

Stochastic Asset Flow Equations: Interdependence of Trend and Volatility

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Abstract

The complex interaction between the volatility and the price trend is examined. We model stochastic asset prices using the asset flow model with randomness arising directly from supply and demand. We show that the volatility is smallest at the extrema of the expected logarithm of the price. Linearizing the stochastic differential equation (SDE) about equilibrium, we obtain an exact relation for the autocovariance function, and relate it to the (3 by 3) Jacobian of the linearized SDE. In particular, we find the conditions under which one has a pair of complex conjugate eigenvalues of the Jacobian resulting in oscillations. The frequency of the oscillations depends only on the imaginary part of the complex pair, while the decay rate depends only on the real eigenvalue and the real parts of the complex pair. For the deterministic system, oscillations typically decay rapidly. However, randomness induces oscillations to continue indefinitely with a frequency that depends on the parameters of the deterministic system. The computations and analytical results presented here demonstrate that volatility increases when traders place greater emphasis on trend, confirming a generally held belief among practitioners.

Keywords: autocovariance, random supply, demand, stochastic differential equations, behavioral effects, price oscillations

1. Introduction

In this paper we build on two sets of ideas in order to construct a model of asset price dynamics with randomness incorporated in a natural way via supply and demand. One of these is the asset flow model of Caginalp and Balenovich, 1999 [3] which utilizes the approach that was introduced by Caginalp and Ermentrout, 1990 [7]. The key components in this deterministic model are the following. First, in the decision making process, traders are influenced not only by the value of the asset – which classical finance asserts is ultimately the only consideration – but by the price trend as well (see discussion by Gippel, 2013 [15]). Other motivations are also easily incorporated. Second, the total assets in the system are finite – so that there is no infinite arbitrage as one assumes in classical finance. It is less obvious that this second factor is crucial, but the assumption of infinite arbitrage, a central pillar of classical finance, essentially eliminates the effect of trend-based traders. In other words, suppose there is a huge amount of capital controlled by very knowledgeable investors who base their decisions solely on the value of the asset. Then even if there are traders focussed on trend or other features of prices, classical finance theories assert that their trading will be exploited quickly by the savvy investors, and will lead to small random perturbations in price about the expected growth of the asset.

The other set of ideas involves an examination of the source of randomness in markets. The classical approach centers around the stochastic differential equation

$$P^{-1}dP = \mu dt + \sigma dW \quad (1)$$

for price P in terms of the expected return, μ , and the variance, σ^2 . Here $W(t)$ is Brownian motion, so that (1) asserts that in a small time interval, Δt , the change, ΔP , in price will be a deterministic term, $\mu\Delta t$, plus a random term, $\sigma\Delta W$. By definition of Brownian motion, ΔW will be normal with mean zero and variance Δt . The classical assumptions thus include (i) the expected return and variance are independent, (ii) the randomness in price is normally distributed. The latter assumption is based loosely on empirical observation and mathematical convenience. This approach has its roots in the thesis of Bachelier, 1900 [1] which, after about a half-century, evolved into the cornerstone of much of mathematical finance, particularly options pricing and risk (see e.g., Hull, 2009 [17]).

An immediate implication of the classical approach is that observing the variance of price change gives us no information about the expected price change. Fundamentally, it invokes the question of how the randomness in price arises. We have argued that if one knows how the supply and demand will change, then one knows with certainty how price will change. This is almost a tautology since the price change is a function of supply and demand. In a practical sense, the market makers and short term traders for each stock have extensive experience in observing similar changes in supply and demand, and can be expected to optimize for their own trading.

Thus, the key aspect of our approach is to assume randomness in supply and demand in order to explore fluctuations in price as a consequence. As discussed below (see Section 2.4), the general equation, in terms of an arbitrary supply, S , demand, D , that we have derived (Caginalp and Caginalp 2019 [4], [5]) is

$$P^{-1}dP = f(D/S)dt + \sigma v(D/S)dW. \quad (2)$$

The function f is a strictly increasing function of its argument that must satisfy Condition 1 of [4] (p. 809). With our choice of $f(x) = x - x^{-1}$ as a prototype, we obtain $v(x) = xf'(x) = x + x^{-1}$ (see [4]), which implies

$$P^{-1}dP = f(D/S)dt + \sigma \frac{D}{S} f'(D/S)dW. \quad (3)$$

In this equation, the supply and demand are (possibly stochastic) functions of time that need to be specified. Equation (3) has some important consequences, one of which is that it provides an endogenous explanation for *fat tails*, whereby the density decays as power law rather than exponentially, as (1) implies. Another is the prediction that marginal volatility (i.e., limit of small time interval) peaks as the expected relative price change is largest, and is at a minimum when the expected log-price is at an extremum. The latter conclusion is a strong departure from the classical perspective of (1) in which price change and volatility are independent.

In this paper, instead of making explicit assumptions about the dynamics of demand and supply, we integrate this approach with the asset flow equations described above, which provide a natural description of the dynamics of D and S for a group of traders with prescribed trading sentiments.

The results of this paper are as follows. In Sections 2 and 3 we define three quantities that express the relative volatility in a time interval Δt and the limiting, or marginal, volatility as Δt approaches zero. We derive relationships between the volatility and the autocorrelation function. Also, we evaluate and compare the definitions in terms of the quantities in the price equation. Finally, we examine the volatility as a function of the time interval, Δt . In an example of autocovariance that is a damped sinusoidal, we find that the maximum of volatility in terms of Δt varies only slightly with time. In the stationary case (i.e., where $VarP(t)$ is constant), we obtain additional relationships between the correlation between $P(t)$ and $P(t + \Delta t)$ and the variance.

In Section 4, we examine the extrema of price and volatility and the relation between them in terms of the model (2).

Section 5 features an important new development consisting of integrating the stochastic ideas with the deterministic asset flow equations. This allows us to examine the interaction between randomness and momentum trading. We show that increased focus on price trend interacts with the randomness so that variance increases as the coefficient of momentum trading increases. An explanation of this is that when randomness causes a small increase in price, it spurs further buying due to momentum trading. This is then a self-feeding process that pushes up prices until either negative random terms or deviation from equilibrium price (which depends on valuation and liquidity price) terminates the small uptrend. Analogously, shortening the time scale of the trend motivation (i.e., making c^{-1} smaller) leads to higher volatility.

In Section 6 and 7 we describe the equilibrium state of the system and explore fluctuations around the equilibrium using standard theory of SDEs [28]. We examine the autocovariance function of the linearized system analytically and show that increasing momentum trading strength promotes autocorrelation of fluctuations around the equilibrium and increases price variance.

After examining deterministic examples of price peaks in Section 8, we perform numerical computations on the SDEs in Section 9, and show that when momentum traders are involved, oscillatory behavior in the system persists even when the deterministic equivalent has a stable equilibrium. We also show that the marginal volatility is highest at the points of greatest relative change in the price, and lowest near the extrema of price. The numerical computations also provide a great deal of insight into the interaction between price and volatility, the possibility of using volatility as a timing tool for market peaks, and the assessment of risk.

At a more general level, our methodology can be utilized in a broad spectrum of problems in which there is an interplay between two stochastic factors. Buying due to the trend-based trading motivation is an example of a self-feeding process while selling due to overvaluation acts as a control mechanism. The two processes are connected through conservation laws, which in our case is the conservation of cash and shares.

Future research may be directed toward market timing issues that could improve, for example, options and futures pricing in finance, and analogous issues in other disciplines. In another direction, which can be regarded as a control problem, one can explore the impact of various trading features [29] such as short selling, margin buying, etc., on the size of bubbles, or the magnitude of oscillations. For example, the impact on the US pandemic stimulus packages can be investigated with these methods.

2. Variance, Volatility and Autocovariance

2.1. Volatility and marginal volatility

In practice, volatility is the variation of trading price over time. It can be measured directly as the variance of the difference $P(t + \Delta t) - P(t)$, or as a relative quantity by dividing by $P(t)$ or $\mathbb{E}P(t)$. Alternatively, one can define volatility as the variance of a logarithmic return $\log(P(t + \Delta t)/P(t))$. A general discussion of volatility can be found in [17]. See also [20, 31, 36, 34, 14, 9] and references therein. Here we analyze and compare three possible definitions:

Definition. The following expressions for *volatility* are defined for the interval $(t, t + \Delta t)$:

$$\hat{V}(t, \Delta t) := \text{Var} \left[\frac{P(t + \Delta t) - P(t)}{\mathbb{E}P(t)} \right], \quad (4)$$

$$V(t, \Delta t) := \text{Var} \left[\frac{P(t + \Delta t) - P(t)}{P(t)} \right], \quad (5)$$

$$\tilde{V}(t, \Delta t) := \text{Var} [\log P(t + \Delta t) - \log P(t)]. \quad (6)$$

The difference between the last two definitions will be made more precise in Section 2.2 below, using a refinement of the approximation

$$\log P(t + \Delta t) - \log P(t) = \log [P(t + \Delta t)/P(t)] \cong P(t + \Delta t)/P(t) - 1.$$

The quantity $\hat{V}(t, \Delta t)$ is the simplest of these measures, and has the convenient feature that the normalization factor $\mathbb{E}P(t)$ is not random and can be factored. Also, one can obtain a direct expression for \hat{V} in terms of the quantity in terms of the asset flow equations. Using these definitions one can write the marginal, or instantaneous, volatilities below.

Definition. The *marginal, or limiting, volatilities*, $\hat{V}(t)$, $\mathbb{V}(t)$ and $\tilde{V}(t)$ are defined as follows if the limits exist:

$$\hat{V}(t) := \lim_{\Delta t \rightarrow 0} (\Delta t)^{-1} \hat{V}(t, \Delta t), \quad (7)$$

$$\mathbb{V}(t) := \lim_{\Delta t \rightarrow 0} (\Delta t)^{-1} V(t, \Delta t), \quad (8)$$

$$\tilde{V}(t) := \lim_{\Delta t \rightarrow 0} (\Delta t)^{-1} \tilde{V}(t, \Delta t). \quad (9)$$

We have thus established three definitions of volatility and marginal volatility that will be useful in the study of stochastic differential equations in subsequent sections, and related to autocovariance below.

2.2. Relationship between volatility and autocovariance

For arbitrary random variables, R_1 and R_2 , that are not necessarily independent, one has the general relationship :

$$\text{Var} [R_1 - R_2] = \text{Var}R_1 + \text{Var}R_2 - 2\text{Cov}(R_1, R_2).$$

Defining the autocovariance of price as

$$R(t, s) := Cov(P(t), P(s)),$$

we can then relate the volatility to the covariance as

$$Var[P(t + \Delta t) - P(t)] = VarP(t + \Delta t) + VarP(t) - 2R(t + \Delta t, t). \quad (10)$$

Dividing by $(\mathbb{E}P(t))^2$ and utilizing the definition of $\hat{V}(t, \Delta t)$ one has then

$$\hat{V}(t, \Delta t) = (\mathbb{E}P(t))^{-2} [VarP(t + \Delta t) + VarP(t) - 2R(t + \Delta t, t)]. \quad (11)$$

Upon dividing by Δt , one then has

$$\hat{V}(t) = \lim_{\Delta t \rightarrow 0} \frac{VarP(t + \Delta t) + VarP(t) - 2R(t + \Delta t, t)}{\Delta t (\mathbb{E}P(t))^2} \quad (12)$$

if the limit exists.

For the second formulation of volatility we use the definition of $V(t, \Delta t)$, i.e., (8) and the approximate relation for the variance of a ratio of two random variables [22], which requires $VarR_i$ to be small compared to $(\mathbb{E}R_i)^2$,

$$Var \left[\frac{R_1}{R_2} \right] \cong \left(\frac{\mathbb{E}R_1}{\mathbb{E}R_2} \right)^2 \left[\frac{VarR_1}{(\mathbb{E}R_1)^2} + \frac{VarR_2}{(\mathbb{E}R_2)^2} - 2 \frac{Cov(R_1, R_2)}{\mathbb{E}R_1 \mathbb{E}R_2} \right]$$

to obtain

$$V(t, \Delta t) \cong \left(\frac{\mathbb{E}P(t + \Delta t)}{\mathbb{E}P(t)} \right)^2 \left[\frac{VarP(t + \Delta t)}{(\mathbb{E}P(t + \Delta t))^2} + \frac{VarP(t)}{(\mathbb{E}P(t))^2} - 2 \frac{R(t + \Delta t, t)}{\mathbb{E}P(t + \Delta t)\mathbb{E}P(t)} \right] \quad (13)$$

which can be compared with the exact limiting relation (11).

For the third definition, the approximation [22], i.e.,

$$Var[\log(R)] \cong (\mathbb{E}R)^{-2} VarR$$

implies

$$\tilde{V}(t, \Delta t) = Var \left[\log \frac{P(t + \Delta t)}{P(t)} \right] \cong \left(\mathbb{E} \left[\frac{P(t + \Delta t)}{P(t)} \right] \right)^{-2} Var \left[\frac{P(t + \Delta t)}{P(t)} \right]$$

With the help of another approximation [22],

$$\mathbb{E} \left[\frac{R_1}{R_2} \right] \cong \frac{\mathbb{E}R_1}{\mathbb{E}R_2} \left(1 - \frac{Cov(R_1, R_2)}{\mathbb{E}R_1 \mathbb{E}R_2} + \frac{VarR_2}{(\mathbb{E}R_2)^2} \right),$$

we can use (13) to obtain

$$\tilde{V}(t, \Delta t) \cong \frac{\frac{VarP(t + \Delta t)}{(\mathbb{E}P(t + \Delta t))^2} + \frac{VarP(t)}{(\mathbb{E}P(t))^2} - 2 \frac{R(t + \Delta t, t)}{\mathbb{E}P(t + \Delta t)\mathbb{E}P(t)}}{\left(1 - \frac{R(t + \Delta t, t)}{\mathbb{E}P(t + \Delta t)\mathbb{E}P(t)} + \frac{VarP(t)}{(\mathbb{E}P(t))^2} \right)^2},$$

which, in the first order, becomes

$$\tilde{V}(t, \Delta t) \cong \frac{VarP(t + \Delta t)}{(\mathbb{E}P(t + \Delta t))^2} + \frac{VarP(t)}{(\mathbb{E}P(t))^2} - 2 \frac{R(t + \Delta t, t)}{\mathbb{E}P(t + \Delta t)\mathbb{E}P(t)}. \quad (14)$$

This last relation should be compared with the exact relation (11) or the approximate relation (13). In each case the volatility is a linear combination of the price variances $VarP(t)$, $VarP(t + \Delta t)$ and the autocovariance $R(t + \Delta t, t)$, with the former two increasing the volatility and the later decreasing it. The coefficients are strictly dependent on the price expectations, and are identical if the average price is constant in time. When the average price is increasing, we have $\tilde{V}(t, \Delta t) < V(t, \Delta t)$.

2.3. The stationary case

Suppose we assume that $P(t)$ is a stationary process with $\mathbb{E}P(t) = \mu = \text{const}$ and $\text{Var}P(t) = \sigma^2 = \text{const}$. Using the identity (10) and the definition of correlation, $\rho(R_1, R_2)$, as

$$\rho(R_1, R_2) = \frac{\text{Cov}(R_1, R_2)}{(\text{Var}R_1)^{1/2} (\text{Var}R_2)^{1/2}},$$

one obtains,

$$\text{Var}[P(t + \Delta t) - P(t)] = 2\sigma^2 [1 - \rho(P(t), P(t + \Delta t))].$$

In other words, as the correlation between $P(t)$ and $P(t + \Delta t)$ increases, the variance of the difference decreases proportionately.

In particular, in the identity above, we can divide by μ^2 and obtain

$$\text{Var}\left[\frac{P(t + \Delta t) - P(t)}{\mathbb{E}P(t)}\right] = \frac{2\sigma^2}{\mu^2} [1 - \rho(P(t), P(t + \Delta t))].$$

Since $P(t)$ is assumed stationary, the covariance and volatility are independent of t , i.e., $\rho(P(t), P(t + \Delta t)) = \rho_1(\Delta t)$, $\hat{V}(t, \Delta t) = \hat{V}_1(\Delta t)$ and we can write

$$\hat{V}_1(\Delta t) = \frac{2\sigma^2}{\mu^2} [1 - \rho_1(\Delta t)]. \quad (15)$$

Upon taking the limit of the ratio of $\hat{V}_1(\Delta t)/\Delta t$, we can write in the stationary case,

$$\hat{V}(t) = \frac{2\sigma^2}{\mu^2} \lim_{\Delta t \rightarrow 0} \frac{1 - \rho_1(\Delta t)}{\Delta t}.$$

Hence, the marginal volatility is proportional to the negative derivative of autocorrelation with respect to Δt .

With the same assumptions, the second and third definitions of volatility are also independent of t , i.e., $V(t, \Delta t) = V_1(\Delta t)$ and $\tilde{V}(t, \Delta t) = \tilde{V}_1(\Delta t)$, and since $\mathbb{E}P(t + \Delta t) = \mathbb{E}P(t) = \mu$ and $\text{Var}P(t + \Delta t) = \text{Var}P(t) = \sigma^2$, the relations (13) and (14) reduce to

$$V_1(\Delta t) = \tilde{V}_1(\Delta t) = \hat{V}_1(\Delta t) = \frac{2\sigma^2}{\mu^2} [1 - \rho_1(\Delta t)].$$

In summary, for stationary processes the three definitions of volatility are identical within the first order of approximation.

2.4. Comparison of volatilities derived from price change equation

Consider now a model of the form (2) where the ratio D/S is given by some (possibly stochastic) function of time $x(t)$, let $f(t) = f(x(t))$ and $v(t) = v(x(t))$, and for the sake of generality assume that σ is a function of time.

$$P^{-1}dP = f(t)dt + \sigma(t)v(t)dW.$$

By Ito's formula, the following version is equivalent

$$d \log P = \left(f(t) - \frac{1}{2}\sigma(t)^2 v(t)^2 \right) dt + \sigma v(t) dW.$$

The integral formulas for the two versions are

$$P(t + \Delta t) - P(t) = \int_t^{t+\Delta t} P f d\tau + \int_t^{t+\Delta t} \sigma P v dW(\tau) \quad (16)$$

$$\log P(t + \Delta t) - \log P(t) = \int_t^{t+\Delta t} \left(f - \frac{1}{2}\sigma^2 v^2 \right) d\tau + \int_t^{t+\Delta t} \sigma v dW(\tau) \quad (17)$$

Note that the relation $\mathbb{E} \left[\int g(\tau) dW(\tau) \right] = 0$ from Ito calculus implies, with $P(0)$ deterministic,

$$\mathbb{E}P(t) = P(0) + \mathbb{E} \left[\int_0^t P f d\tau \right] = P(0) + \int_0^t \mathbb{E} [P(\tau) f(\tau)] d\tau \quad (18)$$

$$\mathbb{E}(\log P(t)) = \log P(0) + \mathbb{E} \left[\int_0^t \left(f - \frac{1}{2} \sigma^2 v^2 \right) d\tau \right] = \log P(0) + \int_0^t \mathbb{E} \left[f - \frac{1}{2} \sigma^2 v^2 \right] d\tau \quad (19)$$

and hence $\mathbb{E} \log P(t) = \log P(0) \exp \left(\int_0^t \mathbb{E} \left[f - \frac{1}{2} \sigma^2 v^2 \right] d\tau \right)$ and, if $f(t)$ is deterministic, $\mathbb{E}P(t) = P(0) \exp \left(\int_0^t f d\tau \right)$.

We now use the different definitions of volatility, the integral equations above, and the relation $\mathbb{E} \left[\left(\int g(\tau) dW(\tau) \right)^2 \right] = \int \mathbb{E} [g(\tau)^2] d\tau$ from Ito calculus, to obtain

$$\begin{aligned} [\mathbb{E}P(t)]^2 \hat{V}(t, \Delta t) &= \text{Var} \left[\int_t^{t+\Delta t} P f d\tau + \int_t^{t+\Delta t} \sigma P v dW(\tau) \right] \\ &= \text{Var} \left[\int_t^{t+\Delta t} P f d\tau \right] + 2 \mathbb{E} \left[\int_t^{t+\Delta t} P f d\tau \int_t^{t+\Delta t} \sigma P v dW(\tau) \right] \\ &\quad + \int_t^{t+\Delta t} \mathbb{E} [\sigma^2 P^2 v^2] d\tau. \end{aligned}$$

Similarly,

$$\begin{aligned} V(t, \Delta t) &= \text{Var} \left[P(t)^{-1} \int_t^{t+\Delta t} P f d\tau + P(t)^{-1} \int_t^{t+\Delta t} \sigma P v dW(\tau) \right] \\ &= \text{Var} \left[P(t)^{-1} \int_t^{t+\Delta t} P f d\tau \right] + 2 \mathbb{E} \left[P(t)^{-2} \int_t^{t+\Delta t} P f d\tau \int_t^{t+\Delta t} \sigma P v dW(\tau) \right] \\ &\quad + \int_t^{t+\Delta t} \mathbb{E} [P(t)^{-2} \sigma^2 P^2 v^2] d\tau. \end{aligned}$$

Likewise,

$$\begin{aligned} \tilde{V}(t, \Delta t) &= \text{Var} \left[\int_t^{t+\Delta t} \left(f - \frac{\sigma^2 v^2}{2} \right) d\tau + \int_t^{t+\Delta t} \sigma v dW(\tau) \right] \\ &= \text{Var} \left[\int_t^{t+\Delta t} \left(f - \frac{\sigma^2 v^2}{2} \right) d\tau \right] + 2 \mathbb{E} \left[\int_t^{t+\Delta t} \left(f - \frac{\sigma^2 v^2}{2} \right) d\tau \int_t^{t+\Delta t} \sigma v dW(\tau) \right] \\ &\quad + \int_t^{t+\Delta t} \mathbb{E} [\sigma^2 v^2] d\tau. \end{aligned}$$

Assuming boundedness, in each of the expressions above the first term is $O[(\Delta t)^2]$, the second $O[(\Delta t)^{3/2}]$, and the last $O[\Delta t]$. Upon dividing by Δt and taking the limit as $\Delta t \rightarrow 0$, we obtain marginal volatilities

$$\hat{V}(t) = \frac{\mathbb{E} [\sigma(t)^2 P(t)^2 v(t)^2]}{[\mathbb{E}P(t)]^2}, \quad (20)$$

$$V(t) = \mathbb{E} [\sigma(t)^2 v^2(t)], \quad (21)$$

$$\tilde{V}(t) = \mathbb{E} [\sigma(t)^2 v^2(t)]. \quad (22)$$

It follows that the second and third definitions of volatility have the same corresponding marginal volatilities which differ from the marginal volatility of the first definition.

3. Extrema in volatility as a function of Δt

3.1. Volatility in the limit of small σ

The volatility per unit time interval can be easily expressed by dividing the corresponding relation (4)–(6) by Δt . From a trading perspective, it is important to understand how the quantities $(\Delta t)^{-1}\hat{V}(t, \Delta t)$, $(\Delta t)^{-1}V(t, \Delta t)$, or $(\Delta t)^{-1}\tilde{V}(t, \Delta t)$ depend on the interval Δt . In particular, it is of interest to establish the value of Δt for which those quantities attain extrema, as traders are typically interested in the variance in relative price change within a particular time period.

In Section 2.4 we derived three expressions for volatility, and each of these is a sum of three terms. The first of these expressions can be expressed as

$$\begin{aligned} [\mathbb{E}P(t)]^2 \hat{V}(t, \Delta t) &= \text{Var} \left[\int_t^{t+\Delta t} P f d\tau + \int_t^{t+\Delta t} \sigma P v dW(\tau) \right] \\ &= \text{Var} \left[\int_t^{t+\Delta t} P f d\tau \right] + 2\mathbb{E} \left[\int_t^{t+\Delta t} P f d\tau \int_t^{t+\Delta t} \sigma P v dW(\tau) \right] \\ &\quad + \int_t^{t+\Delta t} \mathbb{E} [\sigma^2 P^2 v^2] d\tau, \\ [\mathbb{E}P(t)]^2 \hat{V}(t, \Delta t) &= I_1 + I_2 + I_3. \end{aligned}$$

The terms have the following dependence on σ and Δt : $I_1 \sim (\Delta t)^2$, $I_2 \sim (\Delta t)^{3/2} \sigma$ and $I_3 \sim (\Delta t) \sigma^2$.

We assume that $\sigma^2 \ll \Delta t$ so that $I_1 \gg I_3$, and also $I_1 \sim (\Delta t)^2 \gg (\Delta t)^{3/2} \sigma \sim I_2$.

With the above constraint in mind, in view of the results in Section 2.4, the volatility expressions are given by

$$\frac{\hat{V}(t, \Delta t)}{\Delta t} \cong \frac{1}{\Delta t [\mathbb{E}P(t)]^2} \text{Var} \left[\int_t^{t+\Delta t} P f d\tau \right]. \quad (23)$$

$$\frac{V(t, \Delta t)}{\Delta t} \cong \frac{1}{\Delta t} \text{Var} \left[P(t)^{-1} \int_t^{t+\Delta t} P f d\tau \right]. \quad (24)$$

$$\frac{\tilde{V}(t, \Delta t)}{\Delta t} \cong \frac{1}{\Delta t} \text{Var} \left[\int_t^{t+\Delta t} f d\tau \right]. \quad (25)$$

To find extrema, we differentiate these expressions. For future use note that for a random function $g(t, \omega)$, where we will omit the ω for brevity below, we have

$$\begin{aligned} \partial_{\Delta t} \text{Var} \left[\int_t^{t+\Delta t} g d\tau \right] &= \partial_{\Delta t} \left\{ \mathbb{E} \left(\int_t^{t+\Delta t} g d\tau \right)^2 - \left(\mathbb{E} \int_t^{t+\Delta t} g d\tau \right)^2 \right\} \\ &= 2\mathbb{E} \left[\left(\int_t^{t+\Delta t} g d\tau \right) g(t + \Delta t) \right] - 2 \left(\mathbb{E} \int_t^{t+\Delta t} g d\tau \right) \mathbb{E} g(t + \Delta t) \\ &= 2\text{Cov} \left[\int_t^{t+\Delta t} g d\tau, g(t + \Delta t) \right]. \end{aligned}$$

Here we have used the fundamental theorem of calculus to write $\partial_{\Delta t} \int_t^{t+\Delta t} g(\tau) d\tau = g(t + \Delta t)$. Therefore, using the bilinearity of covariance, we can write

$$\begin{aligned} \partial_{\Delta t} \left(\frac{1}{\Delta t} \text{Var} \left[\int_t^{t+\Delta t} g d\tau \right] \right) &= -\frac{1}{(\Delta t)^2} \int_t^{t+\Delta t} d\tau \int_t^{t+\Delta t} d\tau' \text{Cov} [g(\tau), g(\tau')] \\ &\quad + \frac{2}{\Delta t} \int_t^{t+\Delta t} d\tau \text{Cov} [g(t + \Delta t), g(\tau)] \\ &= -I_1 + I_2. \end{aligned}$$

For a stationary process $g(t)$, using the abbreviation $C(\tau - \tau') = \text{Cov}[g(\tau), g(\tau')]$, the integrals can be expressed as

$$\begin{aligned} I_1 &= \frac{1}{(\Delta t)^2} \int_t^{t+\Delta t} d\tau \int_t^{t+\Delta t} d\tau' C(\tau - \tau'), \\ I_2 &= \frac{2}{\Delta t} \int_t^{t+\Delta t} d\tau C(t + \Delta t - \tau). \end{aligned} \quad (26)$$

For I_1 , the region of integration is $R := [0, \Delta t] \times [0, \Delta t]$, while for I_2 it is (by symmetry) any side of R . We split R into four triangles via the diagonals. Let \tilde{I}_1 be the left triangle which has vertices $(0, 0)$, $(\Delta t/2, \Delta t/2)$, $(0, \Delta t)$, and hence, by symmetry, $I_1 = 4\tilde{I}_1$.

We transform the integration variables for I_1 using the new coordinates (v, u) by

$$\begin{aligned} v &:= \frac{s' + s}{\sqrt{2}}, \quad u := \frac{s' - s}{\sqrt{2}}, \\ s &= \frac{v - u}{\sqrt{2}}, \quad s' = \frac{v + u}{\sqrt{2}} \end{aligned}$$

which has Jacobian 1. In terms of the (v, u) coordinates the vertices of the left triangle are now $(0, 0)$, $(0, 2^{-1/2}\Delta t)$, $(2^{-1/2}\Delta t, 2^{-1/2}\Delta t)$.

The region of integration for I_1 in (s, s') is $\{(s, s') : 0 \leq s \leq \Delta t/2, s \leq s' \leq \Delta t - s\}$ while for the (v, u) coordinates it is

$$\{(v, u) : 0 \leq v \leq 2^{-1/2}\Delta t, 0 \leq u \leq v\}.$$

We evaluate the integrals in two examples and then draw some conclusions in general.

The two integrals can now be written as, with $I_1 = 4\tilde{I}_1$

$$\begin{aligned} I_2(\Delta t) &= \frac{1}{\Delta t} \int_0^{\Delta t} C(\Delta t - s) ds \\ \tilde{I}_1(\Delta t) &= \frac{1}{(\Delta t)^2} \int_0^{\Delta t/\sqrt{2}} \left(\int_0^v C(\sqrt{2}u) du \right) dv. \end{aligned}$$

3.2. Example

These relations can be examined in the context of an exponentially damped sinusoidal autocovariance. Numerical computations to be discussed in Section 9 below show that the asset flow SDEs do in fact exhibit such an autocovariance function. We let $C(u) = e^{bu} \cos(au)$, for $b \in (-\infty, 0)$ and $a \in (-\infty, \infty)$, and integrate in the same way, obtaining

$$\begin{aligned} I_2(\Delta t) &= \frac{1}{\Delta t} \int_0^{\Delta t} e^{b(\Delta t - s)} \cos(a(\Delta t - s)) ds \\ &= \frac{(\Delta t)^{-1}}{a^2 + b^2} \left[e^{b\Delta t} (b \cos a\Delta t + a \sin a\Delta t) - b \right]. \end{aligned}$$

Next, we recall that the integral over the left triangle in $R = [0, \Delta t] \times [0, \Delta t]$ for I_1 is \tilde{I}_1 and $I_1 = 4\tilde{I}_1$ by symmetry, and

$$\tilde{I}_1(\Delta t, a, b) = \frac{1}{(\Delta t)^2} \int_0^{\Delta t/\sqrt{2}} \left(\int_0^v e^{\sqrt{2}bu} \cos(\sqrt{2}au) du \right) dv.$$

Computing this integral yields

$$\begin{aligned}\tilde{I}_1(\Delta t, a, b) &= \frac{(\Delta t)^{-2} e^{b\Delta t}}{2(a^2 + b^2)^2} \left\{ (b^2 - a^2) \cos a\Delta t + 2ab \sin a\Delta t \right\} \\ &\quad + \frac{(\Delta t)^{-2} (a^2 - b^2)}{2(a^2 + b^2)^2} - \frac{b}{2(\Delta t)(a^2 + b^2)}.\end{aligned}$$

One has directly from the definitions, the identities:

$$\partial_{\Delta t} \left(\frac{\tilde{V}(t, \Delta t)}{\Delta t} \right) = -4\tilde{I}_1(\Delta t, a, b) + 2I_2(\Delta t, a, b),$$

$$\begin{aligned}\partial_{\Delta t} \left(\frac{\tilde{V}(t, \Delta t)}{\Delta t} \right) &= \frac{2e^{bx}}{x^2(a^2 + b^2)^2} \left((a^2 - b^2) \cos ax - 2ab \sin ax \right) \\ &\quad + \frac{2(b^2 - a^2)}{x^2(a^2 + b^2)^2} + \frac{2}{x(a^2 + b^2)} \left(e^{bx} (b \cos ax + a \sin ax) \right).\end{aligned}$$

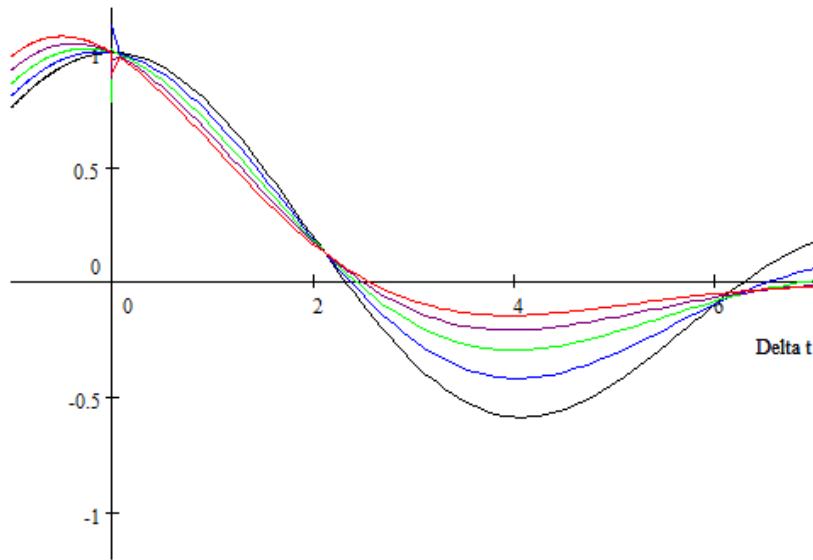


Figure 1. Examples of $\partial_{\Delta t} \left(\frac{\tilde{V}(t, \Delta t)}{\Delta t} \right)$ for $a = 1$, and values of b starting at 0 (black, no damping), decreasing by -0.1 to $b = -0.4$ (red).

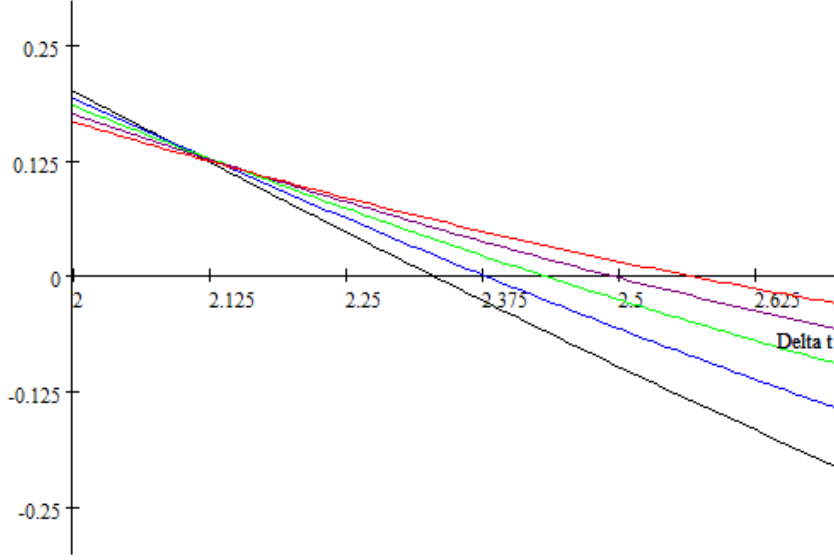


Figure 2. A close up of the region in Figure 1 where the graphs cross the x-axis. The intercepts decrease as b increases.

Hence we see that there is a maximum time interval Δt over which the volatility per unit time is maximum (see Figure 1). There is relatively little movement in the maximum value of Δt as a function of damping (see Figure 2).

4. Price peaks for stochastic equations

4.1. General relations.

We let $X := D/S$, assume $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies Condition 1 of [4], and consider the stochastic equation (3). The objective is to relate the extrema of $\mathbb{E} \log P(t)$ to the volatility within a small time interval Δt and to the limiting volatility. In particular, the extrema of $\mathbb{E} \log P(t)$ can be determined by differentiating (19)

$$\frac{d}{dt} \mathbb{E} \log P(t) = \mathbb{E} \left[f(x(t)) - \frac{\sigma^2(t)}{2} (x(t) f'(x(t)))^2 \right] = 0. \quad (27)$$

Using the expression for the limiting volatility, $\mathbb{V}(t) = \tilde{\mathbb{V}}(t) = \mathbb{E} [\sigma(t)^2 v^2(t)]$, one has the relation

$$\mathbb{E} [f(x(t))] = \tilde{\mathbb{V}}(t) / 2.$$

Thus, the peak in $\mathbb{E} \log P(t)$ occurs when the expected value of the price function, $f(x(t))$, equals one half the limiting volatility. We note also that for small Δt , one has the approximation

$$\begin{aligned} (\Delta t)^{-1} \tilde{\mathbb{V}}(t, \Delta t) &= (\Delta t)^{-1} \text{Var} [\log P(t + \Delta t) - \log P(t)] \\ &\simeq (\Delta t)^{-1} \text{Var} [\Delta P/P(t)] = (\Delta t)^{-1} V(t, \Delta t). \end{aligned}$$

Thus, the volatility on a small time scale can be examined, and compared with the maximum of the expected logarithm of price.

4.2. Relations for deterministic $X(t)$

Let σ be a positive constant. For $f(x) := x - x^{-1}$, $xf'(x)$ has a minimum at $x = 1$, and on the interval $[a, 1]$ it has a maximum at $x = a$ and on $[1, b]$ a maximum at $x = b$. Now suppose that $x(t)$ is a deterministic function such that $x(0) < 1$, peaks at $x(t_m) = x_m > 1$ and declines to $1 = x(t_p)$. Thus, the ratio of demand to supply as a function of t , i.e., $X(t) = D(t)/S(t)$, peaks at t_m and declines until supply and demand are equal at t_p . Note that even though this ratio is declining during the time period (t_m, t_p) , the excess demand is still positive so that prices can be expected to rise until the excess demand vanishes at t_p . (See also [4].)

The maximum of $\mathbb{V}(t)$ is attained at t_m since $X(t)$ has its maximum at that time. The minimum of $\mathbb{V}(t)$ is attained at t_p since $X(t_p) = 1$, which corresponds to the minimum of $xf'(x)$.

Next, consider the maximum of $\mathbb{E} \log P$. Since $X(t)$ is assumed deterministic, (27) implies

$$\frac{d}{dt} \mathbb{E} \log P(X(t)) = f(X(t)) - \frac{\sigma^2}{2} (X(t) f'(X(t)))^2 = 0 \quad (28)$$

as the condition for a maximum for $\mathbb{E} \log P(X(t))$.

The right hand side of (28) for $f(x) := x - x^{-1}$ and thus $xf'(x) = x + x^{-1}$ can be written as

$$x - x^{-1} - \frac{\sigma^2}{2} (x + x^{-1})^2 = 0.$$

For small σ , one has an approximate solution $x \cong 1 + \sigma^2/2$. Hence, the maximum of $\mathbb{E} \log P(X(t))$ occurs when x is slightly greater than 1.

Example. Consider $f(x) = x - x^{-1}$ so $f'(x) = 1 + x^{-2}$ and $xf'(1) = 2$ and define

$$X(t) = -(t-1)^2 + 2.$$

The maximum of $\mathbb{E} \log P(X(t))$ will occur when $X(t) = 1 + \sigma^2$, i.e., at the value of t such that

$$1 + \sigma^2 = -(t-1)^2 + 2.$$

The larger of the two roots of this quadratic is thus at $t \cong 2 - \sigma^2/2$.

Hence, the expected log price peaks at $t \cong 2 - \sigma^2/2$, while the marginal volatility, $\mathbb{V}(t)$, given by (21), peaks when x is furthest from $x = 1$, which, in this case is at $t_m := 1$ on the interval $[0, 2]$. In other words, when $X(t) = D(t)/S(t)$ is at a maximum (i.e., the relative excess demand $\{D(t) - S(t)\}/S(t)$ is also at a maximum), the marginal volatility is at its peak.

4.3. Stochastic $X(t) = D(t)/S(t)$ and extrema.

The expressions (27) and (21) imply that the extrema of $\mathbb{E} \log P$ occur when

$$\mathbb{E} f(X(t)) = \tilde{\mathbb{V}}(t)/2$$

which, for $\sigma(t)$ deterministic yields

$$\mathbb{E} f(X(t)) = \frac{\sigma^2(t)}{2} \mathbb{E} [(X(t) f'(X(t)))^2].$$

A necessary condition for inflection points of $\mathbb{E} \log P(t)$ is

$$0 = \frac{d^2}{dt^2} \mathbb{E} \log P(t) = \frac{d}{dt} \left(\mathbb{E} f(X(t)) - \frac{1}{2} \tilde{\mathbb{V}}(t) \right).$$

From this equation we have the result that the limiting volatility $\tilde{V}(t)$ has an extremum when $\frac{d}{dt}\mathbb{E}f(X(t)) = \frac{d^2}{dt^2}\mathbb{E}\log P(t)$. In other words, when $\mathbb{E}\log P(t)$ has oscillatory behavior, the minima and maxima of $\tilde{V}(t)$ coincide with inflection points of $\mathbb{E}\log P(t)$.

Next, we assume that $\sigma \in \mathbb{R}$, i.e., constant in time, and define in place of $\tilde{V}(t) = \mathbb{E}[\sigma(t)(v(X)^2)]$ the function $\tilde{V}_1(X)$, by

$$\tilde{V}_1(X) := \mathbb{E}[\sigma^2 v(X)^2] = \sigma^2 \mathbb{E}[X^2 + 2 + X^{-2}]$$

so that the extrema as a function of X occur when $0 = \partial_X \mathbb{E}[v(X)^2]$, i.e., when

$$\mathbb{E}[X] = \mathbb{E}[X^{-3}].$$

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function, e.g., $g(x) = x^{-3}$. By Taylor series we have

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{1}{2}g''(x_0)(x - x_0)^2 + \dots$$

Let $x := X$ and $x_0 := \mathbb{E}X$ yielding

$$\begin{aligned} g(X) &= g(\mathbb{E}X) + g'(\mathbb{E}X)(X - \mathbb{E}X) + \frac{1}{2}g''(\mathbb{E}X)(X - \mathbb{E}X)^2 + \dots \\ \mathbb{E}g(X) &= g(\mathbb{E}X) + g'(\mathbb{E}X)\mathbb{E}(X - \mathbb{E}X) + \frac{1}{2}g''(\mathbb{E}X)\mathbb{E}(X - \mathbb{E}X)^2 + \dots \end{aligned}$$

Hence we have for $g(x) = x^{-3}$, $g'(x) = -3x^{-4}$, $g''(x) = 12x^{-5}$, so,

$$\mathbb{E}[X^{-3}] - (\mathbb{E}X)^{-3} \cong 6(\mathbb{E}X)^{-5} \text{Var}X,$$

yielding a polynomial equation for the value of $\mathbb{E}X$ that provides an extremum of \tilde{V}_1 .

In the next section, we will see that $X(t) = D(t)/S(t)$ can be expressed as $(k(t)/(1 - k(t))(L/P(t))$ where $k(t)$ will be determined by the asset flow equations and will depend on the motivations such as valuation and price trend, and $L \in \mathbb{R}$ is the liquidity value. Thus, under conditions for which $|\mathbb{E}X|^{-5} \text{Var}X \ll 1$ we have then

$$\mathbb{E}X \cong (\mathbb{E}X)^{-3}, \text{ so, } \mathbb{E}X \cong 1.$$

This means that, under these conditions, the minimum volatility, $\mathbb{V}(X) = \mathbb{V}\left(\frac{k}{1-k} \frac{L}{P}\right)$ occurs when

$$\mathbb{E}\left[\frac{k}{1-k} \frac{L}{P}\right] \cong 1.$$

5. Derivation of the Stochastic Differential Equations Model

5.1. Deterministic Asset Flow

Along the lines of Caginalp and Balenovich [3], we first describe the (deterministic) asset flow equations in the closed system in which there are a fixed number of shares N and cash M . Let $P(t)$ be the price of the shares. We let $B(t)$ represent the fraction of assets that are in the stock, i.e., $B = NP/(NP + M)$. Clearly, $1 - B$ is the fraction in cash. In the asset flow equations we consider, a key quantity is the transition rate, $k \in (0, 1)$, that is the probability that a unit of cash is submitted to the market as a buy order. We assume that $1 - k$ is the analogous transition rate for a sell order. Within this formalism, the supply, S , and demand, D , are given by

$$D = k(1 - B), \quad S = (1 - k)B$$

i.e., the demand is the product of the fraction of assets in cash times the transition rate. An important variable that has been characterized is the liquidity¹, $L(t)$, defined as the ratio of M to N . In closed systems L is again a constant. Furthermore, it follows from the definition of B that

$$\frac{L}{P} = \frac{(1-B)}{B}, \quad (29)$$

and hence the liquidity to price ratio is a convenient representative of the ratio of demand to supply:

$$\frac{D}{S} = \frac{k}{1-k} \frac{1-B}{B} = \frac{k}{1-k} \frac{L}{P}.$$

The price change is then given by

$$P^{-1} \frac{dP}{dt} = f\left(\frac{D}{S}\right) = f\left(\frac{k}{1-k} \frac{L}{P}\right), \quad (30)$$

where f is an increasing function satisfying $f(1) = 0$ that depends on the nature of the market (see Condition 1 p. 809 of [4]). In the standard model we use the prototype², $f(x) := x - x^{-1}$, and hence

$$P^{-1} \frac{dP}{dt} = \frac{D}{S} - \frac{S}{D} = \frac{k}{1-k} \frac{L}{P} - \frac{1-k}{k} \frac{P}{L}. \quad (31)$$

Let $\zeta \in \mathbb{R}$ denote the trading sentiment so that a large positive value indicates a strong inclination to buy. We consider the two basic components; ζ_1 , based on trend, and ζ_2 , based on valuation, so $\zeta = \zeta_1 + \zeta_2$. These values are mapped onto the transition rate via the function

$$k(\zeta_1, \zeta_2) = \frac{1}{2} + \frac{1}{2} \tanh(\zeta_1 + \zeta_2), \quad (32)$$

ensuring that $k \in (0, 1)$. Both components of the sentiment are crucial as under-valuation is a primary (and classical) reason for buying an asset, while trend is essentially a behavioral component that encompasses the motivation to avoid selling an asset that appears to be increasing in price. There is also considerable empirical evidence that trend is a factor in purchasing (e.g. Poterba and Summers, 1988 [30] and Caginalp et al., 2014 [6] and references therein). The formulation that has been used for ζ_1 is

$$\zeta_1(t) := q_1 c_1 \int_{-\infty}^t e^{-c_1(t-\tau)} \frac{1}{P(\tau)} \frac{dP(\tau)}{d\tau} d\tau, \quad (33)$$

where c_1^{-1} is a time scale that characterizes the aggregate traders, while q_1 is simply an amplitude that establishes the magnitude of this component. In a similar way, one defines c_2^{-1} and q_2 and ζ_2 by

$$\zeta_2(t) := q_2 c_2 \int_{-\infty}^t e^{-c_2(t-\tau)} \frac{P_a(\tau) - P(\tau)}{P_a(\tau)} d\tau, \quad (34)$$

where $P_a(t)$ is the fundamental, or intrinsic, value of the asset. Measuring the value of a stock or other asset is a classical problem that dates back to Ben Graham in the 1930's (see e.g., Graham, 1965 [16]) who used tools such as the expected dividend stream, the price earning ratio, etc., to calculate a fundamental value in order that an investor can find stocks that are undervalued and likely to return a profit at some point. This type of investment strategy is known as value investing. Note that the asset flow equations can incorporate any motivation that can be quantified.

The integral formulation for ζ_1 and ζ_2 can be converted to the differential equations:

$$\frac{d\zeta_1}{dt} = c_1 q_1 \frac{1}{P} \frac{dP}{dt} - c_1 \zeta_1, \quad (35)$$

¹In an experimental setting, e.g., Porter and Smith, 1994 [29], we see that there are two immediate variables with units of price per share: the trading price established by traders and the payout defined by the experimenters. However, as Caginalp and Balenovich, 1999 [3] noted, there is a third key variable, L , with these units obtained by dividing the total number of dollars in the system by the total number of shares.

²One can also use functions (see [5]) that correspond to a particular density decay such as a mollification of $f(x) = \text{sign}(x - x^{-1})|x - x^{-1}|^{1/q}$ with $q > 0$.

$$\frac{d\zeta_2}{dt} = c_2 q_2 \frac{P_a - P}{P_a} - c_2 \zeta_2. \quad (36)$$

The system of ordinary differential equations (31), (35) and (36) forms the deterministic asset flow model for one group of identical traders. Equations similar to these have been studied as a deterministic system in numerous works, many of which have focussed on stability properties in the ODE sense. In general, small q_1 and c_1 favor stability since there is less emphasis on trend and the time scale is large (see DeSantis et al., 2018 [10], Merdan and Alisen [24], and Merdan and Cakmak [25], and references therein). The concept of investor sentiment has long been recognized as an important construct among market professionals. In 1990, it was introduced in a mathematical framework [7]. This concept was also utilized recently in [19] where it was incorporated into SDE's within a model involving Multivariate Adaptive Regression Splines, which is a method for regression using classification with interactive variables.

5.2. Stochastic Asset Flow Equations

As noted in the Introduction, the key idea in our approach is that randomness arises through the quotient of supply and demand. A probabilistic analysis of this quotient yields the following stochastic equation (see [4] and [5])

$$\frac{1}{P} dP = f\left(\frac{k}{1-k} \frac{L}{P}\right) dt + \sigma v\left(\frac{k}{1-k} \frac{L}{P}\right) dW. \quad (37)$$

where the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly increasing function of its argument that must satisfy Condition 1 of [4] (p. 809), and $v(x) := x f'(x)$.

Substitution of this expression into (35), which can be equivalently written as

$$d\zeta_1 = c_1 q_1 \frac{1}{P} dP - c_1 \zeta_1 dt,$$

implies the stochastic ζ_1 equation,

$$d\zeta_1 = \left\{ c_1 q_1 f\left(\frac{k}{1-k} \frac{L}{P}\right) - c_1 \zeta_1 \right\} dt + c_1 q_1 \sigma v\left(\frac{k}{1-k} \frac{L}{P}\right) dW. \quad (38)$$

The stochastic ζ_2 equation has no additional stochastic component and remains as

$$d\zeta_2 = \left\{ c_2 q_2 \left(1 - \frac{P}{P_a}\right) - c_2 \zeta_2 \right\} dt. \quad (39)$$

As mentioned above, we use the basic prototype, $f(x) = x - x^{-1}$ implying $v(x) = x + x^{-1}$ throughout the paper (see [4]). In principle, one can include an additional stochastic term in $d\zeta_2$ that may be correlated with dW .

5.3. Existence and uniqueness

A standard definition is the class $H_2[0, T]$ as the set of \mathcal{F}_t adapted³ functions $f(t, \omega)$ such that

$$\int_0^T E f^2(s, \omega) ds < \infty.$$

We say that the initial value problem (37), (38), (39) with initial conditions $P(0) := P_0$, $\zeta_1(0) := \zeta_{10}$ and $\zeta_2(0) := \zeta_{20}$ has a solution in the Ito sense if for $\vec{x} := (P, \zeta_1, \zeta_2)$ one has (i) $\vec{x} := (P, \zeta_1, \zeta_2) \in H_2[0, T]$, and (ii) the stochastic differential equations [i.e., (37), (38), (39)] and initial conditions hold for almost all $\omega \in \Omega$.

The existence and uniqueness of solutions of this stochastic system are a consequence of the standard theory. A basic theorem ([33] p. 94) states that if the coefficients in (37), (38), (39) are all uniformly Lipschitz continuous

³ \mathcal{F}_t adapted consists of functions that are independent of events that occur after time t . See p. 33 of Schuss [33] for precise definition.

except possibly on a set of measure zero in the probability space, then there exists a unique solution to the initial value problem. The solution $\vec{x} := (P, \zeta_1, \zeta_2)$ is continuous in t with probability one.

In conclusion, we have derived a set of stochastic differential equations based on supply/demand considerations incorporating the finiteness of assets and cash. A variety of motivations can be considered in the supply and demand. At present we consider value-based and trend-based investing which are the key components for oscillations and stability. We will utilize these equations to understand the complex relationship between the volatility, trend and market tops and bottoms.

6. Equilibrium and Stability for the Deterministic Equations

6.1. Equilibrium

We first write the conditions for equilibrium in the deterministic setting. Setting the derivatives of P , ζ_1 and ζ_2 to zero in the ordinary differential equations (31), (35) and (36), and denoting by P_E the equilibrium price, ζ_{1E} and ζ_{2E} the equilibrium values of the sentiments, we have

$$0 = \frac{1}{P} \frac{dP}{dt} = \frac{k}{1-k} \frac{L}{P} - \frac{1-k}{k} \frac{P}{L}$$

which implies

$$\frac{P_E}{L} = \frac{k_E}{1-k_E} = e^{2(\zeta_{1E} + \zeta_{2E})}. \quad (40)$$

Since $dP/dt = 0$ and $d\zeta_1/dt = 0$ one has the equilibrium value $\zeta_1 = \zeta_{1E} := 0$. Finally, from $d\zeta_2/dt = 0$ we have $c_2 q_2 \frac{P_a - P}{P_a} = c_2 \zeta_2$ from (36), yielding the equilibrium value $\zeta_2 = \zeta_{2E} := q_2 \left(1 - \frac{P_E}{P_a}\right)$.

By putting all results together we obtain a transcendental equation for P_E :

$$\frac{P_E}{L} = e^{2q_2 \left(1 - \frac{P_E}{P_a}\right)}. \quad (41)$$

Note that the left hand side of (41) is linearly increasing with P_E starting from 0 while the right hand side is monotonically decreasing with P_E , starting at e^{2q_2} for $P_E = 0$ and converging to 0 as $P_E \rightarrow \infty$. By the Intermediate Value Theorem the system has a unique positive equilibrium P_E for any choice of the parameters L, q_2, P_a . Note that the equilibrium of the system does not depend on the timing constants c_1, c_2 and neither it does depend on q_1 .

In summary, P_E is a solution of (41) and

$$\zeta_{1E} = 0, \quad \zeta_{2E} = \frac{1}{2} \ln \frac{P_E}{L} = q_2 \left(1 - \frac{P_E}{P_a}\right). \quad (42)$$

One also has

$$k_E = \frac{1}{2} + \frac{1}{2} \tanh \left(q_2 \left(1 - \frac{P_E}{P_a}\right) \right). \quad (43)$$

Note that (41) and (43) imply the following:

(i) $k_E > 1/2$ is equivalent to $1 > \frac{P_E}{P_a}$, and implies $L < P_E < P_a$.

(ii) $k_E = 1/2$ is equivalent to $1 = \frac{P_E}{P_a}$, and implies $P_a = P_E = L$.

(iii) $k_E < 1/2$ is equivalent to $1 < \frac{P_E}{P_a}$, and implies $P_a < P_E < L$.

Thus, the equilibrium value P_E lies between P_a and L , if these two quantities differ.

6.2. Stability

Stability of the deterministic equations and related systems has been studied in several papers (see [26, 10, 11] and references therein). For the system of equations (31), (35) and (36) we will now examine its stability about the equilibrium $(P_E, \zeta_{1E}, \zeta_{2E})$ discussed in the earlier section. For convenience, we will retain P_E as a parameter and express L as a quantity dependent on the chosen equilibrium via (41).

The Jacobian of the system in equilibrium is

$$J = \begin{pmatrix} -2 & 4P_E & 4P_E \\ -\frac{2c_1q_1}{P_E} & c_1(4q_1 - 1) & 4c_1q_1 \\ -\frac{c_2q_2}{P_a} & 0 & -c_2 \end{pmatrix}. \quad (44)$$

The characteristic polynomial of J is given by $p_J(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ where

$$\begin{aligned} a_2 &= 2 + c_1 + c_2 - 4c_1q_1 \\ a_1 &= 2(c_1 + c_2) + c_1c_2 + 4c_2q_2\frac{P_E}{P_a} - 4c_1c_2q_1 \\ a_0 &= 2c_1c_2\left(1 + 2q_2\frac{P_E}{P_a}\right). \end{aligned}$$

The Routh-Hurwitz theorem implies that the system has eigenvalues with negative real parts if and only if the following conditions hold: $a_2 > 0$, $a_0 > 0$ and $a_1a_2 > a_0$. Since the second condition is always satisfied, the stability of the system is determined by the first and third conditions. First note that if $c_1 = 0$ or $q_1 = 0$, then the three conditions are satisfied and hence the system is stable. We assume $c_1 > 0$ and $q_1 > 0$ below.

In view of the first condition, stability requires that either $q_1 \leq 1/4$, or $q_1 > 1/4$ and c_1 is sufficiently small, i.e.,

$$q_1 < \frac{1}{4} + \frac{2 + c_2}{4c_1}. \quad (45)$$

The third condition requires that not only $a_1 > 0$, i.e.,

$$q_1 < \frac{1}{4} + \frac{c_1 + c_2 + 2c_2q_2\frac{P_E}{P_a}}{2c_1c_2}, \quad (46)$$

but also that q_1 needs to be sufficiently far away from the boundaries of the admissible regions defined by the inequalities (45) and (46). Unfortunately, $q_1 \leq 1/4$ does not guarantee the satisfaction of the third condition for all parameter choices, but smallness of the parameter q_1 does. Indeed, the third condition can be written in the form $Ac_1^2q_1^2 - Bc_1q_1 + C > 0$ where the three terms A, B, C are all positive and independent of q_1 ,

$$\begin{aligned} A &= 16c_2 \\ B &= 4\left(2c_1(c_2 + 1) + 4c_2 + c_2^2 + 4c_2q_2\frac{P_E}{P_a}\right) \\ C &= (2 + c_2)\left(c_1^2 + c_1(c_2 + 2) + 2c_2 + 4c_2q_2\frac{P_E}{P_a}\right) \end{aligned}$$

and hence the third condition is satisfied for small enough q_1 for any value of c_1 . A closer examination also reveals that C is positive for $c_1 = 0$, and hence we can conclude that the third condition is satisfied for small enough c_1 for any value of q_1 . In general, a large trend coefficient, q_1 , characteristic of traders focusing on momentum, favors instability. Large c_1 corresponds to a small time scale that also favors instability as traders focus on short term trends that may well be a consequence of randomness. Thus, randomness that moves prices higher would be interpreted as an uptrend, so that momentum trading would enhance the price movement. A very small c_1 means that traders are looking at trend on a very large time scale so that short term trends do not create instability.

In addition to stability considerations, it is important to note that the Jacobian has complex eigenvalues (and hence the system has oscillatory behavior when approaching the equilibrium) if either $3a_1 - a_2^2 > 0$ (i.e., $p_J(\lambda)$ has no local

maxima or minima), $p_J(\lambda_1) < 0$ (i.e., local maximum of $p_J(\lambda)$ is negative), or $p_J(\lambda_2) > 0$ (i.e., local minimum of $p_J(\lambda)$ is positive), where

$$\lambda_{1,2} = \frac{1}{3} \left(-a_2 \mp \sqrt{a_2^2 - 3a_1} \right). \quad (47)$$

The dependence of stability of the equilibrium on the parameters c_1 and q_1 is summarized in Figure 3.

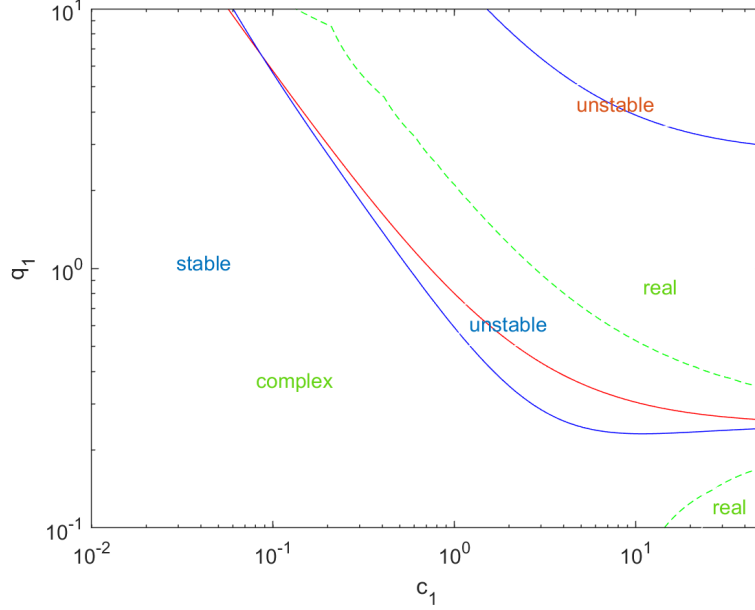


Figure 3: A numerical example of dependence of the stability of the equilibrium on (c_1, q_1) is shown above for the following choices of parameters $(L, P_a, c_2, q_2) = (1, 0.7, 0.2, 10)$, which correspond to the equilibrium $P_E = 0.7119$. The area below and to the left of the red curve has $a_2 > 0$, while the area below and to the left of the lower blue curve (and above the upper blue curve) has $a_1 a_2 > a_0$ and hence corresponds to stable equilibria. Parameters in the area to the left of the green dashed curve correspond to $3a_1 - a_2^2 > 0$ and hence result in complex eigenvalues.

7. Analysis of Stochastic Equations

We have seen in the previous section, particularly for a stationary process, that any measure of the volatility of a random process is closely related to its autocovariance (or autocorrelation). In this section we compute the autocovariance in price for the stochastic asset trading model defined by equations (37), (38), (39).

7.1. Linearization

We linearize the stochastic asset flow equations (37), (38), (39) with initial conditions $P(0) =: P_0$, $\zeta_1(0) =: \zeta_{10}$ and $\zeta_2(0) =: \zeta_{20}$ about the equilibrium values $P_E, \zeta_{1E}, \zeta_{2E}$ derived in (41) and (42) as functions of the parameters q_2, L and P_a . As in Section 6.2 we retain P_E as a parameter and consider L a dependent quantity. The Jacobian J of the system in this notation was given in (44). In the linearization of the stochastic system, the coefficients of the dW terms in (37) and (38) will be given by the leading order terms. Noting that

$$v \left(\frac{k}{1-k} \frac{L}{P} \right) |_{eq} = v \left(e^{2(\zeta_{1E} + \zeta_{2E})} \frac{L}{P_E} \right) = v(1) = 2,$$

one has that the coefficients of the dW terms are given by

$$\sigma P_E v \left(\frac{k}{1-k} \frac{L}{P} \right) |_{eq} = 2\sigma P_E, \quad c_1 q_1 \sigma v \left(\frac{k}{1-k} \frac{L}{P} \right) = 2c_1 q_1. \quad (48)$$

We introduce variables $p = P - P_E$, $\eta_1 = \zeta_1 - \zeta_{1E}$, and $\eta_2 = \zeta_2 - \zeta_{2E}$ to describe the departure of the system from equilibrium. Replacing dW by dW_1 and setting $dW_2 = dW_3 = 0$, we can thus write the linearized SDE in the form

$$\begin{pmatrix} dp \\ d\eta_1 \\ d\eta_2 \end{pmatrix} = J \begin{pmatrix} p \\ \eta_1 \\ \eta_2 \end{pmatrix} dt + \begin{pmatrix} 2\sigma P_E & 0 & 0 \\ 2\sigma c_1 q_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \\ dW_3 \end{pmatrix}. \quad (49)$$

Let S be defined as the 3×3 matrix above, and let $d\vec{W} := (dW_1, dW_2, dW_3)$, so we can write

$$S d\vec{W} := \begin{pmatrix} 2\sigma P_E & 0 & 0 \\ 2\sigma c_1 q_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} dW_1 \\ dW_2 \\ dW_3 \end{pmatrix} = \begin{pmatrix} 2\sigma P_E \\ 2\sigma c_1 q_1 \\ 0 \end{pmatrix} dW_1,$$

Subsequently, we drop the subscript 1 in dW_1 . Using $\vec{X}(t) := (p(t), \eta_1(t), \eta_2(t))$ the linearized SDE can be written in a more compact form as

$$d\vec{X} = J\vec{X}dt + S d\vec{W}. \quad (50)$$

The exact solution for (50) is easily established as (see p. 71 of Pavliotis, 2014 [28]):

$$\vec{X}(t) = e^{Jt} X(0) + \int_0^t e^{J(t-\tau)} S d\vec{W}(\tau) \quad (51)$$

7.2. The autocovariance function

The expectation of the solution (51) is given by

$$\mathbb{E}\vec{X}(t) = \mathbb{E}\left[e^{Jt} X(0)\right] = e^{Jt} \vec{X}(0), \quad (52)$$

where the last equality depends on the initial conditions being deterministic. Thus, we have

$$\vec{X}(t) - \mathbb{E}\vec{X}(t) = \int_0^t e^{J(t-\tau)} S d\vec{W}(\tau). \quad (53)$$

By definition of the autocovariance we have

$$\begin{aligned} R(t, s) &:= Cov(\vec{X}(t), \vec{X}(s)) = \mathbb{E}\left[\left(\vec{X}(t) - \mathbb{E}\vec{X}(t)\right)\left(\vec{X}(s) - \mathbb{E}\vec{X}(s)\right)^T\right] \\ &= \mathbb{E}\left[\left(\int_0^t e^{J(t-\tau)} S d\vec{W}(\tau)\right)\left(\int_0^s e^{J(s-\tau)} S d\vec{W}(\tau)\right)^T\right] \\ &= \int_0^{\min\{t,s\}} e^{J(t-\rho)} Z e^{J^T(s-\rho)} d\rho, \end{aligned}$$

where $Z := S S^T$. In terms of components, with $i, j \in \{1, 2, 3\}$, we have, with $s \leq t$,

$$R(t, s)_{ij} = \int_0^s \left(e^{J(t-\rho)} Z e^{J^T(s-\rho)} \right)_{ij} d\rho. \quad (54)$$

The variance, $R(t, t)$ will satisfy the differential equation

$$\frac{dR(t, t)}{dt} = JR(t, t) + R(t, t)J^T + Z \quad (55)$$

which has a steady state solution

$$R^\infty = \int_0^\infty e^{-J\rho} Z e^{-J^T\rho} d\rho \quad (56)$$

that satisfies the Lyapunov equation

$$JR^\infty + R^\infty J^T + Z = 0. \quad (57)$$

7.3. Steady state autocovariance

With $Q_1 := c_1 q_1 / P_E$ and $Q_2 := c_2 q_2 / P_a$, the Jacobian J in (44) and matrix Z reduce to:

$$J := \begin{pmatrix} -2 & 4P_E & 4P_E \\ -2Q_1 & 4Q_1 P_E - c_1 & 4Q_1 P_E \\ -Q_2 & 0 & -c_2 \end{pmatrix}$$

$$Z := 4\sigma^2 P_E^2 \begin{pmatrix} 1 & Q_1 & 0 \\ Q_1 & Q_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lyapunov equation (57) is a system of six linear equations for the six entries of the symmetric matrix R^∞ . We can rewrite it as a six-dimensional system $C \vec{R}^\infty = c$ for the vector $\vec{R}^\infty = (R_{11}^\infty, R_{12}^\infty, R_{13}^\infty, R_{22}^\infty, R_{23}^\infty, R_{33}^\infty)^T$ where

$$C := \begin{pmatrix} -4 & 8P_E & 8P_E & 0 & 0 & 0 \\ -2Q_1 & 4P_E Q_1 - c_1 - 2 & 4P_E Q_1 & 4P_E & 4P_E & 0 \\ -Q_2 & 0 & -c_2 - 2 & 0 & 4P_E & 4P_E \\ 0 & -4Q_1 & 0 & 8P_E Q_1 - 2c_1 & 8P_E Q_1 & 0 \\ 0 & -Q_2 & -2Q_1 & 0 & 4P_E Q_1 - c_2 - c_1 & 4P_E Q_1 \\ 0 & 0 & -2Q_2 & 0 & 0 & -2c_2 \end{pmatrix}$$

and $c = 4\sigma^2 P_E^2 (1, Q_1, 0, Q_1^2, 0, 0)^T$. Of course, the system has a unique solution when $\det(C) \neq 0$ where

$$\det(C) = 16c_1(c_2 + 2P_E Q_2)D$$

with

$$D = 16c_2 P_E^2 Q_1^2 - 4(2c_1 + 4c_2 + c_2^2 + 2c_1 c_2 + 4Q_2 P_E)P_E Q_1 + 4(c_2 + 2)Q_2 P_E + (c_2 + 2)(c_1 + 2)(c_1 + c_2).$$

For the R_{11}^∞ component we then have the solution

$$R_{11}^\infty = \frac{16P_E^2 \sigma^2 c_1 F}{\det(C)} = \frac{P_E^2 \sigma^2}{c_2 + 2P_E Q_2} \frac{F}{D} \quad (58)$$

where

$$F = -4c_2(c_2(c_1 + 2) + 4Q_2 P_E)P_E Q_1 + 16Q_2^2 P_E^2 + 4(c_1^2 + c_1 c_2 + 2c_1 + c_2^2 + 4c_2)Q_2 P_E + c_2(c_2 + 2)(c_1 + 2)(c_1 + c_2).$$

When $q_1 = 0$, the steady state variance R^∞ simplifies to

$$R_{11}^\infty|_{q_1=0} = \frac{P_E^2 \sigma^2}{(c_2 + 2)(c_2 + 2P_E Q_2)} \begin{pmatrix} c_2(c_2 + 2) + 4P_E Q_2 & 0 & -Q_2 c_2 \\ 0 & 0 & 0 \\ -Q_2 c_2 & 0 & Q_2^2 \end{pmatrix},$$

and hence the variance of the price, R_{11}^∞ , is positive. For small q_1 , the first order expansion of R_{11}^∞ is given by

$$R_{11}^\infty = \frac{P_E^2 \sigma^2}{c_2(c_2 + 2)} \left(1 + 2q_2 \frac{P_E}{P_a}\right) (G_0 + q_1 G_1) + O(q_1^2)$$

where

$$G_0 = c_2 \left(c_2 + 2 + 4q_2 \frac{P_E}{P_a} \right)$$

$$G_1 = 4c_1 \frac{16c_2^2 q_2^2 \left(\frac{P_E}{P_a}\right)^2 + 4c_2 q_2 \frac{P_E}{P_a} (2c_1 + 4c_2 + 2c_1 c_2 + c_2^2) + c_2 (c_2 + 2)^2 (c_1 + c_2)}{(c_2 + 2) \left((c_1 + c_2)(c_1 + 2) + 4c_2 q_2 \frac{P_E}{P_a} \right)}.$$

Since $G_0, G_1 > 0$, we can conclude that R_{11}^∞ increasing with q_1 for small values of q_1 and hence positive in that range.

For $q_1 > 0$, the sign and magnitude of R_{11}^∞ depend on the signs of the terms F and D in (58). Note that F is linear and decreasing in Q_1 (and hence in q_1) with positive intercept, while D is quadratic in Q_1 with positive quadratic coefficient, negative linear term and a positive constant term. Furthermore, D is nonnegative at the value q_{1F} at which F reaches zero, as

$$D|_{q_{1F}} = \frac{8P_E Q_2 (c_1 - c_2)^2 (c_2 + 2P_E Q_2) (2c_1 + 2c_2 + 4P_E Q_2 + c_1 c_2 + c_1^2)}{c_2 (2c_2 + 4P_E Q_2 + c_1 c_2)^2}$$

while F is positive at the value $q_{1Dmin} = (2c_1 + 4c_2 + 4P_E Q_2 + 2c_1 c_2 + c_2^2) / (8c_1 c_2)$ at which D reaches minimum, as

$$F|_{q_{1Dmin}} = 8Q_2^2 P_E^2 + 2((c_1 - c_2)^2 + (c_1 + 2)(c_1 + c_2))Q_2 P_E + \frac{1}{2}c_2(c_1 + 2)(c_2^2 + 2c_1).$$

In summary, for any choice of the model parameters in the linearized model (49), the steady state price variance R_{11}^∞ is positive and increasing on the interval $q_1 \in [0, q_1^*)$ where q_1^* is the smaller root of D :

$$q_1^* = \frac{2c_1 + 4c_2 + 2c_1 c_2 + c_2^2 + 4Q_2 P_E}{8c_1 c_2} - \frac{\sqrt{16Q_2^2 P_E^2 + 8(2c_1 c_2 + 2c_1 - c_2^2)Q_2 P_E + (c_2^2 + 2c_1)^2}}{8c_1 c_2}.$$

The variance, R_{11}^∞ , diverges as $q_1 \rightarrow q_1^*$. Not surprisingly, q_1^* is also the value at which the equilibrium of the system loses stability, since it is the value at which the determinant of the matrix C (and hence determinant of J) becomes zero. Thus, it is consistent with results discussed in Section 6.2.

We conclude that for the linearized system the variance in price, which is non-zero in the steady state, increases as the momentum coefficient, q_1 , increases. In other words, a larger amplitude in momentum trading has a persistent effect in terms of increasing the variance in price.

7.4. General autocovariance

Next, we calculate the autocovariance function (54) explicitly for arbitrary s , not just $s = \infty$, so the system need not be at steady state. We specify below the notation for the linear algebraic systems.

Eigenvalue, eigenvector notation. Let $J \in \mathbb{R}^{3 \times 3}$ have eigenvalues $d_1 \in \mathbb{R}$ and complex conjugates, d_2 and d_3 , with corresponding unit eigenvectors, e_1, e_2 and e_3 . Let $U \in \mathbb{C}^{3 \times 3}$ such that $J = UDU^{-1}$ where D is the diagonal matrix with entries d_1, d_2 and d_3 . Then we have the following.

A. The eigenvector e_1 can be chosen to be real, and e_2 and e_3 to be complex conjugates. The e_i are column vectors that we label as

$$e_1 = \begin{pmatrix} e_{11} \\ e_{21} \\ e_{31} \end{pmatrix}, \quad e_2 = \begin{pmatrix} e_{12} \\ e_{22} \\ e_{32} \end{pmatrix}, \quad e_3 = \begin{pmatrix} e_{13} \\ e_{23} \\ e_{33} \end{pmatrix},$$

so that

$$U = (e_1, e_2, e_3) = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

and $U_{jk} = e_{jk}$. Thus we have $e_2 = \bar{e}_3$, i.e., for $j \in \{1, 2, 3\}$ one has

$$e_{1k} \in \mathbb{R}, \quad e_{j2} = \bar{e}_{j3}. \quad (59)$$

B. Let g_1, g_2 and g_3 be defined as the rows of $G = U^{-1}$ (which exists since the e_i are independent). We label these row vectors g_i as

$$g_1 = (g_{11}, g_{12}, g_{13}), \quad g_2 = (g_{21}, g_{22}, g_{23}), \quad g_3 = (g_{31}, g_{32}, g_{33})$$

so that $G_{jk} = g_{jk}$ and

$$G = U^{-1} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}.$$

Then the first vector, g_1 , is real, and the second and third row are complex conjugates, i.e., $g_2 = \bar{g}_3$. So for $k \in \{1, 2, 3\}$ we have

$$g_{1k} \in \mathbb{R}, \quad g_{2k} = \bar{g}_{3k}. \quad (60)$$

In the general case with $c_1 \in [0, \infty)$ and $q_1 \in [0, \infty)$ the matrix S has two nonzero entries, i.e.,

$$S = \begin{pmatrix} 2\sigma P_E & 0 & 0 \\ 2c_1 q_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With $\mu := 2\sigma P_E$ and $\nu := 2c_1 q_1$ we see that $Z := S S^T$ is given by

$$Z = \begin{pmatrix} 4\sigma^2 (P_E)^2 & 4\sigma c_1 q_1 P_E & 0 \\ 4\sigma c_1 q_1 P_E & 4c_1^2 q_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mu^2 & \mu\nu & 0 \\ \mu\nu & \nu^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (61)$$

Subsequently, we will write $(A^T)_{ij}$ as just $A_{ij}^T = A_{ji}$ and $(A^{-1})_{ij}$ as A_{ij}^{-1} . For any 3×3 matrix, Z , with $Z_{i3} = 0$, $Z_{3j} = 0$ one has

$$\begin{aligned} (WZ W^T)_{ij} &= \sum_{p,q=1}^3 W_{ip} Z_{pq} (W^T)_{qj} = \sum_{p=1}^2 \sum_{q=1}^2 W_{ip} Z_{pq} W_{jq} \\ &= W_{i1} Z_{11} W_{j1} + W_{i1} Z_{12} W_{j2} + W_{i2} Z_{21} W_{j1} + W_{i2} Z_{22} W_{j2}. \end{aligned} \quad (62)$$

In particular, with Z as defined by (61) we have

$$(WZ W^T)_{ij} = \mu^2 W_{i1} W_{j1} + \nu^2 W_{i2} W_{j2} + \mu\nu (W_{i1} W_{j2} + W_{i2} W_{j1}). \quad (63)$$

Using the diagonalization (as earlier), we have

$$R(t, s)_{ij} = \int_0^s \sum_{k,r=1}^3 U_{ik} e^{d_k(t-\rho)} \left(U^{-1} Z (U^{-1})^T \right)_{kr} e^{d_r(s-\rho)} U_{rj}^T d\rho.$$

Recalling earlier definitions $U_{ij} = e_{ij}$ and $U_{ij}^{-1} = g_{ij}$, we define also

$$\begin{aligned} f_{kr} &:= \left(U^{-1} Z (U^{-1})^T \right)_{kr} = \mu^2 g_{k1} g_{r1} + \nu^2 g_{k2} g_{r2} + \mu\nu (g_{k1} g_{r2} + g_{k2} g_{r1}) \\ c_{kr}^{(i,j)} &= U_{ik} U_{jr} f_{kr} = e_{ik} e_{jr} f_{kr}. \end{aligned} \quad (64)$$

Performing the integral leads to

$$\begin{aligned} R(t, s)_{ij} &= \sum_{k,r=1}^3 U_{ik} U_{jr} f_{kr} \int_0^s e^{d_k(t-\rho) + d_r(s-\rho)} d\rho \\ &= \sum_{k,r=1}^3 c_{kr}^{(i,j)} \int_0^s e^{d_k(t-\rho) + d_r(s-\rho)} d\rho. \end{aligned}$$

Upon integrating we have

$$R(t, s)_{ij} = \sum_{k,r=1}^3 \frac{c_{kr}^{(i,j)}}{d_k + d_r} e^{d_k t} (e^{d_r s} - e^{-d_k s}). \quad (65)$$

Properties of e_{ij} , g_{ij} , f_{kr} and $c_{kr}^{(i,j)}$. For $\alpha, \beta \in \{1, 2, 3\}$, we define the "prime" function that takes 1 to 1 and reverses 2 and 3. That is, functions $\alpha' : \alpha \mapsto \{1, 2, 3\}$ and $\beta' : \beta \mapsto \{1, 2, 3\}$ are defined by

$$\alpha' := \begin{cases} 1 & \text{if } \alpha = 1 \\ 3 & \text{if } \alpha = 2 \\ 2 & \text{if } \alpha = 3 \end{cases}, \quad \beta' := \begin{cases} 1 & \text{if } \beta = 1 \\ 3 & \text{if } \beta = 2 \\ 2 & \text{if } \beta = 3 \end{cases}.$$

In this notation we can re-express

(i) $\bar{e}_{\alpha\beta} = e_{\alpha\beta'}$, $\bar{g}_{\alpha\beta} = g_{\alpha'\beta}$.

From the definition of f_{kr} which utilizes the g_{ij} and real constants, one obtains immediately,

(ii) $f_{\alpha\beta} = f_{\beta\alpha}$.

Next, from the expression directly, taking the complex conjugate, we have

$$\begin{aligned} \bar{f}_{kr} &= \mu^2 \bar{g}_{k1} \bar{g}_{r1} + \nu^2 \bar{g}_{k2} \bar{g}_{r2} + \mu\nu (\bar{g}_{k1} \bar{g}_{r2} + \bar{g}_{k2} \bar{g}_{r1}) \\ &= \mu^2 g_{k'1} g_{r'1} + \nu^2 g_{k'2} g_{r'2} + \mu\nu (g_{k'1} g_{r'2} + g_{k'2} g_{r'1}) \\ &= f_{k'r'}. \end{aligned}$$

Thus, we have,

(iii) $\bar{f}_{\alpha\beta} = f_{\alpha'\beta'}$.

(iv) Combining these into $c_{\alpha\beta}^{(i,j)}$ one has a relation involving complex conjugates:

$$\overline{c_{\alpha\beta}^{(i,j)}} = \bar{e}_{i\alpha} \bar{e}_{j\beta} \bar{f}_{\alpha\beta} = e_{i\alpha'} e_{j\beta'} f_{\alpha'\beta'} = c_{\alpha'\beta'}^{(i,j)}.$$

Note that one has trivially, the symmetry relation (though we will not use it),

(v) $c_{kr}^{(i,j)} = e_{ik} e_{jr} f_{kr} = e_{ik} e_{jr} f_{rk} = c_{kr}^{(j,i)}$, so $c_{kr}^{(i,i)} = c_{rk}^{(i,i)}$.

Now applying these to (65), we can group the k, r in the following way (note that the (i, j) will not vary throughout): $\{(1, 1)\}, \{(1, 2), (1, 3)\}, \{(2, 2), (3, 3)\}, \{(2, 3), (3, 2)\}$. The first group is all real, and the others are complex conjugates.

(a) The $(k, r) = (1, 1)$ term of (65) is

$$\frac{c_{11}^{(i,j)}}{2d_1} e^{d_1 t} (e^{d_1 s} - e^{-d_1 s}) = \frac{c_{11}^{(i,j)}}{d_1} e^{d_1 t} \sinh(d_1 s).$$

(b) The $(k, r) = (1, 2)$ and $(1, 3)$ terms, which are complex conjugates in (65) add up to:

$$\begin{aligned} &\frac{c_{12}^{(i,j)}}{d_1 + d_2} e^{d_1 t} (e^{d_2 s} - e^{-d_1 s}) + \frac{c_{13}^{(i,j)}}{d_1 + d_3} e^{d_1 t} (e^{d_3 s} - e^{-d_1 s}) \\ &= 2e^{d_1 t} \operatorname{Re} \left[\frac{c_{12}^{(i,j)}}{d_1 + d_2} (e^{d_2 s} - e^{-d_1 s}) \right]. \end{aligned}$$

(c) The $(k, r) = (2, 3)$ and $(3, 2)$ terms add up to (using $d_2 = a + ib$, $d_3 = a - ib$)

$$\begin{aligned} &\frac{c_{23}^{(i,j)}}{d_2 + d_3} e^{d_2 t} (e^{d_3 s} - e^{-d_2 s}) + \frac{c_{32}^{(i,j)}}{d_3 + d_2} e^{d_3 t} (e^{d_2 s} - e^{-d_3 s}) \\ &= \frac{1}{2 \operatorname{Re} d_2} 2 \operatorname{Re} \left[c_{23}^{(i,j)} e^{d_2 t} (e^{d_3 s} - e^{-d_2 s}) \right] \\ &= \frac{e^{a(t+s)}}{a} \operatorname{Re} \left[c_{23}^{(i,j)} (e^{ib(t+s)} - e^{ib(t-s)}) \right]. \end{aligned}$$

(d) The $(k, r) = (2, 2)$ and $(3, 3)$ terms add up to

$$\begin{aligned} & \frac{c_{22}^{(i,j)}}{2d_2} e^{d_2 t} (e^{d_2 s} - e^{-d_2 s}) + \frac{c_{33}^{(i,j)}}{2d_3} e^{d_3 t} (e^{d_3 s} - e^{-d_3 s}) \\ &= 2 \operatorname{Re} \left[\frac{c_{22}^{(i,j)}}{2d_2} e^{d_2 t} (e^{d_2 s} - e^{-d_2 s}) \right] \\ &= e^{a(t+s)} \operatorname{Re} \left[\frac{c_{22}^{(i,j)}}{d_2} e^{ib(t+s)} - e^{ib(t-s)} \right]. \end{aligned}$$

Combining all of the terms (a) – (d) we have

$$\begin{aligned} R(t, s)_{ij} &= \sum_{k,r=1}^3 \frac{c_{kr}^{(i,j)}}{d_k + d_r} e^{d_k t} (e^{d_r s} - e^{-d_k s}) \tag{66} \\ &= \frac{c_{11}^{(i,j)}}{d_1} e^{d_1 t} \sinh(d_1 s) + 2e^{d_1 t} \operatorname{Re} \left[\frac{c_{12}^{(i,j)}}{d_1 + d_2} (e^{d_2 s} - e^{-d_1 s}) \right] \\ &\quad + \frac{e^{a(t+s)}}{a} \operatorname{Re} \left[c_{23}^{(i,j)} (e^{ib(t+s)} - e^{ib(t-s)}) \right] + e^{a(t+s)} \operatorname{Re} \left[\frac{c_{22}^{(i,j)}}{d_2} (e^{ib(t+s)} - e^{ib(t-s)}) \right] \\ &= \frac{c_{11}^{(i,j)}}{d_1} e^{d_1 t} \sinh(d_1 s) - 2e^{d_1(t-s)} \operatorname{Re} \left[\frac{c_{12}^{(i,j)}}{d_1 + d_2} \right] + 2e^{d_1 t + a s} \operatorname{Re} \left[\frac{c_{12}^{(i,j)}}{d_1 + d_2} e^{ib s} \right] \\ &\quad + \frac{e^{a(t+s)}}{a} \operatorname{Re} \left[c_{23}^{(i,j)} (e^{ib(t+s)} - e^{ib(t-s)}) \right] + e^{a(t+s)} \operatorname{Re} \left[\frac{c_{22}^{(i,j)}}{d_2} (e^{ib(t+s)} - e^{ib(t-s)}) \right]. \end{aligned}$$

Thus, the first two terms are pure decay with decay exponent d_1 while the last two are oscillatory decay with decay exponent $\operatorname{Re} d_2 = a$ and frequency $\operatorname{Im} d_2 = b$.

Thus, the frequency of the oscillations in the autocovariance are dependent solely on the imaginary part of the complex pair of eigenvalues. The decay rate is given by a combination of the first eigenvalue (which is real) and the real part of the complex pair of eigenvalues. The coefficients are given by terms that can all be calculated from the properties of the original 3×3 matrix. Hence, if we do not need the coefficients for these terms, but just the decay and oscillation rates, all we need are the eigenvalues.

Another feature is that all of the autocovariance components oscillate and decay with the same parameters. We will examine below how the $b = \operatorname{Im} d_2$ varies with the parameters of the system.

7.5. The steady state variance

The steady state variance of the price is given by $\lim_{t \rightarrow \infty} R(t, t)_{11}$ in the expression (65) calculated above:

$$\begin{aligned} R(t, t)_{ij} &= \sum_{k,r=1}^3 \frac{c_{kr}^{(i,j)}}{d_k + d_r} (e^{(d_k + d_r)t} - 1), \\ \lim_{t \rightarrow \infty} R(t, t)_{ij} &= \sum_{k,r=1}^3 -\frac{c_{kr}^{(i,j)}}{d_k + d_r}. \end{aligned}$$

It can also be calculated from (57) above.

8. Deterministic examples with peaks

Before considering the stochastic model we discuss price extrema under several conditions when there is no randomness. Recall that liquidity price, L , is given by the ratio of total cash in a system divided by the number of shares. In addition to the trading price per share and the fundamental value, P_a , or any other assessment of intrinsic value, the definition of L provides another variable with dimensions of price per share. In a deterministic setting, mathematical studies have shown the importance of this concept in the magnitude of bubbles [3]. These were also confirmed by experimental studies [8] in which the peak price of the bubble was strongly influenced by L .

We consider prototype cases in which L and P_a are constants and show that any peak in prices will be in excess of the equilibrium price. Later we will study computationally the stochastic differential equations in which the difference between the deterministic oscillations and the stochastic is explored.

Suppose that $P(t)$ is strictly increasing on $[0, t_1)$ and $P'(t_1) = 0$, and that $P(t) \leq P_E$ for $t \leq t_1$. We claim that this yields a contradiction. Indeed, we note:

(i) From the integral formulation (33) we see that $\zeta_2(t_1) > \zeta_{2E}$ since

$$\frac{P_a - P(t)}{P_a} \leq \frac{P_a - P_E}{P_a} \text{ if } t \leq t_1.$$

(ii) From the integral formulation (34) and the assumption that $P'(t) > 0$ we obtain $\zeta_1(t_1) > 0 = \zeta_{1E}$.

Thus, we obtain from the definition, (32), the inequality $k(t_1) > k_E$ so that comparison with (40) yields

$$\frac{k(t_1)}{1 - k(t_1)} > \frac{k_E}{1 - k_E} = \frac{P_E}{L}.$$

The basic price equation then implies

$$P^{-1}P'(t_1) = \frac{k(t_1)}{1 - k(t_1)} \frac{L}{P(t_1)} - \frac{1 - k(t_1)}{k(t_1)} \frac{P(t_1)}{L},$$

so that if $P(t_1) \leq P_E$ then $P'(t_1) > 0$, so that one cannot have a maximum at $P(t_1) \leq P_E$. Hence, either $P(t)$ never attains a maximum, and increases asymptotically to P_E , or it satisfies $P'(t_1) = 0$ for $P > P_E$. In the latter case, P must have a maximum at t_1 since it ultimately reaches P_E as $t \rightarrow \infty$.

9. Numerical studies.

We report here on numerical studies of an example system (37), (38), (39) to demonstrate the findings made in earlier sections of the paper, namely the presence of autocovariance in price trajectories and the dependence of volatility on parameter values.

For a reference scenario we choose parameters as follows: $L = 1$, $P_a = 0.7$, $c_2 = 0.2$, and $q_2 = 10$. We choose the initial price, $P(0)$, below both L and P_a so that there is initial undervaluation, leading prices to move higher due to value based buying. We shall vary the parameters $c_1 \in [0, 12]$ and $q_1 \in [0, 0.8]$ to study the effect of momentum traders on the stability of the system and magnitude of fluctuations both during dynamical changes in price and in equilibrium.

The parameters are chosen so that the equilibrium is linearly stable with a pair of complex eigenvalues, which gives rise to damped oscillatory decay (see Figure 4 below). Thus in a deterministic regime with $\sigma = 0$, the price has several oscillations before approaching the equilibrium price, $P_E = 0.7119$. When we introduce a small $\sigma > 0$, the price exhibits oscillations indefinitely. The oscillations appear to have the same periodicity as the approach to equilibrium. The standard deviation as a function of time converges to a fixed value.

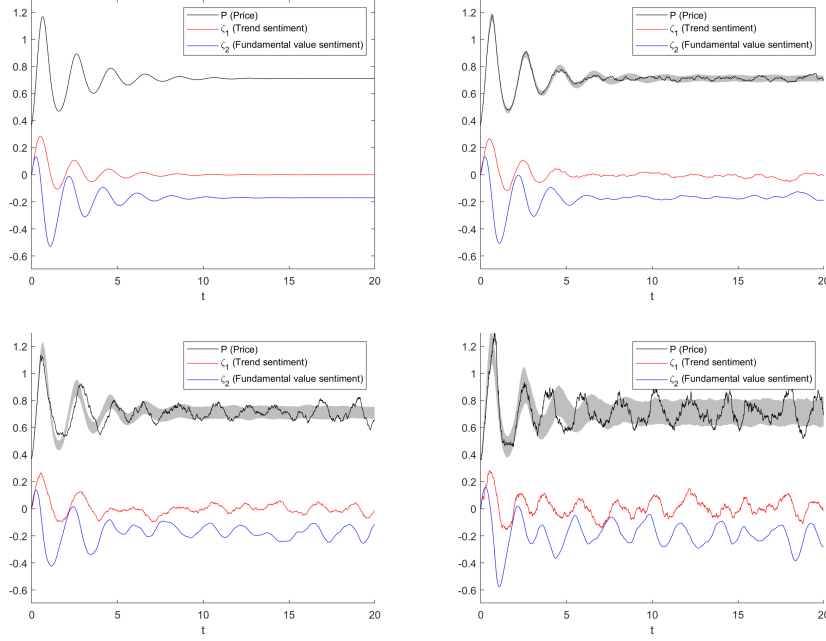


Figure 4. Dependence of the sample paths of the system on the strength of the stochastic contribution σ : $\sigma = 0$ (top left), $\sigma = 0.02$ (top right), $\sigma = 0.04$ (bottom left), $\sigma = 0.08$ (bottom right). Here $(L, P_a, c_1, q_1, c_2, q_2) = (1, 0.7, 1.2, 0.3, 0.2, 10)$, which corresponds to the equilibrium $P_E = 0.7119$. The initial state in each case is $(P(0), \zeta_1(0), \zeta_2(0)) = (0.3679, 0, 0)$. The gray area depicts the population behavior of the system by showing ± 1 standard deviation departure from expected price.

In the Figure 5 below, the initial conditions on the left correspond to price starting at the liquidity value while both sentiments start at zero, and the initial conditions on the right correspond to initial state equal to the equilibrium values. The autocovariance of the fluctuations in price is clearly visible in both figures. Note that the oscillations induced by stochasticity and trend persist regardless of the initial conditions, even when the system starts in equilibrium, and therefore are inherent component of the behavior of the system and not simply induced by the initial perturbation.

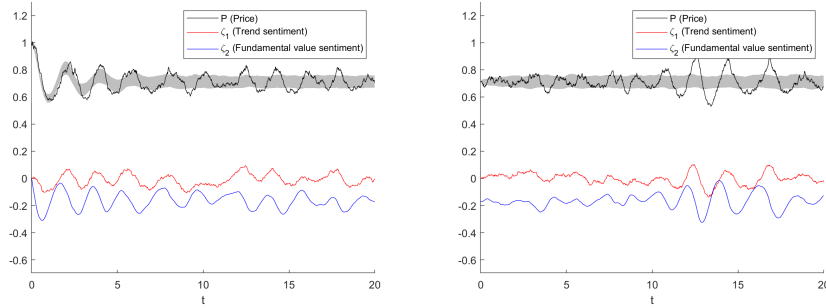


Figure 5. Dependence of the sample paths of the system on the initial state: $(P(0), \zeta_1(0), \zeta_2(0)) = (1, 0, 0)$ (left) and $(P(0), \zeta_1(0), \zeta_2(0)) = (P_E, \zeta_{1E}, \zeta_{2E}) = (0.7119, 0, -0.1699)$ (right). Here $(L, P_a, c_1, q_1, c_2, q_2, \sigma) = (1, 0.7, 1.2, 0.3, 0.2, 10, 0.04)$. The gray area depicts the population behavior of the system by showing ± 1 standard deviation departure from expected price.

A possible mechanism for this is that a small positive contribution from ΔW moves prices upward. If the time scale is small, i.e., $c_1 \gg 1$, and q_1 significant, then ζ_1 turns positive, while ζ_2 is small since the price is close to P_a .

This starts an uptrend. The random events continue, but the price momentum and positive ζ_2 are often enough to keep prices moving higher. Note that on this time scale there appear to be fewer fluctuations as it crosses the equilibrium value. In this region, while the random fluctuation terms (i.e., ΔW) continue, the trend effect is considerably stronger.

Figure 6 shows the autocorrelation of one sample path of the system. Since the system is time-independent, once it reaches an equilibrium, its trajectory becomes a stationary process and the autocorrelation $\rho(s, t)$ depends on the difference $t - s$ only. One can observe that the autocorrelation is consistent with sustained oscillations.

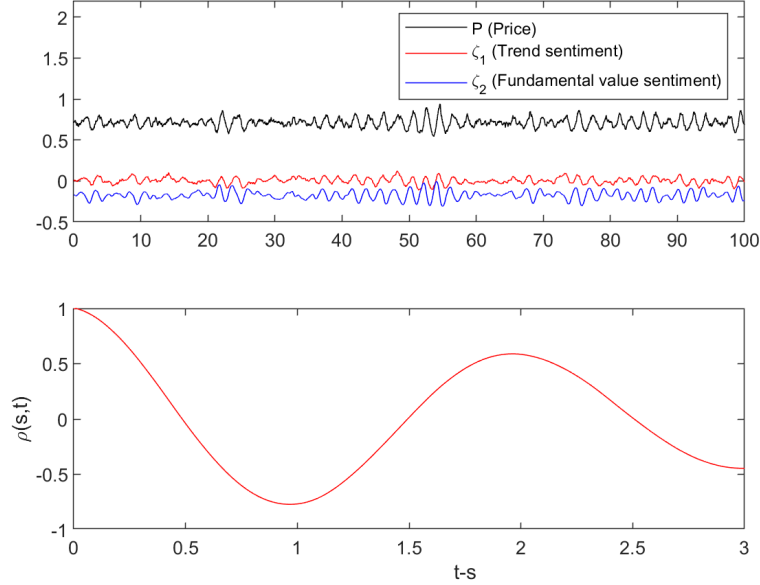


Figure 6. Autocorrelation (bottom) of a sample path (top) of the system near equilibrium. Here $(L, P_a, c_1, q_1, c_2, q_2, \sigma) = (1, 0.7, 1.2, 0.3, 0.2, 10, 0.04)$.

A key conclusion arising from the numerical results is that randomness triggers oscillations whose frequency is determined by the parameters of the linearized deterministic system, while the amplitude is largely governed by the magnitude of σ .

Obtaining an understanding of large oscillations can be further studied by using optimal control and Markov regime-switching models [32] and other control and timing methods [27].

The concept of stochastically induced oscillations can be considered in other contexts where a competition between an impulse and restraint. In our case the impulse is generated by seeing rising prices, while the restraint involves resisting the purchase of overvalued assets. It is possible that the model can be integrated with related methodologies in other disciplines, such as operation research techniques in neuroscience (see, for example, [23]).

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Appendix A. Differences in modeling asset prices.

A standard model for modeling stochastic asset prices, $P(t)$, is through the use of geometric Brownian motion,

$$P^{-1}dP = \mu dt + \sigma dW(t). \quad (\text{A.1})$$

Ito's formula for a stochastic process $dX = a dt + b dW$, where a, b are functions of t and ω , is expressed as

$$df(X(t), t) = \left(\frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} + \frac{1}{2} b^2 \frac{\partial^2 f}{\partial x^2} \right) dt + b \frac{\partial f}{\partial x} dW(t).$$

Applying this to (A.1) yields

$$d \log P(t) = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dW(t). \quad (\text{A.2})$$

Using Ito's formula with $f(x) := \log x$, one can show that a solution to this equation is

$$P(t) = P(0) e^{\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W(t)}. \quad (\text{A.3})$$

An alternative approach is to start with the stochastic process

$$P^{-1} dP = \left(\mu + \frac{1}{2} \sigma^2 \right) dt + \sigma dW(t), \quad (\text{A.4})$$

which is equivalent to

$$d \log P(t) = \mu dt + \sigma dW(t), \quad (\text{A.5})$$

with solution

$$P(t) = P(0) e^{\mu t + \sigma W(t)}. \quad (\text{A.6})$$

The two approaches represent the same growth in deterministic systems, but differ essentially by a term of order σ^2 when there is randomness. The difference is essentially due to the mechanism whereby the randomness arises.

Note that in the standard stochastic approach described above, one has, e.g., for constant μ and σ ,

$$\Delta P \cong \mu P \Delta t + \sigma P \Delta W$$

where $\Delta P(t) := P(t + \Delta t) - P(t)$ so that on a small time interval, the fractional change in price due to randomness is added onto the deterministic change, which seems intuitively reasonable.

Appendix B. Consistent initial conditions for ζ_1 and ζ_2 .

The equations for $\zeta_{1,2}$ were originally defined for time integrals that extended to $-\infty$. However, we usually start at $t = 0$ or some other finite time. One way to approach this is to assume

$$\begin{aligned} P(t) &= P(0) \quad \text{for } t \leq 0, \\ P_a(t) &= P_a(0) \quad \text{for } t \leq 0. \end{aligned}$$

We can then use the original integral formulations. Starting with ζ_1 we write

$$\zeta_1(t) := q_1 c_1 \int_0^t e^{-c_1(t-\tau)} \frac{1}{P(\tau)} \frac{dP(\tau)}{d\tau} d\tau.$$

Clearly, we also have $\zeta_1(0) = 0$ by substitution. Differentiating $\zeta_1(t)$ we obtain the same differential equation subject to this initial condition:

$$\frac{d\zeta_1}{dt} = c_1 q_1 \frac{1}{P} \frac{dP}{dt} - c_1 \zeta_1, \quad \zeta_1(0) = 0.$$

Similarly, for ζ_2 we use the definition above for $P_a(t)$ for $t \leq 0$, and write

$$\begin{aligned} \zeta_2(t) &:= q_2 c_2 \int_{-\infty}^0 e^{-c_2(t-\tau)} \frac{P_a(0) - P(0)}{P_a(0)} d\tau \\ &+ q_2 c_2 \int_0^t e^{-c_2(t-\tau)} \frac{P_a(\tau) - P(\tau)}{P_a(\tau)} d\tau. \end{aligned}$$

The first integral simplifies and we have

$$\zeta_2(t) = \zeta_2(0) + q_2 c_2 \int_0^t e^{-c_2(t-\tau)} \frac{P_a(\tau) - P(\tau)}{P_a(\tau)} d\tau$$
$$\zeta_2(0) = q_2 \frac{P_a(0) - P(0)}{P_a(0)} =: \zeta_{20}.$$

Once again, we can differentiate this integral expression and obtain the differential equation

$$\frac{d\zeta_2}{dt} = c_2 q_2 \frac{P_a - P}{P_a} - c_2 \zeta_2, \quad \zeta_2(0) = q_2 \frac{P_a(0) - P(0)}{P_a(0)}.$$

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