\mathbf{j}}$ and $\mathbf{j}_2$ is exactly like $\mathbf{j}_1$ except that $\mathbf{j}_2$(x) = d) then $Val_{\mathbf{j}_2}$(Ax) = T.
> In other words, a sentence ∀xAx is true on a certain Q-interpretation just in case the ingredient sentence Ax is true on *every* Q-interpretation that can be obtained from the given one by re-assigning x a value in the domain—and keeping fixed the domain and interpretations of other variables (and constants as well).

Picture **11A-1** lists all the rows that are relevant, by this recipe, to calculating a value for each of the four universal quantifications of Awxyz.

11A-1 PICTURE. *(Assignment-table with four variables)*

					Given	**Calculated**		
	w	x	y	z	Awxyz	∀wAwxyz	∀xAwxyz	∀yAwxyz
	1*	1	3	2	T			
	2	1	1*	2	F			
	2	1	2*	2	T			
	2	1	3	1*	F			
j	2*	1*	3*	2*	T	T	F	F
	2	1	3	3*	F			
	2	2*	3	2	T			
	2	3*	3	2	F			
	3*	1	3	2	T			

For any one quantifier, you need only three rows: the target row plus two others, one for each re-assignment (from $Domain_{\mathbf{j}}$) to the variable in question. Since we are playing with four quantifiers, you will need two extra rows for each of them.[2] In other words, the "other" rows are obtained by varying the target row in exactly one column, so that in addition to the target row, there are eight others. The nine

[2]Symbolize—in your head—the sentence "you will need two extra rows for each quantifier." Note that "each" forces wide scope (is major).

rows are arranged in “numerical” order, but, since we are picturing a mere nine rows out of 243, the vertical spacing between the various rows means nothing.

You will note that every listed row (other than the target row) is exactly like the target row, except in one place; and you can see that we have marked that place with a star (*).

Picture **11A-1** shows the *given* pattern of truth values under Awxyz for these rows. This “given” column is just an example, for illustration. But based in the pattern of truth values for Awxyz in these nine rows, the truth values (in the target row) of the four universal-quantifier statements can be *calculated*. How?

- For ∀wAwxyz to be true in the target row, we must consult the three rows with a star under w. These three rows are exactly like the target row except (perhaps) with respect to w, and under w we get each of the three possible entities from $Domain_{\mathtt{j}}$. Checking then delivers that $Val_{\mathtt{j}}$(∀wAwxyz) = T.
- For ∀xAwxyz to be true in the target row, we must consult the three rows with a star under x. These three rows are exactly like the target row except (perhaps) with respect to x , and under x we get each of the three possible entities from $Domain_{\mathtt{j}}$. Checking now tells us that $Val_{\mathtt{j}}$(∀xAwxyz) = F.
- For ∀yAwxyz to be true in the target row, we must consult the three rows with a star under y. These three rows are exactly like the target row except (perhaps) with respect to y, and under y we get each of the three possible entities from $Domain_{\mathtt{j}}$. As above, we may be sure that $Val_{\mathtt{j}}$(∀yAwxyz) = F.
- For ∀zAwxyz to be true in the target row, we must consult the three rows with a star under z. These three rows are exactly like the target row except (perhaps) with respect to z, and under z we get each of the three possible entities from $Domain_{\mathtt{j}}$. Checking those three rows with a star under x yields that $Val_{\mathtt{j}}$(∀zAwxyz) = F. (This column is omitted from the picture.)

11B Existentials in assignment tables

Note that each of the four companion *existential* statements must automatically be true in the particular target row $\mathtt{j}$ of this example. Why? Since the target row itself gives Awxyz the value T (i.e., $Val_{\mathtt{j}}$(Awxyz) = T), the target row must also give T to every existential quantification of Awxyz (e.g. $Val_{\mathtt{j}}$(∃wAwxyz) = T), by the rule of existential generalization. To offer an "interesting" example of calculating a value for existential quantifications, we should need to change the example, considering another sentence, say Bwxyz, that has an F in the target row (i.e., Val_{j}(Bwxyz) = F). Then we should be forced to look at rows other than the target row (the very same "other" rows that are pictured) in order to calculate whether, for instance, Val_{j}(∃wBwxyz) = T.

Exercise 83 *(Calculating existentials from an assignment table)*

Optional (since the section itself is optional). Suppose the "Given" truth-value entry for Awxyz in row **j** is changed from T to F (and the rest of the "Given" column is left alone). Calculate the truth values of prefixing Awxyz with an existential quantifier using (in turn) each of w, x, y, and z. For example, what is Val_{j}(∃wAwxyz)?

▷ . . . . ◁

Chapter 12

Appendix: The criteria for being a good definition

12A Criteria of eliminability and conservativeness

The standard criteria for good definitions are those of "eliminability" and "conservativeness." Where do these criteria come from? Why should we pay attention to them?

We can derive a motivation for these criteria from the concept of definitions as *explanations of the meanings of words* (or other bits of language). Under the concept of a definition as explanatory, (1) a definition of a word should explain *all* the meaning that a word has, and (2) it should do *only* this and nothing more.

1. That a definition should explain *all* the meaning of a word leads to the criterion of eliminability.
2. That a definition should *only* explain the meaning of the word leads to the criterion of conservativeness.

We discuss the two criteria in order.

12A.1 Criterion of eliminability

One may approach the criterion of eliminability from the direction of the "use" picture of meaning, with its picture-slogan, "meaning is use." Then to explain *all* of the meaning of a word is to explain *all* its uses, that is, its use in *every* context. (We intend that this recipe neglect ambiguity.) The advance is this: The metaphorical quantifier in "*all* the meaning" is cashed out in terms of a non-metaphorical (if still imprecise) quantifier over contexts. But what shall we mean by a "context"?

By §8A.5 we are considering only first-order languages. Accordingly, the criterion of eliminability requires that for *each* first-order sentential context B in the New language (i.e., the vocabulary of B is confined to $\mathbf{C}\cup\{\mathbf{c}\}$), the definition gives enough information to allow formation of an equivalent piece of language B′ in the Old language (i.e., the vocabulary of B′ is confined to **C**). (The definition **D** constitutes a bridge between New language and Old.) Then, and only then, will we be sure that we have explained *all* the meaning of the word to be defined—whether our purpose is to explain an existing meaning of a word already in use or to give a new meaning to a fresh piece of vocabulary. The move from B in the New language (i.e., *with* the definiendum **c**) to the equivalent B′ in the Old language (i.e., *without* **c**) is what you should think of as "eliminating" the definiendum.

It should be clarifying to use our logical concepts in order to sharpen this idea. The criterion of eliminability requires the following:

> For every sentence A of the New language (so A may be expected to contain **c**) there is a sentence A′ of the Old language (so A′ definitely does not contain **c**) such that [1]
>
> $$\mathbf{G}\cup\{\mathbf{D}\}\vDash_{Q} A\leftrightarrow A'.$$

It is the universal quantifier "For every sentence A of the New language" on the front of [1] that entitles us to say that the definition **D** explains *all* the meaning of the definiendum **c**.

Let's consider some examples of definitions that *do* and some definitions that *don't* satisfy the criterion of eliminability.

12A-1 EXAMPLE. *(Piety)*

Piety is what the gods all love (Euthyphro to Socrates, in Plato's dialogue *Euthyphro*). In notation: ∀x[x is pious ↔ the gods all love x].

This definition permits the elimination of "_ is pious" in all first-order contexts. How? By the rule that we have called "UI+RE" (Rule **3E-7**(2) on p. 131). Since Example **12A-1** is a universally generalized biconditional, the rule UI+RE applies exactly: It tells us that given either of the following as a premiss,

. . . t is pious . . . [2]

. . . all the gods love t . . . [3]

we are entitled to infer the other—regardless of the symbolic context represented by the ". . .". For example, under the definition, we could use UI+RE to show that the following are interchangeable:

- *Euthyphro's act of prosecuting his father* is pious.
- All the gods love *Euthyphro's act of prosecuting his father*.

So far so good. But it is to be carefully noted that UI+RE is worthless if the "context" represented by ". . ." in [2] and [3] is not first-order (quantifiers plus truth functions). Socrates gives a telling example of the importance of this qualification. Socrates asked Euthyphro to consider the following plausible premiss.

If x is pious, then all the gods love x *because* it is pious.

If we tried to use Euthyphro's Example **12A-1** to eliminate the second occurrence of "pious"—the one in the scope of "because"—we should find that we were led to the following silly conclusion.

If x is pious, then all the gods love x *because* all the gods love x.

This leads us to see that the limitation of our account to first-order contexts as in §8A.5 has real bite: Example **12A-1** suffices to explain "pious" in first-order contexts, but fails miserably for other philosophically important contexts such as clauses beginning with "because."

In addition to "becausal" contexts, we should keep in mind quotation contexts, and also psychological contexts involving rapt attention such as "he was turning over in his mind the question of whether prosecution of his father was pious." If we do then we will be more likely to keep remembering that *no* definition will permit elimination in absolutely *every* context. Such remembering will perhaps disincline us to ask more of a definition than it can very well deliver. But perhaps not. Some people respond by cooking up an account of quotes or of so-called "propositional attitudes" on which they do not count as "contexts." This seems an unhelpful obfuscation of the general principle. What *is* helpful is to realize that definitions like Example **12A-1** will help eliminate a defined term in all (and only) first-order contexts.

Exercise 84 *(Criterion of eliminability)*

Let the Old language be the language of Peano arithmetic with (say) just 0, $'$, + and $\times$. Suppose we enter the following as a definition **D**: $\forall x \forall y[x \leqslant y \leftrightarrow \exists z(x+z)=y]$. How could you eliminate $\leqslant$ from the context "$0 \leqslant 0$"? That is, find a sentence A' in the *Old* language such that $\mathbf{D} \vDash_Q (0 \leqslant 0) \leftrightarrow A'$. *Prove* that your particular candidate for A' works by providing a Fitch proof with premiss **D** (that is, with premiss $\forall x \forall y[x \leqslant y \leftrightarrow \exists z(x+z)=y]$) and with conclusion $(0 \leqslant 0) \leftrightarrow A'$.

Just to be sure that you have the idea firmly in mind, show how you could use the same definition **D** to eliminate $\leqslant$ from the context $0' \leqslant 0$. Find the appropriate A', and prove that it works.[1]

▷ . . . . ◁

It is good to replace the slightly informal account of the criterion of eliminability with a sharp definition. You will see, indeed, that all we are doing is changing the status of the "requirements" [1] for eliminability into clauses of a definition of the very kind we are discussing in this chapter. Since there are four key entities involved in explaining the criterion of eliminability, we define a four-place predicate, as follows.

12A-2 DEFINITION. *(Eliminability)*

D, *qua definition of* **c**, *satisfies the criterion of eliminability relative to* **C** *and* **G** iff each of the following.

1. **D** is a sentence, **c** is a constant, **C** is a set of constants, and **G** is a set of sentences.
2. The definiendum **c** does not belong to **C**, but all constants in **D** except **c**, and all constants in **G**, belong to **C**.
3. For every sentence A all of whose constants are in $\mathbf{C} \cup \{\mathbf{c}\}$ (so A is in the New language), there is a sentence A' all of whose constants are in **C** (so A' is in the Old language) such that

$$\mathbf{G} \cup \{\mathbf{D}\} \vDash_Q (A \leftrightarrow A').$$

[1] Probably unnecessary warning: Our use of $'$ for the successor operator in Peano arithmetic has nothing to do with our use of $'$ in making up a new sentence-variable A'.

12A.2 Criterion of conservativeness

On the standard view a good definition (in the sense of an explanation of meaning) should not only explain *all* the meaning of the word, as required by the criterion of eliminability. It should *only* do this. It should be empty of assertional content beyond its ability to explain meaning. If it were to go beyond the job assigned, say by claiming that Euthyphro's act of prosecuting of his father is pious, it might indeed be doing something useful. It would, however, no longer be entirely satisfactory as a *definition*. In this perfectly good sense, a definition should be neither true nor false (whether explaining a word already in use or introducing a fresh word): Whether or not it has a truth value qua definition (we might argue about that), it should make no claims beyond its explanation of meaning.

There is a special case: Clearly a definition should not take you from consistency to inconsistency; that would be a dreadful violation of conservativeness. The older theoreticians of definitions in mathematics were, however, insufficiently severe when they suggested that consistency is the *only* requirement on a mathematical definition.

Terminology for this criterion is a little confusing. Some folks use "conservative" to mean something like "does not permit the proof of anything we couldn't prove before." This accounts for calling the second criterion that of conservativeness: A definition satisfying the second criterion is conservative in that very sense. Other folks use "creative" to mean something like "permits proof of something we couldn't prove before." It is with this sense in mind that the second criterion is sometimes called "the criterion of noncreativity." We opt for the former name merely to avoid the negative particle. Under either name, the criterion demands that the New theory $\mathbf{G}\cup\{\mathbf{D}\}$ (with the definition) not have any consequences in the Old language that were not obtainable already from the Old theory (without the definition).[2] If before the definition of "sibling" we could not establish that Euthyphro's act is pious, then it should not be possible to use the definition of "sibling" to show that Euthyphro's act is pious. Were we able to do so, the definition would manifestly contain more information that a mere explanation of meaning: It would not be conservative. Neither in explaining the meanings of words already in use nor in introducing fresh words should one use the cover of definitions to smuggle in fresh assertions in the Old language. It's bad manners.

[2]Watch out: "Old" is critical here. The definition **D** itself will of course be in the New language, since it will have to contain the definiendum **c**. And it will also have numerous other consequences in the New language. Understanding conservativeness requires that you distinguish the Old and New languages in order to appreciate that what is forbidden is consequences of the New theory $\mathbf{G}\cup\{\mathbf{D}\}$ in the Old language unless they are already consequences of the Old theory **G**.

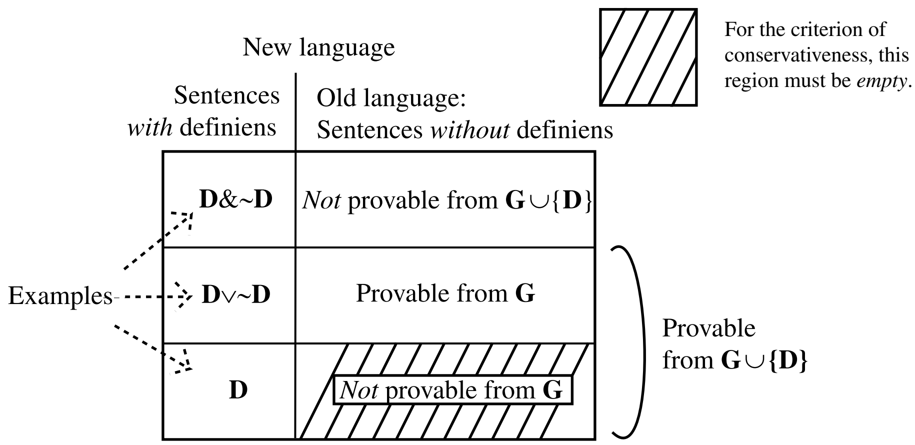

Figure 12.1: A picture of the criterion of conservativeness

Again it should be helpful to use our logical concepts to enhance clarity. The criterion of conservativeness requires the following:

> For any sentence A, if A is in the Old language, then if $\mathbf{G}\cup\{\mathbf{D}\}\vDash_Q A$ then $\mathbf{G}\vDash_Q A$. [4]

That is, anything you can derive in the Old language using the definition you could already derive without the definition. Some people are more at home with the contrapositive: If A is in the Old language, then if $\mathbf{G}\nvDash_Q A$ then $\mathbf{G}\cup\{\mathbf{D}\}\nvDash_Q A$ (adding the definition does not permit the derivation of anything in the Old language that isn't already derivable in the Old theory). See Figure 12.1 for a picture.

12A-3 EXAMPLE. *(Noid)*

Suppose that Wordsmith introduces "noid" by announcing that all the guys on the other side of Divider Street are noids (a sufficient condition) and that all noids are born losers (a necessary condition). In notation, his candidate definition **D** is the following: $\forall x[Dx\rightarrow Nx]$ & $\forall x[Nx\rightarrow Lx]$.

Wordsmith's plan appears innocent: He wishes to help his gang understand "noid" by squeezing it between two understood ideas. Wordsmith's introduction of "noid," however, may or may not be innocent. If the gang and Wordsmith have *already* committed themselves to the proposition that all guys on the other side of Divider Street are born losers, (in notation: if $\forall x[Dx \rightarrow Lx]$ is already derivable from the Old theory **G**, without the definition) then Wordsmith has not by this "definition" **D** violated conservativeness. But if Wordsmith has not done this, if he has not committed himself and the gang to the born-loserness of all guys across Divider (if, that is, $\forall x[Dx \rightarrow Lx]$ is *not* already derivable from the Old theory **G**), then he has violated conservativeness—just because his "definition" **D**, i.e. $\forall x[Dx \rightarrow Nx]$ & $\forall x[Nx \rightarrow Lx]$, logically implies $\forall x[Dx \rightarrow Lx]$ (that is, $\mathbf{G} \cup \{\mathbf{D}\} \vDash_Q \forall x[Dx \rightarrow Lx]$). The latter is in the Old language since it does not involve the definiendum "noid," so there is no excuse for smuggling in its assertion under cover of an act of definition.

The following example of a violation of conservativeness is just a little trickier. Let us consider two speech acts, one by Sarah and one by Janet.

12A-4 EXAMPLE. *(Empty set)*

1. Suppose Sarah says "Axiom: $\varnothing$ is a set and $\varnothing$ has no members" ($\varnothing$ is a set & $\forall x[x \notin \varnothing]$). (Compare Axiom **4A-12**.)
2. Suppose Janet says "Definition: $\varnothing$ is a set and $\varnothing$ has no members" ($\varnothing$ is a set & $\forall x[x \notin \varnothing]$).

Observe that in either case, "there is a set with no members" (in notation, $\exists z$[z is a set and $\forall x[x \notin z]$]) follows logically. The key feature here is that this logical consequence, since it does not involve the constant $\varnothing$, is in the Old language. But there remains a difference between (1) and (2).

First, Sarah's (1). Sarah does not purport to offer a definition at all, so conservativeness is simply not an issue. There is nothing wrong with an axiom that governs New notation having consequences such as $\exists z$[z is a set and $\forall x[x \notin z]$] that do not involve that notation.

Furthermore, if Janet's community has *already* fixed on a theory **G** that implies $\exists z$[z is a set and $\forall x[x \notin z]$] (without using Janet's definition), she is not in violation of conservativeness. But since this "definition" obviously permits her to prove that there is a set with no members (by existential generalization), she violates conservativeness if she couldn't prove it before. That is, the sentence "$\exists z$[z is a set and $\forall x[x \notin z]$]" would constitute a violation of conservativeness since (a) it is in the Old language (since it does not contain the defined constant $\varnothing$), (b) but (c) it is not a consequence of only the Old theory **G** without the use of Janet's definition. She would therefore be using the "definition" to smuggle in an existence claim to which she is not entitled.

Suppose the worst, however, that Janet's community has a theory **G** according to which no set has no members.[3] Suppose she goes ahead and introduces $\varnothing$ in accord with Example **12A-4**. What should we say about her? What did she denote by $\varnothing$? Probably no one cares, but there are cases like Janet's about which philosophers do seem to care. For example Leverrier introduced "Vulcan" as the planet responsible for certain astronomical anomalies. Later it was learned that relativistic considerations sufficed for the anomalies; there was no such planet. What shall we then say about the meaning of "Vulcan" and of all the discourse using this term? Our own view is this. You should say that the answer to "Is there a unique planet responsible for the astronomical anomalies at issue?" is "No." You should say that the answer to "Is the definition defective?" is "Yes." You should say that some of what we prize in science and other activities does not rest on answers to these eminently clear questions, since we often manage to get along very well when presuppositions fail. You should add that nevertheless *there are no general policies for what to do or what to say in the presence of defective definitions*. Just about anything you remark beyond this will be either artificial or unclear, and is likely only to add to the general babble of the universe.[4]

It is easy to imagine languages that allow nonconservative or creative definitions in the following sense: In them a single speech act both introduces the new terminology and makes the assertion needed to justify that introduction. In fact English, even scientific English, is like that. Our practices are lax. We suppose that is one thing people mean when, like Quine, they advise us to abandon the analytic-synthetic distinction.

We should not, however, follow this advice. Those philosophers who wish to be clear should follow Carnap in maintaining the analytic-synthetic distinction to the maximum extent possible. That is, when confronted with a complex speech act such as that of Leverrier, it is good to idealize it as consisting of two distinct components: the assertion of the existence and uniqueness of an anomaly-causing planet, and the definition of Vulcan as this very planet. There is, we think, no other way to continue with a *clear* conversation about the matter. For example, we do not think we can otherwise make a clear distinction between revealed and concealed assertions. And we should do so. Speech acts that conceal assertions are (whatever

[3]The logician Lésniewski promoted such a theory.

[4]A signal exception to this rule is the careful treatment of confusion in Camp (2002).

their intentions) misleading. We should try to avoid them whenever we are clear enough to be able to do so.

Here is the sharp definition that, in parallel to Definition **12A-2**, replaces our slightly informal account [4] of the criterion of conservativeness. As there, we define a four-place predicate.

12A-5 DEFINITION. *(Conservativeness)*

D, *qua definition of* **c**, *satisfies the criterion of conservativeness relative to* **C** *and* **G** iff each of the following.

1. **D** is a sentence, **c** is a constant, **C** is a set of constants, and **G** is a set of sentences.
2. The definiendum **c** does not belong to **C**, but all constants in **D** except **c**, and all constants in **G**, belong to **C**.
3. For every sentence A, if every constant in A is in **C** (so A is in the Old language), then

   if $\mathbf{G} \cup \{\mathbf{D}\} \vDash_{Q} A$ then $\mathbf{G} \vDash_{Q} A$.

The diagram of Figure 12.1 on p. 281 should help sort this out for you.

Exercise 85 *(Conservativeness)*

Suppose that the Old vocabulary **C** consists in the standard constants for Peano arithmetic (§7B): 0, ′, +, and ×. Let the Old theory **G** be the Peano axioms Axioms **7B-1**. Let the definiendum be the predicate constant "is interesting." Let the candidate definition **D** be ∀x∀y[x is interesting ↔ x = y)]. Your job is to show the candidate definition violates conservativeness by finding a sentence in the Old language that can be derived (by a proof) from the candidate definition. (Hint: The example is silly, but the proof is easy.)

▷ . . . . ◁

12A.3 Joint sufficiency of criteria for a "good definition"

In concluding this preliminary account, we add the following. The proposal is that each of the criteria of eliminability and conservativeness are necessary for a good definition in the sense of an explanation of meaning; and that together they are sufficient (so that their conjunction provides a good definition of "good definition"). The pros and cons of necessity are easily discussed through examples. In contrast, we do not much know how to defend the claim to sufficiency other than by a rhetorical question: You've asked for a good definition of this word, and now we've given you a procedure for eliminating it from every intended context in favor of something you yourself grant is entirely equivalent; and we have done nothing else (i.e., we haven't slipped any further statements past you); so what more do you want?

We enshrine this view in (what else?) a definition:

12A-6 DEFINITION. *(Good definition)*

D, *qua definition of* **c**, *is a good* Q-*definition* relative to **C** *and* **G** iff each of the following.

1. **D**, *qua definition of* **c**, satisfies the criterion of eliminability relative to **C** *and* **G** (Definition **12A-2**).
2. **D**, *qua definition of* **c**, satisfies the criterion of conservativeness relative to **C** *and* **G** (Definition **12A-5**).

A little more needs to be said. It is common to think of an entire set of sentences **G*** (instead of a single sentence **D**) as expressing an "implicit definition" of some constant **c** taken as definiendum in terms of some Old constants **C**. Here the entire, presumably infinite, set of sentences **G*** is taken as definiens. Since our account Definition **12A-6** of "good definition" presupposes a *single-sentence* definition **D**, it simply does not apply to this important case. Instead, analysis suggests that the right idea is ask if **G*** together with the Old theory **G** forces a unique interpretation on **c** in the following sense: Whenever the interpretations of all Old constants **C** are fixed in some way as to make all the sentences of **G** true, there is a *unique* interpretation of **c** on which every member of **G*** comes out true. If so, the implicit definition **G*** is good; if not, not. We do not, however, have to worry much about good implicit definitions for this reason: Beth's Theorem—which is one of the important theorems of the science of logic—tells us that whenever there is a good infinite set-of-sentences *implicit* definition **G*** of **c**, there is also a good single-sentence *explicit* definition **D** of **c**—a single-sentence definition **D** of **c** that satisfies all the rules for and all the criteria of "good definition."

12A.4 Conditional definitions and reduction sentences

This section may be skipped by those concentrating on the art of logic. Here we use our discussion of definitions in order to help in some philosophical work. We start with an example that revolves around the fact that division by zero makes no sense. Here you must imagine that the Old vocabulary includes + and $\times$ and 0 and 1, and that the Old theory is not mere Peano arithmetic, but includes the arithmetic of the rational numbers (fractions) as well. We are trying to define "the inverse of x," usually written as "1/x," in terms of the Old vocabulary.

12A-7 EXAMPLE. *(Inverse)*

Take **D** as $\forall x[x \neq 0 \rightarrow \forall y[\text{Inverse}(x) = y \leftrightarrow (x \times y) = 1]]$, with Inverse(_) the definiendum. Let **C** and **G** be as in Peano arithmetic, §7B.

Thus the definition takes Inverse(_) as the standard multiplicative inverse, $(1 \div \_)$. And Inverse(x) "is defined" only when x is not zero so that the problem of division by zero is avoided. Such a definition is often called a "conditional definition," the *condition* being that $x \neq 0$.

Evidently **D** does not satisfy eliminability, since you cannot use it to eliminate all uses of Inverse(0), for example, its use in "Inverse(0) = 3."

Exercise 86 *(Inverse)*

What happens when you instantiate the definition with 0 for x?

▷ . . . . ◁

It does, however, satisfy a "partial eliminability," since **D** does permit elimination of Inverse(t) for any term t such that $\mathbf{G} \cup \{\mathbf{D}\} \vDash_Q t \neq 0$. More generally, **D** permits elimination of Inverse(x) in any sentence $\forall x B(x)$ [or $\exists x B(x)$] such that $\mathbf{G} \cup \{\mathbf{D}\} \vDash_Q$

$\forall$xB(x) $\leftrightarrow$ $\forall$x((x$\neq$0)$\rightarrow$B(x)) [or $\mathbf{G}\cup\{\mathbf{D}\}\vDash_Q \exists$xB(x) $\leftrightarrow$ $\exists$x((x$\neq$0)&B(x))]. Since these are known *in advance* as the only contexts of Inverse(t) or Inverse(x) that we care about (nobody wants an Inverse(0)), such partial eliminability is conceptually satisfying.

We know that Frege thought otherwise; he believed in total eliminability. Our own view is that Frege's belief (or practice) arose from a certain slightly misdirected compulsiveness—the same compulsiveness, however, that gave the world its most glorious standard of uncompromising rigor. Many of us, perhaps almost equally compulsive, avoid conditional definitions for what we announce as mere technical convenience. Our alternative device is to use "don't-care clauses," e.g. defining Inverse(0) as equal to 0 (or perhaps as equal to 14), and then remarking that this bit of the definition is a don't-care. That is, we would replace the definition of Example **12A-7** with the following "good definition":

$$\forall x\forall y[\text{Inverse}(x)=y \leftrightarrow ((x\neq 0\rightarrow((x\times y)=1))\ \&\ (x=0\rightarrow y=0))]. \qquad [5]$$

Now if you let the definition **D** be [5], you can eliminate every occurrence of Inverse(__). Since you can use [5] to prove "Inverse(0) = 0," when you come to a sentence A with an occurrence of Inverse(0), just choose A′ as the result of putting 0 in place of Inverse(0), and you will be sure that $\mathbf{G}\cup\{\mathbf{D}\}\vDash_Q A\leftrightarrow A'$, as required for eliminability.

If one studies our inferential practices, however, one sees that we never *use* the don't-care clauses, and that we counsel others to avoid relying on them for conceptual interest. So practically speaking, conditional definitions such as Example **12A-7** and don't-care clauses such as used in [5] come to the same thing.

Not all conditional definitions, however, are created equal. Carnap envisaged two kinds of predicates for science: An Observation predicate, e.g., "dissolves in water (when put into water)," is such that whether or not it holds of an object is a matter of direct observation. In contrast, a Theoretical predicate, e.g. a so-called dispositional predicate such as "is soluble," is only indirectly tied to observation via some theory. Carnap suggested that what he called a "reduction sentence" could serve as a kind of partial meaning-explanation of a Theoretical predicate in terms of Observation predicates. Here is an example.

12A-8 EXAMPLE. *(Solubility)*

Let $\mathbf{D}_1$ be as follows: $\forall$x[x is put in water $\rightarrow$ (x is soluble $\leftrightarrow$ x dissolves)]. Take the definiendum **c** to be the Theoretical predicate constant, "is soluble," take the Old vocabulary **C** to include only Observation predicates, and identify the Old theory **G** as common-sense generalizations using only Observation predicates (e.g., perhaps "If you put a sample of copper in water, it doesn't dissolve").

$\mathbf{D}_1$ is exactly what Carnap called a "reduction sentence." He was surely thinking that the definition of Example **12A-8** gave us a "partial" definition of "soluble." Suppose the Old theory contains "Sam is put in water." Then clearly the definition $\mathbf{D}_1$ of Example **12A-8** permits elimination of "soluble" in the context "Sam is soluble," for it is easy to see that $\mathbf{G} \cup \{\mathbf{D}\} \vDash_{\mathrm{Q}}$ Sam is soluble $\leftrightarrow$ Sam dissolves. That is the sort of thing that Carnap had in mind. On the other hand, however, we couldn't use $\mathbf{D}_1$ to eliminate the defined symbol from "Mary is soluble" unless "Mary is put in water" was a consequence of our Old theory **G**. But suppose that in fact it is *false* that Mary is put in water? What then?

The superficial form of a reduction sentence is the same as that of a conditional definition. We think this resemblance misleads some of those who rely on reduction sentences: Reduction sentences such as Example **12A-8** and conditional definitions such as Example **12A-7** are entirely different. In the case of conditional definitions, we do not *want* an account of the newly introduced constant as applied to arguments not satisfying the condition. We do not want, for example, an account of Inverse(0). In the case of the reduction sentence for solubility, however, the samples whose presence in water we cannot deduce from the theory are *precisely* those samples whose solubility we care about. For example, it would be ridiculous to provide a don't-care clause for the condition, adding the following to Example **12A-8**: "and if x is not put in water then x is a rotten apple." That this would be foolish is obvious from examination of the inferential contexts in which one finds the concept of solubility. We infer that Carnap's terminology represents a wrong turn. Reduction sentences do not give partial explanations of meaning; unless, on analogy, one wishes to take "counting to three" as "partially counting to infinity."

Turning now to the other criterion, the reduction sentence $\mathbf{D}_1$ of Example **12A-8** is, if taken as a definition, doubtless conservative. Suppose, however, that we enrich that definition by adding a conjunct.

12A-9 EXAMPLE. *(Solubility, another try)*

Let $\mathbf{D}_2$ now be taken as a definition of solubility, where $\mathbf{D}_2$ is as follows: ($\mathbf{D}_1$ & $\forall x \forall y$[x and y are of the same natural kind $\rightarrow$ (x is soluble $\leftrightarrow$ y is soluble)]).

Adopting $\mathbf{D}_2$ would doubtless enlarge the cases in which we could eliminate "soluble," making it in that respect more definition-like. But notice that we can now conclude that each thing of the same natural kind as Sam, if put in water, dissolves if and only if Sam does. Unless our Old theory **G** already committed us to this view, our definition would have smuggled in new assertional content and would not be conservative. A healthy respect for conservation regards such sneaking as reprehensible. Thus again, reduction sentences are not much like definitions after all. They do not reduce, or not enough to count.

What, then, do reduction sentences do? It is perfectly all right to suggest that they describe or establish "meaning relations" between the Theoretical mystery vocabulary to which "soluble" belongs and the less mysterious Observation language of water and the like. The only thing that is wrong is to think of them as "reducing" the mysterious to the less mysterious, even partially. Only the terminology is wrong. Here is an example that early Carnap considered on his way to reduction sentences, though we add a moral of which only the later Carnap would have approved.

12A-10 EXAMPLE. *(Solubility; yet another try)*

Suppose we let $\mathbf{D}_3$ be as follows: $\forall$x[x is soluble $\leftrightarrow$ (x is put in water $\rightarrow$ x dissolves)].

This definition of solubility is bound to satisfy both criteria (it is a good definition in the sense of Definition **12A-6**), regardless of the Old theory **G**. Given that the connectives $\leftrightarrow$ and $\rightarrow$ are truth-functional, however, it does have the strange (but still conservative) consequence that whatever is not put in water is soluble, so that as *stipulative* it is not a happy choice and as *lexical* it is doubtless false to our usage. One can see here a motive for a "modal" theory of "if" that brings into play the idea of *possibility*, and accordingly a motive for not identifying "logic" with the logic that we are presently studying.

12A.5 Relaxing the criteria

The standard criteria and rules must be counted as logic of the greatest interest. They are, we should say, always to be respected as devices to keep us from addle. On the other hand, one can easily find cases in which good thinking bids us moderate their force. Here is an illustration concerning noncircularity.

Gupta and Belnap (1993), building on earlier work by Gupta, establishes the following.

1. Some concepts are essentially circular. (Normal results of "inductive definitions," e.g. multiplication in Peano arithmetic, are *not* examples of circular concepts.)

2. The standard account of definitions says nothing useful about circular concepts (We suppose it denies their existence).

3. One obtains a powerful theory of circular concepts by reworking the theory of definitions to admit circular definitions.

4. Truth (in e.g. English) is a circular concept.

5. The ideas of the reworked definitional theory, when applied to truth, make fall into place both the ordinary and the extraordinary (pathological, paradoxical) phenomena to which philosophers have called attention.

These notions are extended and defended in *The revision theory of truth*, Gupta and Belnap (1993), which we will reference as RTT. It is not possible to summarize even the main ideas here; we only make two remarks relating the RTT work to the standard account of definitions.

RTT fully *accepts conservativeness*. The norm remains: If it's a definition, then it should not provide camouflage for a substantive assertion. We shan't say anything more on this topic.

RTT *abandons eliminability, and in particular noncircularity*. The indispensable idea is that a definitional scheme cannot illuminate a circular concept except with the help of a circular definition, and that circularity intrinsically prevents eliminability. For example, the RTT definition of truth permits its elimination from "'Euthyphro is pious' is true," but not from "This very sentence is true." An especially fascinating part of this story is that RTT furnishes something worthwhile to say about circular concepts such as truth even in those cases in which eliminability fails. It does so by attending to their *patterns of revision* under different hypotheses. In this way forms of pathologicality can be distinguished one from the other in a conceptually satisfying way. You can *see* the difference between "This very sentence is true" and "This very sentence is false" by watching the unfolding of their altogether different patterns of revision.

12B Coda

Two items remain. The first is to express regret that these remarks have been too coarse to capture the fine structure of even the standard account of definitions. The second is to make explicit that the general lesson has the usual banal form. On the one hand, the standard criteria and rules are marvelous guides. They have point. They are not to be taken lightly. Every philosopher and indeed we should say everyone engaged in conceptual work ought to know and understand them. He or she who unknowingly breaks with the standard criteria or the rules risks folly. On the other hand, there is no single aspect of these criteria or rules that we should consider unalterable.

Chapter 13

Appendix: Future discussion of decidability, etc.

This chapter is what it is and is not another thing.

13A Effective, constructive, prospective

(This section is not intended as self-standing; it presupposes outside explanation.) From Myhill (1952):

> A character is *effective* if human beings may be so trained as to respond differentially to its absence and its presence.
>
> A character is *constructive* if human beings may be so trained as to execute a program in the course of which a replica of every object having that property will appear, barring the untimely death or distraction from his purpose of that human being. By a "replica" in this connection is meant an object not differing from the original in any way which could confer upon it or detract from it the character declared by this definition to be constructive.
>
> A character is *prospective* if it is neither effective nor constructive, and if nonetheless some human being clearly entertains an idea of that character.

A character is *effective* iff it is possible to provide a routine and mechanical and finite "decision procedure" for that character, a procedure that involves neither luck nor ingenuity. Since the definiens did not mention Mind, a character can be effective even if nobody knows a decision procedure that will work for that character. A character is *constructive* iff its theory can be formalized. "Constructive" is the same as "formalizable." For Myhill's "prospective," but without its denial of effectivity and constructivity, we just say "clear."

Theory	Effective	Constructive	Clear
Truth-table tautology	**Yes** (Post 1921)	$\rightarrow$	$\rightarrow$
Quantifier validity	**No** (Church 1936)	**Yes** (Gödel 1930)	$\rightarrow$
Arithmetic truth	$\downarrow$	**No** (Gödel 1931)	**Yes** (Gentzen 1936)
Set-theory truth	$\downarrow$	$\downarrow$	**No** (Russell 1902)

Arrows indicate that affirmatives propagate rightward and that negatives propagate downward.

Bibliography

Belnap, N. (1993). On rigorous definitions. *Philosophical studies*, 72:pp. 115–146.

Belnap, N., Perloff, M., and Xu, M. (2001). *Facing the future: Agents and their choices in our indeterminist world*. Oxford University Press, Oxford.

Belnap, N. and Steel, T. (1976). *The logic of questions and answers*. Yale University Press, New Haven.

Camp, J. (2002). *Confusion*. Harvard University Press, Cambridge, MA.

Church, A. (1956). *Introduction to mathematical logic*. Princeton University Press, Princeton, NJ.

Curry, H. (1963). *Foundations of mathematical logic*. McGraw-Hill, Inc., New York.

Curry, H. and Feys, R. (1958). *Combinatory logic*, volume I. North-Holland Publishing Company, Amsterdam.

Fitch, F. B. (1952). *Symbolic logic*. The Ronald Press, New York.

Gentzen, G. (1934). Untersuchungen ber das logische schliessen. *Mathematische Zeitschrift*, 39:176–210, 405–431. English translation (Investigations into logical deduction) by M. E. Szabo in *The collected papers of Gerhard Gentzen*, ed. M. E. Szabo, Amsterdam (North-Holland Publishing Co.), 1969, pp. 68–131.

Gupta, A. and Belnap, N. (1993). *The revision theory of truth*. MIT Press, Cambridge, MA.

Jaśkowski, S. (1934). On the rules of supposition in formal logic. *Studia logica*, 1:5–32. Reprinted in *Polish logic, 1920–1939*, ed. S. McCall, Oxford (Clarendon Press), 1967, pp. 232–258.

Klenk, V. (2002). *Understanding symbolic logic*. Prentice-Hall, Englewood Cliffs, NJ, 4th edition.

Montague, R. (1974). *Formal philosophy: selected papers of Richard Montague*. Yale University Press, New Haven. Edited and with an introduction by R. H. Thomason.

Myhill, J. (1952). Some philosophical implications of mathematical logic. *The review of metaphysics*, 6:165–198.

Suppes, P. (1957). *Introduction to logic*. D. van Nostrand Publishing Company, Princeton, NJ, New York, Toronto, London.

Thomason, R. (1970). *Symbolic logic, an introduction*. The Macmillan Company, New York.

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NOTES ON THE

ART

OF LOGIC

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Draft for use by friends

Contents

Preface

These notes do not constitute a finished book.[†] They represent an incompletely organized collection of materials. For one thing, you will find some places where the text makes little sense without oral supplementation, and for another, in a few cases exercises and examples are not fully in place. If, however, you are using these notes by yourself solely for review, perhaps the most efficient procedure would be to let serve as your agenda the numbered exercises that are scattered throughout the book. **These numbered exercises are listed just after this preface, on p. x.** Then you can read only what you *must* read in order to satisfy yourself that you can carry out these exercises.

The intended user should satisfy one of two criteria: *Either* the user should have had a first and very-elementary course in the art of symbolic logic that emphasized the arts of the logic of truth functions and one-variable quantification, *or* the user should be prepared to master **within the first two weeks, and carrying out an enormous number of exercises,** one of the many texts that teach those arts. Klenk (2002) (*Understanding symbolic logic*) (units 1–16 essential, with 17–18 desirable) is suitable for this purpose—but certainly not uniquely so.

Thanks to M. Dunn and K. Manders for suggestions, to W. O'Donahue and B. Catlin for collecting errors, to M. Kremer for both, and to several groups of students who have helped, namelessly, in nameless ways. I owe yet a third debt to Kremer for permitting use of exercises that he had prepared for his own students, and for some material on symbolization. P. Bartha, A. Crary, and B. Schulz have each made contributions that are at once extensive and valuable, and C. Campbell

[†]This 2009 version of these notes is the result of a medium-sized revision (including moving some material from the main text to some appendices), a procedure that invariably introduces errors. Many have been located by various readers, but it is certain that a number remain. I ask that you let me know about typos, substantive errors, poor or missing exercises, incomprehensible passages, strange formatting, and so on. Your action in this regard will benefit others, since I revise these notes almost each time that I use them.

not only did likewise, but also re-worked a number of exercises that were unsatisfactory, and cheerfully agreed to teach from the book in such a way that his authoritative wisdom contributed manifold suggestions. Gupta helped in particular with chapter 8. C. Hrabovsky helped greatly with preparing the book.

List of exercises