Counting problems can often be harder than those from the last few lectures…

For example…

Combinations with repetition

Repeated choice

SUCCESS
USSECCS
SCSCSEU
…

Permuting indistinguishable items

Permutations with repetition

Recall: r-permutations are ordered collections of r elements drawn from some set

If an r-permutation is drawn from a set of size n without replacement, then there are P(n,r) = n!/(n-r)! possible r-permutations

If we select the elements of a permutation with replacement, then we can use the product rule to count the number of possible r-permutations

How many strings of length r can be created using the 26 English letters?

Let our set S = {A, B, C, …, Z}, with |S| = 26

To count the number of r-length strings, note that:

- 26 ways to choose 1st letter
- 26 ways to choose 2nd letter (not 25)
- 26 ways to choose 3rd letter (not 24)
- …
- 26 ways to choose rth letter (not 26-r+1)

So, there are 26^r possible ways to choose an r-length string from the set S with replacement

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In general: There are n^r possible ways to permute a set of size n if repetition of elements is allowed

Many times, we want to examine combinations of objects in which repeated choices are allowed

Example: How many ways can four pieces of fruit be chosen from a bowl containing at least four apples, four oranges, and four pears? Assume that only the type of fruit chosen matters, not the individual piece.

This is TEDIOUS!!!

Solution #1: Explicit enumeration

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

So, there are 15 possible 4-combinations of a set containing 3 items if repetition is allowed

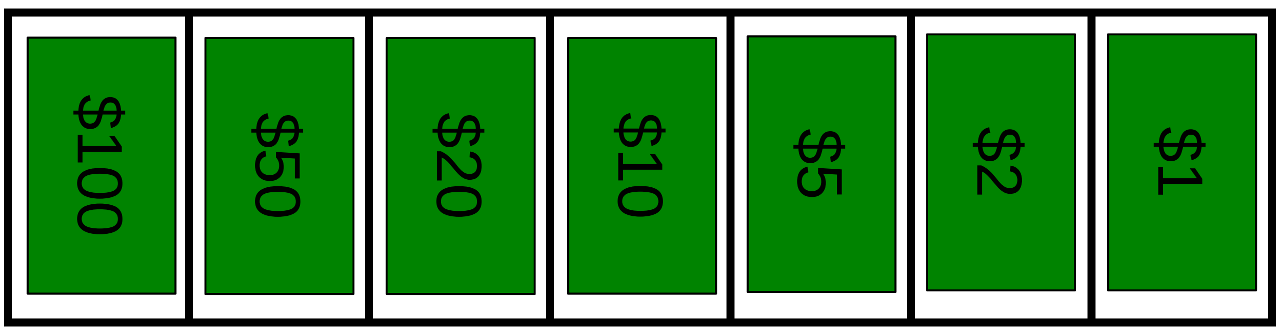
Let's find a nice closed-form expression for counting r-combinations with repetition

Example: Consider a cash box containing \$1 bills, \$2 bill, \$5 bills, \$10 bill, \$20 bills, \$50 bills, and \$100 bills. How many ways are there to choose 5 bills if order does not matter and bills within a single denomination are indistinguishable from one another? Assume that there are at least 5 bills of each denomination.

Observations:

- 7 denominations of bills
- The order that bills are drawn does not matter
- At least 5 bills of each denomination

Implication: We are counting 5-combinations with repetition from a set of 7 items.



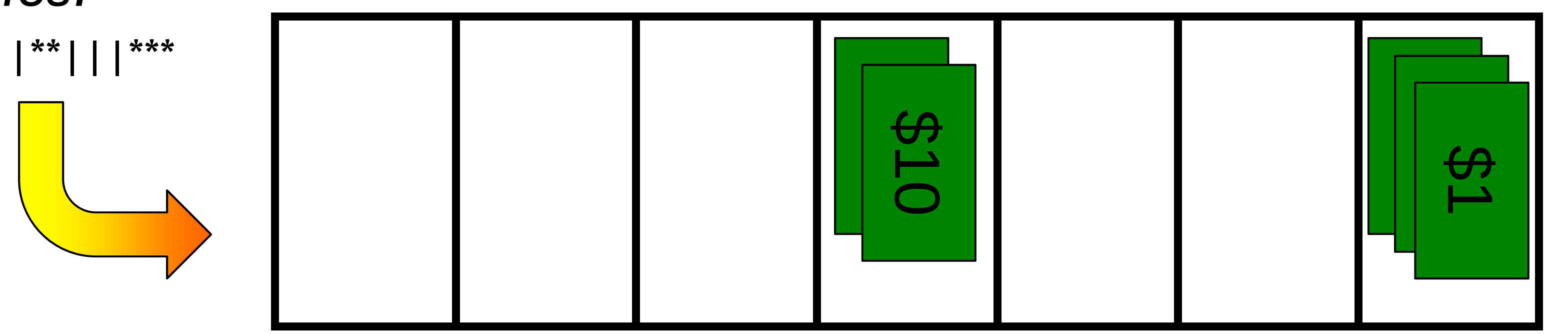
An interesting insight…

Note:

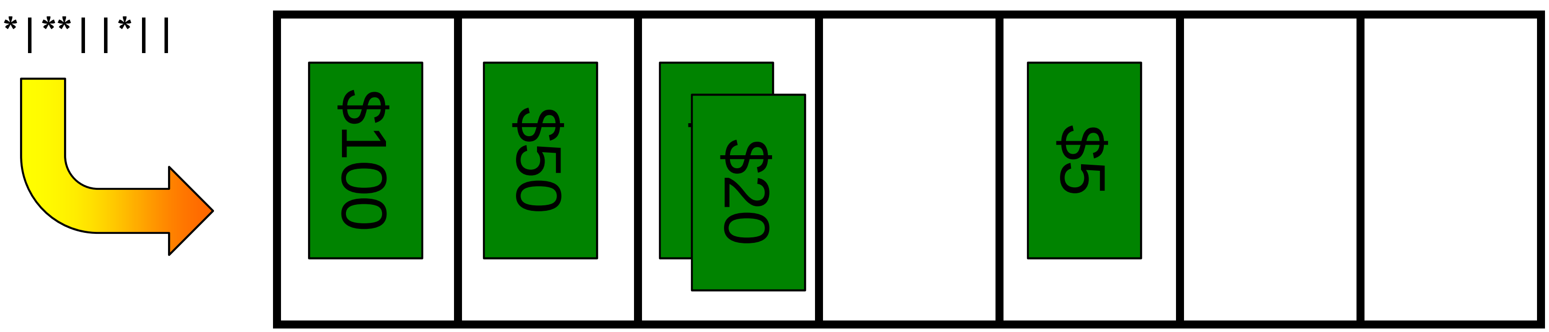
- The cash box has 7 compartments
- These compartments are separated by 6 dividers
- Choosing 5 bills is the same as arranging 5 placeholders (*) and 6 dividers (|)

Examples:

1. |||**|||***



2. *|*|**||*||



This leads us to a nice formula…

Observation: Arranging 5 stars and 6 bars is the same as choosing 5 "places" for the stars out of 11 total "places."

<table><tr><td>*</td><td>|</td><td>*</td><td>*</td><td>|</td><td>|</td><td>*</td><td>|</td><td>*</td><td>|</td><td>|</td></tr></table>

This can be done in C(11, 5) = 462 ways.

General Theorem: There are C(n+r-1, r) r-combinations from a set with n elements when repetition of elements is allowed.

Buying cookies!

Example: How many ways can we choose six cookies at a cookie shop that makes 4 types of cookie? Assume that only the type of cookies chosen matters (not the order in which they are chosen or the individual cookies within a given type).

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Solution #1:

- Need six "stars" since we are choosing six cookies
- Need 3 "bars" to separate the cookies by type
- So, C(9, 6) = 84 ways to choose places to put stars.

Solution #2:

- Since we choose six cookies, r = 6
- Four possible cookie types means n = 4
- So, C(6+4-1, 6) = C(9,6) = 84 ways to choose cookies!

Solving equations

Example: How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have if x_1, x_2, and x_3 are non-negative integers?

Solving equations

Example: How many solutions does the equation $x_1 + x_2 + x_3 = 11$ have if x_1, x_2, and x_3 are non-negative integers?

Observation: Solving this problem is the same as choosing 11 objects from a set of 3 objects such that x_1 objects of type one are chose, x_2 objects of type two are chosen, and x_3 objects of type three are chosen.

Solution:

- n = 3
- r = 11
- So, there are C(3+11-1,11) = C(13, 11) = 78 ways to solve this equation

How do we deal with indistinguishable items?

Example: How many strings can be formed by permuting the letters of the word MOM?

Observation: We can't simply count permutations of the letters in MOM. (Why not?)

Counting permutations leads to an overcount!

- Rewrite MOM as M_1OM_2
- Possible permutations are:
  - M_1OM_2
  - M_1M_2O
  - OM_1M_2
  - M_2OM_1
  - M_2M_1O
  - OM_2M_1

These are really the same!

How do we fix this?

Rather than permuting all letters as a group, arrange identical letters separately

Note: The string MOM contains two Ms and one O.

We can count the distinct strings formed by permuting MOM as follows:

- Set up 3 “slots” for letters
- Count the ways that the 2 Ms can be assigned to the these slots
- Count the ways that the O can be assigned to the remaining slots
- Use the product rule!

This tactic can be stated more generally

Theorem: The number of different permutations of n objects where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …, and n_k indistinguishable objects of type k is:

$$C(n, n_1)C(n - n_1, n_2) \cdots C(n_k, n_k) = \frac{n!}{n_1! n_2! \cdots n_k!}$$

Ways to place objects of type 1

Ways to place objects of type 2

There is always only one way to place objects of type k!

How many strings can be formed by permuting the letters in SUCCESS?

Note: SUCCESS contains

- S × 3
- U × 1
- C × 2
- E × 1

Ways to assign each letter group:

- S: C(7,3)
- U: C(4,1)
- C: C(3,2)
- E: C(1,1)

So, we can form C(7,3) × C(4,1) × C(3,2) × C(1,1) = 7!/(3!2!) = 420 distinct strings using letters from the word SUCCESS

Group Work!

Problem 1: How many ways can we choose 6 donuts from a donut shop that sells three types of donut?

Problem 2: How many distinct strings can be formed by permuting the letters of the word RADAR?

Many counting problems can be solved by “placing items in boxes”

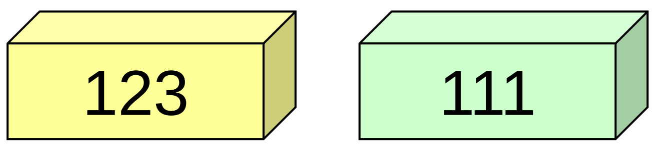
We can consider two types of objects:

1. Distinguishable objects (e.g., “Billy, Chrissy, and Dan”)

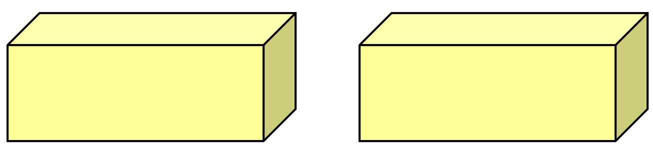
2. Indistinguishable objects (e.g., “three students”)

We can also consider two types of “boxes”:

1. Distinguishable boxes (e.g., “room 123 and room 111”)

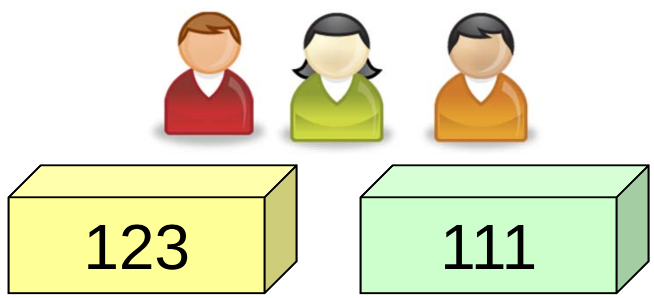


2. Indistinguishable boxes (e.g., “two homerooms”)



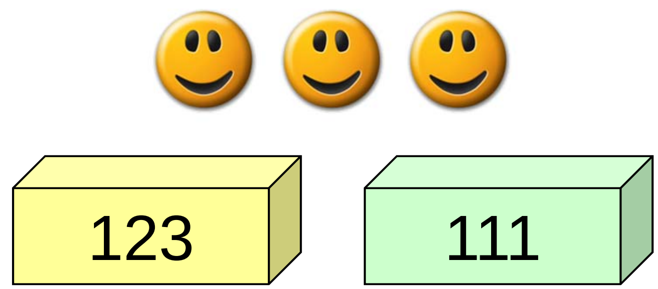
This leads to four classes of problems…

Distinguishable objects / distinguishable boxes



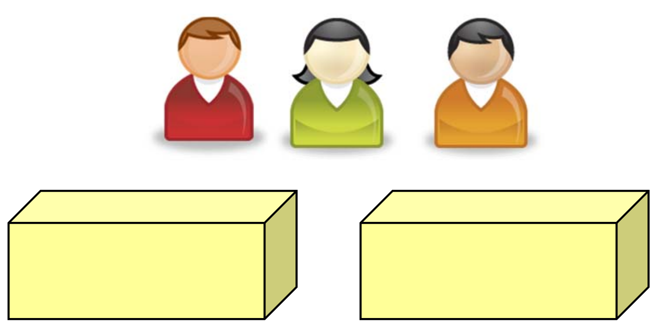
E.g., How many ways can Billy, Chrissy, and Dan be assigned to the homeroom 123 and homeroom 111?

Indistinguishable objects / distinguishable boxes



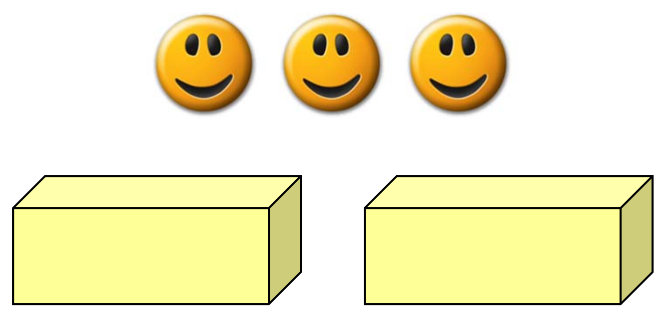
E.g., How many ways can three students be assigned to the homeroom 123 and homeroom 111?

Distinguishable objects / indistinguishable boxes



E.g., How many ways can Billy, Chrissy, and Dan be assigned to two different homerooms?

Indistinguishable objects / indistinguishable boxes



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Counting assignments of distinguishable items to distinguishable boxes

Example: How many ways are there to deal 5-card poker hands from a 52-card deck to each of four players?

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Example: How many ways are there to deal 5-card poker hands from a 52-card deck to each of four players?

Solution:

- Player 1: C(52,5) ways to deal
- Player 2: C(47,5) ways to deal
- Player 3: C(42,5) ways to deal
- Player 4: C(37,5) ways to deal

$$C(52,5)C(47,5)C(42,5)C(37,5)$$
$$= \frac{52!}{47!5!} \cdot \frac{47!}{42!5!} \cdot \frac{42!}{37!5!} \cdot \frac{37!}{32!5!}$$
$$= \frac{52!}{5!5!5!5!32!}$$

Theorem: The number of ways that n distinguishable items can be placed into k distinguishable boxes so that n_i objects are placed into box i ($1 \leq i \leq k$) is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

We can prove this using the product rule!

How can we place n indistinguishable items into k distinguishable boxes?

This turns out to be the same as counting the n-combinations for a set with k elements when repetition is allowed!

Recall: We solved the above problem by arranging placeholders (*) and dividers (|).

To place n indistinguishable items into k distinguishable bins:

1. Treat our indistinguishable items as *s
2. Use | to divide our distinguishable bins
3. Count the ways to arrange n placeholders and k-1 dividers

Result: There are $C(n + k - 1, n)$ ways to place n indistinguishable objects into k distinguishable boxes

Let's see how this works…

Example: How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins?

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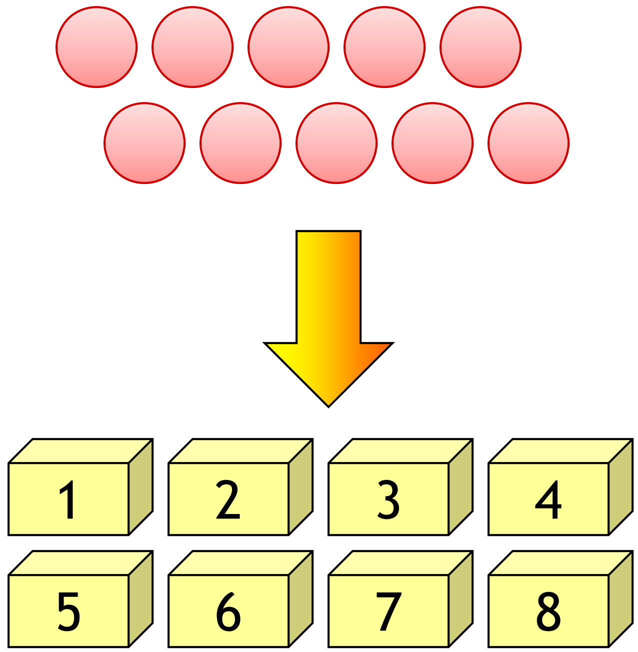

Observation:

1. Treat balls as *s
2. Use 8-1 = 7 dividers to separate bins
3. Pick 10 positions out of a total 17 to place balls (all remaining positions will be bin dividers)

Solution: We have C(10 + 8 – 1, 10) = C(17, 10) = 19,448 ways to arrange 10 indistinguishable balls into 8 distinguishable bins.

Sadly, counting the ways to place distinguishable items into indistinguishable boxes isn't so easy…

Example: How many ways can Anna, Billy, Caitlin, and Danny be placed into three indistinguishable homerooms?

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Example: How many ways can Anna, Billy, Caitlin, and Danny be placed into three indistinguishable homerooms?

Solution:

- Let's call our students A, B, C, and D
- Goal: Partition A, B, C, and D into at most 3 disjoint subsets
- One way to put everyone in the same homeroom
  - {A, B, C, D}
- Seven ways to put everyone in two homerooms
  - {{A, B, C}, {D}}, {{A, B, D}, {C}}, {{A, C, D}, {B}}, {{B, C, D}, {A}}
  - {{A, B}, {C, D}}, {{A, C}, {B, D}}, {{A, D}, {B, C}}
- Six ways to put everyone into three homerooms
  - {{A, B}, {C}, {D}}, {{A, C}, {B}, {D}}, {{A, D}, {B}, {C}}
  - {{B, C}, {A}, {D}}, {{B, D}, {A}, {C}}, {{C, D}, {A}, {B}}
- Total: 14 ways to assign Anna, Billy, Caitlin, and Danny to three indistinguishable homerooms

Is there some simple closed form that we can use to solve this type of problem?

No, but there is a complicated one ☺

S(n,j) is a Stirling number of the second kind that tells us the number of ways that a set of n items can be partitioned into j non-empty subsets.

S(n, j) is defined as follows: $S(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i C(j,i)(j-i)^n$

Result: The number of ways to distribute n distinguishable objects into k indistinguishable boxes is:

$$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k}\frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i C(j,i)(j-i)^n$$

What about distributing indistinguishable objects into indistinguishable boxes?

Example: How many ways can six copies of the same book be packed in at most four boxes, if each box can hold up to six books?

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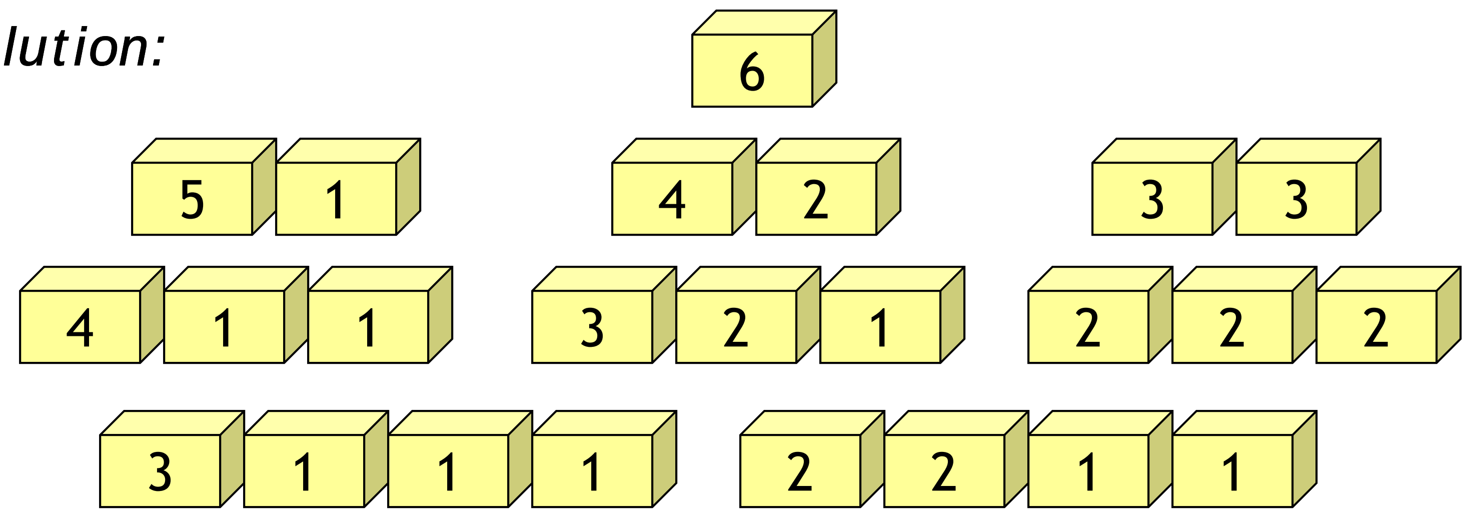
Solution:



Total: There are 9 ways to pack 6 identical books into at most 4 indistinguishable boxes.

That was ugly…

Is there a better way to do this?

Unfortunately, no. 

Here's why: Placing n indistinguishable objects into k indistinguishable boxes is the same as writing n as the sum of at most k positive integers arranged in non-increasing order.

- i.e., $n = a_1 + a_2 + \ldots + a_j$, where $a_1 \geq a_2 \geq \ldots \geq a_j$ and $j \leq k$
- We say that $a_1, a_2, \ldots, a_j$ is a partition of n into j integers

There is no simple closed formula for counting the partitions of an integer, thus there is no solution for placing n indistinguishable items into k indistinguishable boxes.

Final Thoughts

- Many counting problems require us to generalize the simple permutation and combination formulas from last time
- Other problems can be cast as counting the ways to arrange (in)distinguishable objects into (in)distinguishable boxes
- Next time:
  - Probability theory