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Price equations with symmetric supply/demand; implications for fat tails

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HIGHLIGHTS

- A price equation using supply/demand is developed with key symmetry properties.
- The function of supply and demand can be linear or non-linear.
- This function and randomness in supply/demand determine the fat tails exponent.
- If supply, demand are normal, and the function linear, then relative price change falls off as x^{-2}.
- The falloff can be exponential if the function of supply, demand is logarithmic.

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ABSTRACT

Implementing a set of microeconomic criteria, we develop price dynamics equations using a function of demand/supply with key symmetry properties. The function of demand/supply can be linear or nonlinear. The type of function determines the nature of the tail of the distribution based on the randomness in the supply and demand. For example, if supply and demand are normally distributed, and the function is assumed to be linear, then the density of relative price change has behavior x^{-2} for large x (i.e., large deviations). The exponent approaches −1 if the function of supply and demand involves a large exponent. The falloff is exponential, i.e., e^{-x}, if the function of supply and demand is logarithmic.

1. Introduction

Equations for price dynamics have generally fallen into one of two categories: (a) continuum approaches for asset markets that assume infinite arbitrage and stochastics without directly addressing supply and demand, (b) discrete approaches that examine the micro-structure of supply and demand.

The latter approach that is used widely in the mathematical finance community is expressed (see e.g., Bachelier, 1900; Black and Scholes, 1973; Wilmott, 2013) in the continuum form as

$$P^{-1}dP = \mu dt + \sigma dW \tag{1}$$

where $P(t)$ is price at (continuous) time t, while W is Brownian motion, and μ and σ are the mean and standard deviation of the stochastic process. This approach marginalizes the issues involving supply and demand, modeling instead the price change as though (1) were an empirically observed phenomenon. While there is some empirical justification for this equation, there is large discrepancy between the implications for the frequency of unusual events (Fama, 1965; Kemp, 2011; Mandelbrot, 1962; Mandelbrot and Hudson, 2007; Taleb and Daniel, 2011; Taleb, 2005; Xavier, 2009; Gabaix et al., 2006; Daníelsson et al., 2013; Kirchler and Huber, 2007). In particular, if one measures σ for the S&P 500 then (1) would imply that the frequency of a 4% drop, for example, occurs about one in millions of days, instead of about 500 days, the observed frequency. This is a practical implication of the puzzle known as "fat tails" that refers to rare events occurring much more frequently than one might expect from classical results. More

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precisely, the density of relative price changes is observed to fall as a power law rather than exponentially.

The theoretical justification for Eq. (1) is also limited, and its widespread use is largely attributable to mathematical convenience (Champagnat et al., 2013; Caginalp and Caginalp, 2018).

On the other hand, the approach developed by economists, i.e., (b), often called excess demand, is expressed as

$$p_t - p_{t-1} = d - s, \tag{2}$$

Watson and Getz (1981), Henderson and Quandt (1980), Hirshleifer et al. (2005), Schneeweiss (1987) and Plott and Pogorelskiy (2016), with p_t as the price at discrete time, t, with supply, s, and demand, d, at time $t-1$. Eq. (2) must be regarded as a local equation that describes change at a particular set of values of d and s. Clearly, the price change will depend upon the magnitudes of d and s, and not just their differences. One can remedy this feature by normalizing $d-s$ by s, so that the right hand side of (2) is $(d-s)/s$. Similarly, the left hand side of (2) needs to be normalized, for example by dividing by p_{t-1}.

A third approach to price dynamics was built on this perspective to model an actively traded asset or commodity (see e.g. Caginalp and Balevonich, 1999). With active trading one can regard the buy/sell orders as flow. This led to the asset flow equations that were written in continuum form in 1990 (see Caginalp and Balevonich, 1999, and more recent works, e.g., Merdan and Alisen, 2011 and references therein). The price equation has the form

$$\tau_0 \frac{1}{P(t)} \frac{dP(t)}{dt} = \frac{D(t) - S(t)}{S(t)}, \tag{3}$$

Here, τ_0 is a time constant that also incorporates a constant rate factor that can be placed in the right hand side. The difference in the two approaches is due to fact that (1) assumes infinite arbitrage. This means that there is always capital that can take advantage of mispricing of assets. In this way the deviation from realistic value will be small and random.

2. A general symmetric model

Eq. (3) is of course a linearization (in D/S) since relative price change may depend nonlinearly on normalized excess demand. Another feature of the right hand side is that it is not symmetric with respect to supply and demand. This is not significant when supply and demand are approximately equal. However, as $D \to 0$ (with S fixed) we see that the right hand side approaches $-\infty$ but as $S \to 0$ (with D fixed) the right hand side approaches 1.

2.1. Basic requirements

One way to impose symmetry between D and S is to write in place of (3) the equation

$$\tau_0 \frac{1}{P(t)} \frac{dP(t)}{dt} = \frac{1}{2}\left(\frac{D}{S} - \frac{S}{D}\right). \tag{4}$$

Note that the two Eqs. (4) and (3) have the same value for the first term in the perturbation of $D = 1+\delta$ and $S = 1+\varepsilon$ about $\delta = \varepsilon = 0$. Eq. (4) is a basic model that satisfies a number of requirements for a price equation: (*i*) The price derivative vanishes when $D = S$ so that price does not change in equilibrium. When $D > S$, prices rise, and vice-versa. (*ii*) The roles of D and S are anti-symmetric, in the sense that $D/S - S/D = -(S/D - D/S)$. (*iii*) A small change in the positive direction for supply, S, has the same effect as a small change in the negative direction for demand, D. (*iv*) When $D \to \infty$ (with S fixed) the relative price change diverges to ∞; when $S \to \infty$ (with D fixed) it diverges to $-\infty$.

The Eq. (4) is a simple prototype exhibiting the features required for a price adjustment equation. We can consider a more general form by stipulating the requirements for a function $G : \mathbb{R}^+ \to \mathbb{R}$ so $G(D/S)$ replaces the right hand side of (4) i.e.,

$$\frac{d \log P(t)}{dt} = G\left(\frac{D(t)}{S(t)}\right) \tag{5}$$

2.2. Condition G

The function $G : \mathbb{R}^+ \to \mathbb{R}$ is required to be a twice differentiable function satisfying the following:

$$(i)\ \ G(1) = 0,\ (ii)\ G'(x) > 0 \text{ all } x \in \mathbb{R}^+,\ (iii)\ G(x) = -G\left(\frac{1}{x}\right),$$

$$(iv)\ \lim_{x\to\infty} xG'(x) = \infty \text{ and } \lim_{x\to 0+} xG'(x) = \infty.$$

$$(v)\ \left(xG'(x)\right)' \text{ is } \begin{cases} <0 & \text{if } x<1 \\ >0 & \text{if } x>1 \end{cases}.\ ///$$

These properties imply the following:

$$xG'(x) = \frac{1}{x} G'\left(\frac{1}{x}\right). \tag{6}$$

$$\lim_{x\to\infty} G(x) = \infty. \tag{7}$$

The first of these follows from differentiating (*iii*). To prove (7) observe that with $C := G'(1) > 0$ and $x > 1$, condition (iv) implies

$G'(x) > Cx^{-1}$ *if* $x > 1$.

Integrating, we obtain, since $G(1) = 0$,

$$G(x) \geq C \int_1^x s^{-1} ds = C \log x.$$

and so $G(x) \to \infty$ as $x \to \infty$.

Note that since $G'(x) > 0$, we can always normalize so that $G'(1) = 1$ and incorporate the constant into the time variable in the price equation (5).

Conditions (*i*) − (*iii*) are basic requirements for a symmetric price function, while (iv) and (v) are useful symmetry properties for construction of stochastic equations.

2.3. Examples of functions that satisfy condition G

In addition to the function in (4) one can readily verify that the following functions also satisfy this condition:

(*i*) $G(x) = x^q - x^{-q}$ for $q > 0$;

(*ii*) $G(x) = \left(x - x^{-1}\right)^q$ for q an odd positive integer.

3. Fat tails and demand, supply quotient

The price dynamics equations (3) and (4) both involve the quotient D/S. It is reasonable to assume, based on the Central Limit Theorem, that given many agents placing buy and sell orders into the market, the distribution of orders at the market price will be normal (Gaussian). The question of the tail of the distribution then entails the study of a quotient of normals (see, e.g., Dacorogna and Pictet, 1997; Díaz-Francés and Rubio, 2013; Marsaglia, 2006; Tong, 1990). Generally, we expect that S and D will have a negative correlation, and in an idealized setting, they will have correlation -1 as random events that increase supply tend to decrease demand. Earlier work (Caginalp and Caginalp, 2018) on this issue using (4) has produced the result that if D and S are described by a bivariate normal distribution, the density, $f(x)$, will falloff with exponent -2, i.e., $f(x) \sim x^{-2}$ for large x. Moreover, a very simple formula was found for the density in the special case when the correlation between D and S is -1 (anti-correlation). A key theorem proved in Caginalp and Caginalp (2018), on which subsequent results will be based, is stated below.

3.1. Quotient of normals

Theorem. *If $R := D/S$ where D and S are bivariate normal random variables with strictly positive means and variances, $\mu_1, \mu_2, \sigma_1, \sigma_2$, and correlation $-1 < \rho < 1$, then the density of R, falls off as*

$$f_R(x) \sim f_0 x^{-2},$$

where f_0 depends on $\mu_1, \mu_2, \sigma_1, \sigma_2$ and ρ.

For $\rho = -1$ one has the exact expression (for $x \neq -\sigma_1/\sigma_2$)

$$f_R(x) = \frac{\mu_1\sigma_2 + \mu_2\sigma_1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}\left(\frac{\mu_2 x - \mu_1}{\sigma_2 x + \sigma_1}\right)^2}}{(\sigma_2 x + \sigma_1)^2}$$

and $f_R(-\sigma_1/\sigma_2) = 0$.

3.2. Tail behavior

We show below that the $f(x) \sim x^{-2}$ behavior is also valid for the symmetric model (4)

Lemma. *Let R be a random variable with density f_R such that $f_R(x) \sim x^{-p}$ with $p > 1$ for large x. For $q > 0$ (so that $1 - p - q < 0$) one has:*

(a) $f_{R^q}(x) \sim x^{\frac{1-p}{q}-1}$, (b) $f_{R^q - R^{-q}}(x) \sim x^{\frac{1-p}{q}-1}$ and (c) $f_{(R-R^{-1})^q}(x) \sim x^{\frac{1-p}{q}-1}$.

Proof. (a) We note $P\{R^q \le x, R > 0\} = P\left\{R \le x^{1/q}, R > 0\right\} = \int_0^{x^{1/q}} f(s)\,ds$.

The density for R^q, denoted f_{R^q}, is given by

$$f_{R^q}(x) = \partial_x P\left\{R^q \le x, R > 0\right\}, \text{ i.e.,}$$

$$f_{R^q}(x) = \partial_x F\left(x^{1/q}\right) = F'\left(x^{1/q}\right)\frac{1}{q}x^{1/q-1}$$

$$= f\left(x^{1/q}\right)\frac{1}{q}x^{1/q-1} \sim \left(x^{1/q}\right)^{-p} x^{1/q-1} = x^{\frac{1-p}{q}-1}.$$

The remaining parts of the theorem can be obtained by similar methods. ///

Note that the calculations are similar for $0 < p < 1$ (provided we impose $1-p-q < 0$), but the mean does not exist in this range.

Theorem. *Let D and S be bivariate normals with correlation $\rho < 1$, and let $R := D/S$. Then the density of the functions $G_1(R) = R^q - R^{-q}$ and $G_2(R) = (R - 1/R)^q$ both satisfy the large x behavior*

$$f(x) \sim f_0 x^{-1-1/q}.$$

Remark. When $\rho = -1$, we have the exact density for $f_{(D/S)^q}$

$$f_{(D/S)^q}(x) = f\left(x^{1/q}\right)\frac{1}{q}x^{1/q-1}$$

$$= \frac{\mu_1\sigma_2 + \mu_2\sigma_1}{\sqrt{2\pi}q} \frac{e^{-\frac{1}{2}\left(\frac{\mu_2 x^{1/q} - \mu_1}{\sigma_2 x^{1/q} + \sigma_1}\right)^2}}{\left(\sigma_2 x^{1/q} + \sigma_1\right)^2} x^{1/q-1}$$

which, of course, has the $x^{-1-1/q}$ decay.

3.3. Limits

Consider the case $R = D/S$ with D and S normal with arbitrary correlation $\rho < 1$. As noted above, $f_R(x) \sim x^{-2}$, the decay for R^q is $f_{R^q}(x) \sim x^{-1-1/q}$. Recall that the density for both $R^q - R^{-q}$ and $\left(R - R^{-1}\right)^q$ falls off with the same exponent as R^q. Under these conditions we note the following limits.

As $q \to \infty$ the decay goes to x^{-1}. Note that large q means that a change in the demand/supply makes a larger change in relative price, i.e., $P^{-1}dP/dt$. Hence, it appears that, for any correlation, ρ, between D and S one has that as q increases (i.e., prices are very sensitive to supply/demand changes), the decay exponent moves closer to x^{-1}.

As $q \to 0$, i.e., prices do not vary much as supply/demand changes, so that the exponent of $x^{-1-1/q}$ diverges to $-\infty$.

3.4. Logarithmic functions

This last limit suggests that examining $\log(D/S)$ may yield an exponential decay. I.e. we use the equation

$$P^{-1}\frac{dP}{dt} = \log(D/S).$$

Letting $R := D/S$, we know that if D, S are normal, then $f_R(x) \sim x^{-2}$ so that

$$P\{\log R \le x, R > 0\} = P\left\{R \le e^x, R > 0\right\} = \int_0^{e^x} f(s)\,ds.$$

Taking the derivative, we have then

$$f_{\log R}(x) = \partial_x P\{\log R \le x, R > 0\}$$

$$= \partial_x \int_0^{e^x} f(s)\,ds = e^x f\left(e^x\right) \sim e^x\left(e^x\right)^{-2} = e^{-x}.$$

Hence, if D and S are bivariate normal with correlation less than 1, and the relative change in price is proportional to $\log D/S$, then the relative price change has a density that falls off as e^{-x}. Also, if p^{-1} is an odd positive integer, then $G(D/S) = [\log(D/S)]^{1/p}$ yields a decay of $x^{p-1}e^{-x^p}$.

Note that $G(x) = (\log x)^q$, with q an odd integer greater than 1, satisfies Condition G, while $G(x) = \log x$ satisfies only the conditions $(i) - (iii)$. In place of (iv) and (v) it satisfies the symmetry condition $xG'(x) = 1$ for all $x \in \mathbb{R}^+$.

4. Conclusion

The results above establish a link between the relative price changes and the exponent of the fat tails through (5) One of the problems in empirically estimating the likelihood of rare events is that one would need a very long time history, but this often takes us back to a different time period that may be irrelevant. We can use our results to estimate the exponents by examining a much smaller data set and fitting G. To be precise, choose small δt and Δt so that

$$\delta t \ll \Delta t.$$

We approximate the left hand side of (5) as $P^{-1}\delta P/\delta t$ and obtain statistics for all intervals of δt within $(t, t + \Delta t)$. Then we can determine the function G that best fits the data. Using the results of Sections 3.2 and 3.4 one can then ascertain whether the decay in the density falls off exponentially (i.e., G is a logarithmic function) or with fat tails (with a specific exponent).

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