MATH 3550 (ALGEBRAIC GROUPS) – HOMEWORK 2

k denotes an algebraically closed field.

1. Let G be an affine algebraic group, $H \leq G$ a closed subgroup, and $I = \mathcal{I}(H) \trianglelefteq k[G]$. Show that $H = \{g \in G \mid g \cdot I = I\}$. (The action of G on $k[G]$ can be either the left regular, or the right regular action.)

2. Let $\pi : G \to H$ algebraic group morphism and $X \in \mathfrak{g}$. Show that
$$\pi_*(X) : k[H] \to k[H], \quad \pi_*(X) = (\mathrm{id} \otimes \pi_{*,e} X_e) \circ m_H^*.$$

3. Compute the following Lie algebras
   (a) $\mathrm{Lie}(D_n(k))$, $\mathrm{Lie}(U_n(k))$, $\mathrm{Lie}(B_n(k))$;
   (b) $\mathrm{Lie}(\mathrm{O}_n(k))$; $\mathrm{O}_n(k) = \{g \in \mathrm{GL}_n(k) \mid g^t \cdot g = I_n\}$;
   (c) $\mathrm{Lie}(\mathrm{Sp}_{2n}(k))$; $\mathrm{Sp}_{2n}(k) = \{g \in \mathrm{GL}_{2n}(k) \mid g^t \cdot J_{2n} \cdot g = J_{2n}\}$, $J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

4. Consider $\mathrm{Ad} : \mathrm{SL}_n(k) \to \mathrm{GL}(\mathfrak{sl}_n)(k)$ and define $\mathrm{PSL}_n = \mathrm{Ad}(\mathrm{SL}_n)$. Show that
   (a) As abstract group $\ker(\mathrm{Ad}) = Z(\mathrm{SL}_n(k)) = \mathrm{SL}_n(k) \cap (k \cdot I_n)$;
   (b) Compute the Lie algebra of $\mathrm{PSL}_n(k)$;
   (c) If $\mathrm{char}\, k = p$ divides n, then $Z(\mathrm{SL}_p(k)) = \{I_p\}$, but $\mathrm{SL}_p(k)$ and $\mathrm{PSL}_p(k)$ are not isomorphic as algebraic groups.

5. Let $\mathrm{char}\, k = p > 0$ and let $G = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a^p & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a \neq 0 \right\} \leq \mathrm{GL}_3(k)$.
   (a) Compute $\mathrm{Lie}(G)$;
   (b) Show that $\{e\} \subsetneq Z(G) \subsetneq \ker \mathrm{Ad} \subsetneq G$.

6. Let A be a commutative k-algebra, $m : A \otimes_k A \to A$ the multiplication map, and $I = \ker m$. Show that
   (a) $\mathrm{Der}(A, -) : A - mod \to k - mod, \quad M \mapsto \mathrm{Der}_k(A, M), \quad f \mapsto f \circ -$, is a functor;
   (b) I is generated by $\{a \otimes 1 - 1 \otimes a \mid a \in A\}$ as $A \otimes A$-module;
   (c) $\Omega_{A/k} := I/I^2$ is an A-module, and $d : A \to \Omega_{A/k}$, $d(a) = a \otimes 1 - 1 \otimes a + I^2$ is a derivation.

   $\Omega_{A/k}$ is called the module of Kahler differentials.

7. Let A be a commutative k-algebra. Show that the functors $\mathrm{Der}(A, -)$ and $\mathrm{Hom}(\Omega_{A/k}, -)$ are equivalent.

8. Show that $U_n(k)$ is nilpotent and $B_n(k)$ is solvable.

9. Describe all algebraic actions of the group $\mathbb{G}_a$ on the affine variety $\mathbb{A}^1 \setminus \{0\}$.

10. Give an example of a closed subgroup of $\mathrm{GL}_n(k)$ which consists of only semisimple elements but which is not conjugate to a subgroup of $D_n(k)$.