MATH 3550 (ALGEBRAIC GROUPS) – HOMEWORK 1

k denotes an algebraically closed field.

1. Let A be a finite dimensional k-algebra. Show that $\mathrm{Aut}(A)$ is an affine algebraic group.
2. Show that $\mathrm{PGL}_n(k)$ is an affine algebraic group and compute its algebra of regular functions.
3. Assume that $\mathrm{char}\, k = p$. Show that the canonical map $\mathrm{SL}_p(k) \to \mathrm{PGL}_p(k)$ is a bijective algebraic group morphism which is not an isomorphism.
4. Show that the affine algebraic groups $B_n(k)$, $U_n(k)$ and $D_n(k)$ are connected. Compute their algebras of regular functions.
5. Show that a group homomorphism between $\mathbb{G}_a$ and $\mathbb{G}_m$ (in either direction) must be trivial. In particular, $\mathbb{G}_a$ and $\mathbb{G}_m$ are not isomorphic as algebraic groups.
6. Describe the algebraic groups $\mathrm{Aut}(\mathbb{G}_a)$ and $\mathrm{Aut}(\mathbb{G}_m)$.
7. For $i \neq j$, let $\phi_{i,j} : \mathbb{G}_a \to \mathrm{SL}_n(k)$ the algebraic group morphism which maps $x \in k$ to the matrix with all zero entries except for 1's on the diagonal and x in the (i, j) position. Denote by $U_{i,j} = \phi_{i,j}(\mathbb{G}_a)$. Show that $\mathrm{SL}_n(k)$ is generated by the subgroups $U_{i,j}$.
8. Show that $[\mathrm{GL}_n, \mathrm{GL}_n] = \mathrm{SL}_n$.
9. Consider the action of $\mathrm{GL}_2(k)$ on $\mathbb{P}^1$ induced by the canonical action on k^2. Describe the orbit structure of the $\mathrm{GL}_2(k)$ action on $\mathbb{P}^1$, and of the $\mathrm{GL}_2(k)$ diagonal action on $\mathbb{P}^1 \times \mathbb{P}^1$.
10. Let G be a connected affine algebraic group and H finite, normal, algebraic subgroup. Show that $H \subseteq Z(G)$.