Intersecting geodesics on the modular surface

Naser Talebizadeh Sardari (Penn State University)
Representation Theory, L-functions, and Arithmetic

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Modular curve

$$X = PSL_2(\mathbf{Z})\backslash H.$$

DISTRIBUTION OF CLOSED GEODESICS

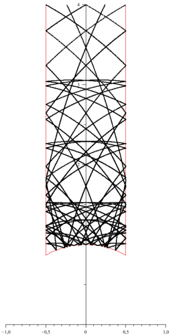

FIGURE 2. The distribution of $\mathcal{G}_{377}$ projected on the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, note that $h(377) = 1$.

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Choose a closed geodesic γ_L of length at most L at random...
$N(\gamma_L)$: the number of self-intersections.
Lalley 2004- Suppose that X has constant negative curvature
$\exists \kappa > 0$,

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[Pollicott-Sharp 2006] Distribution of angles of self-intersection in the negative curvature is

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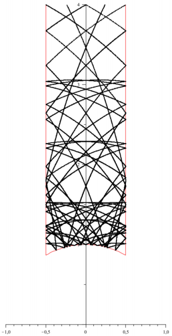

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Modular curve

Let $d > 0$ and $d \equiv 1, 0 \mod 4$ and d is not a square.
C_d : class of primitive binary quadratic forms

$$Q(x, y) = ax^2 + bxy + cy^2,$$

where $\gcd(a, b, c) = 1$ and $b^2 - 4ac = d$.
Class number $h(d) = \#C_d$. Let

$$\phi(Q) := \begin{bmatrix} \frac{t_0 - bu_0}{2} & -cu_0 \\ au_0 & \frac{t_0 + bu_0}{2} \end{bmatrix}.$$

where $t_0^2 - du_0^2 = 4$ is the fundamental solution of Pell's equation.
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Main theorem

Let β, γ be a compact geodesic segments on $\mathbb{X}$. For $0 < \theta_1 < \theta_2 < \pi$, let $I_{\theta_1,\theta_2}(\beta,\gamma)$ be the number of intersection between β and γ with angle between θ_1 and θ_2.

Theorem (J.Jung, N.T.)

Then we have

$$\frac{I_{\theta_1,\theta_2}(\beta, C_d)}{I(\beta)I(C_d)} = \frac{3}{\pi^2}\int_{\theta_1}^{\theta_2} \sin\theta d\theta + O_\epsilon\left(d^{-\frac{25}{3584}+\epsilon}\right),$$

uniformly in $\beta, \theta_1, \theta_2$, under the assumption that

$$\theta_2 - \theta_1 \gg d^{-\frac{25}{7168}}$$

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Theorem (J.Jung, N.T.)

The following estimate holds uniformly in $d_1, d_2 > 0$*, and* $0 < \theta_1 < \theta_2 < \pi$ *such that* $\theta_2 - \theta_1 \gg (d_1 d_2)^{-\frac{25}{3072}}$

$$\frac{I_{\theta_1,\theta_2}(C_{d_1}, C_{d_2})}{I(C_{d_1})I(C_{d_2})} = \frac{3}{\pi^2}\int_{\theta_1}^{\theta_2} \sin\theta d\theta + O_\epsilon\left((d_1 d_2)^{-\frac{25}{6144}+\epsilon}\right).$$

For $\delta > 0$, $k_{\delta,\theta_1,\theta_2} : S\mathbf{H} \times S\mathbf{H} \to \{0,1\}$ is given by

$$k_{\delta,\theta_1,\theta_2}((z_1,\xi_1),(z_2,\xi_2)) = 1$$

if the geodesics segments of length δ starting from z_i and initial angle ξ_i intersect with an angle between θ_1 and θ_2, and 0 otherwise.

We identify SH with $PSL_2(R)$ by sending $g \in PSL_2(R)$ to $g.(i,\pi/2) \in SH$.

$$k(g_1,g_2) = k(gg_1,gg_2).$$

We define

$$K((z_1,\xi_1),(z_2,\xi_2)) := \sum_{\gamma \in SL_2(\mathbf{Z})} k_{\delta,\theta_1,\theta_2}((z_1,\xi_1),\gamma(z_2,\xi_2)).$$

Automorphic Intersection Kernel

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Automorphic Intersection Kernel

Let $\tilde{C}_{d_1}, \tilde{C}_{d_2} \subset S\mathbf{X}$ be the corresponding lift. We have

$$I_{\theta_1,\theta_2}(C_{d_1}, C_{d_2}) = \delta^{-2} \int_{\tilde{C}_{d_1}} \int_{\tilde{C}_{d_2}} K_\delta(s_1, s_2) ds_1 ds_2$$

and

$$I_{\theta_1,\theta_2}(\beta, C_d) = \delta^{-2} \int_{\tilde{\beta}} \int_{\tilde{C}_d} K_\delta(s_1, s_2) ds_1 ds_2 + O(\delta l(C_d)).$$

Duke's theorem

Duke's theorem: C_d becomes equidistributed in $PSL_2(\mathbf{Z})\backslash\mathbf{H}$ as $d \to \infty$.
As noted by Luo-Rudnick-Sarnak (2008) corresponding lift $\tilde{C}_d$ also becomes equidistributed as $d \to \infty$.
In particular for any sufficiently nice continuous function f on $PSL_2(\mathbf{Z})\backslash\mathbf{H}$, we have

$$\frac{\mu_d(f)}{|\tilde{C}_d|} := \frac{1}{|\tilde{C}_d|}\int_{\tilde{C}_d} f(s)ds = \int_{\Gamma\backslash S\mathbf{H}} fdv.$$

Proof for fixed β

Take

$$f(g) := \delta^{-2} \int_{\tilde{\beta}} I_\delta(s, g) ds.$$

Then

$$\int_{\Gamma \backslash \mathbf{H}} f d(vol) = c(\theta_1, \theta_2) I(\beta)$$

for some constant $c(\theta_1, \theta_2) > 0$ indepndent of δ and Γ. By taking δ arbitrary small, we obtain

$$\lim_{d \to \infty} \frac{I_{\theta_1, \theta_2}(\beta, C_d)}{I(\beta) I(C_d)} = \frac{3}{\pi^2} \int_{\theta_1}^{\theta_2} \sin \theta d\theta.$$

Spectral theory-(proof of Duke's theorem for SX)

We consider the Hilbert space

$$F \in L^2(\Gamma \backslash SH) = L^2(\Gamma \backslash PSL_2(R)).$$

$$F(g) = \frac{3}{\pi^2} \int_{S\mathbb{X}} F(g_1) dg_1 + \sum_{\substack{m \geq 0 \\ 2|m}} \sum_{j=1}^{d_m} \sum_{\substack{l \in 2\mathbb{Z} \\ |l| \geq m}} \langle F, \phi_{j,l}^m \rangle_{S\mathbb{X}} \phi_{j,l}^m(g)$$

$$+ \sum_{m \in 2\mathbb{Z}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle F, E_m\left(\cdot, \frac{1}{2} + it\right) \right\rangle_{S\mathbb{X}} E_m\left(g, \frac{1}{2} + it\right) dr,$$

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Period integral formulas

Let

$$\psi_{s,2k}(g) := y(g)^{\frac{s+1}{2}} e(2k\theta)$$

where $g \in PSL_2(R)$ and we identified g with $(x(g), y(g), \theta) \in S\mathbb{H}$. Note that ψ is invariant by the action of the unipotent upper triangular matrices. Let

$$E_{2k}(g, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi_{s,2k}(\gamma g).$$

Period integral formulas

Let

$$d\pi(E^{\pm}) = \pm 2k + 2y\left(\pm i\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right).$$

It is easy to check from the above that

$$d\pi(E^{-})E_{2k}(g,s) = (-2k+1+s)E_{2k-2}(g,s),$$
$$d\pi(E^{+})E_{2k}(g,s) = (2k+1+s)E_{2k+2}(g,s).$$

We have

$$\mu_d(E_{2k}(g,s)) = \eta(2k,s)\mu_d(E_0(g,s))$$

where for $2k \geq 4$, $2|k$,

$$\eta(2k,s) = \eta(-2k,s) = \frac{(1-s_j)(5-s_j)\cdots(2k-3-s_j)}{(3+s_j)(7+s_j)\cdots(2k-1+s_j)}, \qquad (1)$$

and $\eta(2k,s)$ is identically 0 if $k \equiv 1 \pmod 2$.

Period integral formulas

By classical result of Hecke,

$$\mu_d(E_0(g,s)) = \sum_{\text{dics}(q)=d} \int_{C(q)} E_0(g(t),s)dt = L\left(\frac{s+1}{2}, \chi_d\right)\zeta\left(\frac{s+1}{2}\right).$$

The period bound follows from the sub-convexity bound on $L(\frac{1}{2}+it, \chi_d)$, where $s = i2t$. [Heath-Brown 80]

$$L(s, \chi_d) \ll_\epsilon ((|t|+1)d)^{3/16+\epsilon}.$$

Period integral formulas

Similarly, period formulas proved by Waldspurger for holomorphic and by Svetlana Katok and Sarnak Maass cusp forms. By combining the above and smoothing the Intersection kernel we prove

$$\frac{I_{\theta_1,\theta_2}(\beta, C_d)}{I(\beta)I(C_d)} = \frac{3}{\pi^2}\int_{\theta_1}^{\theta_2} \sin\theta d\theta + O_\epsilon\left(d^{-\frac{25}{3584}+\epsilon}\right),$$

Proof for varying D_1, D_2

1)The intersection kernel is not L^2 integrable.
2) Smoothing the kernel (using majorant and minorant) and reducing the problem to small frequencies.

Selberg's pre-trace formula

$$K(g_1, g_2) = \frac{9}{\pi^4} \iint K(g_1, g_2)\, dg_1 dg_2$$

$$+ \sum_{\substack{e \geq 0 \\ 2|e}} \sum_{j=1}^{d_e} \sum_{\substack{m,n \in 2\mathbb{Z} \\ |m|,|n| \geq e}} h\left(k, m, n, \pi_j^e\right) \phi_{j,n}^e (g_1) \overline{\phi_{j,m}^e (g_2)}$$

$$+ \frac{1}{4\pi} \sum_{m,n \in 2\mathbb{Z}} \int_{-\infty}^{\infty} h(k, m, n, t)\, E_n\left(g_1, \frac{1}{2} + it\right) \overline{E_m\left(g_2, \frac{1}{2} + it\right)} dt,$$

where π_j^e is the irreducible unitary representation of $PSL_2(\mathbb{R})$ associated to ϕ_j^e.

generalized Selberg-Harish chandra transform

Let $M_\pi(m,n)(g) := \langle \pi(g) f_m, f_n \rangle$ be a matrix coefficient of π.

$$h\left(k, m, n, \pi_j^e\right) := \left\langle \phi_{j,n}^e, K * \phi_{j,m}^e \right\rangle = \int_{PSL_2(\mathbb{R})} k(u, I) M_\pi(m,n)(u^{-1}) du,$$

$$\left\langle \phi_{j,n}^e, K * \phi_{j,m}^e \right\rangle \ll_{N=N_1+N_2} (1+|m|)^{-N_1}(1+|n|)^{-N_2} \|k\|_{W^{N,1}}.$$

generalized Selberg-Harish chandra transform

We have

$$h(k, m, n, t) = \int_{PSL_2(\mathbb{R})} k_{m,n}(g) y^{\frac{1}{2}+it} e^{-im\theta} dg.$$

and

$$h(k, m, n, t) \ll_{N,C} (1 + |m|)^{-N_1} (1 + |n|)^{-N_2} (1 + |t|)^{-N_3} \|k\|_{W^{N,\infty}}$$

for any $N_1, N_2, N_3 \geq 0$, where $N = N_1 + N_2 + N_3$.

$$k_{m,n}(g) := \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} k(R_{\theta'} g R_\theta) e^{-in\theta' - im\theta} d\theta' d\theta. \qquad (2)$$

Note that

$$k_{m,n}(R_{\theta_1} g R_{\theta_2}) = e^{in\theta_1} k_{m,n}(g) e^{im\theta_2}. \qquad (3)$$

generalized Selberg-Harish chandra transform

We have

$$h\left(k,m,n,t\right)=\int_{PSL_2(\mathbb{R})}k_{m,n}(g)y^{\frac{1}{2}+it}e^{-im\theta}dg.$$

and

$$h\left(k,m,n,t\right)\ll_{N,C}\left(1+|m|\right)^{-N_1}(1+|n|)^{-N_2}(1+|t|)^{-N_3}\|k\|_{W^{N,\infty}}$$

for any $N_1, N_2, N_3 \geq 0$, where $N = N_1 + N_2 + N_3$.

$$k_{m,n}\left(g\right):=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}k\left(R_{\theta'}gR_\theta\right)e^{-in\theta'-im\theta}d\theta'd\theta. \qquad (2)$$

Note that

$$k_{m,n}\left(R_{\theta_1}gR_{\theta_2}\right)=e^{in\theta_1}k_{m,n}\left(g\right)e^{im\theta_2}. \qquad (3)$$

Thank you!