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EFFECTS OF WHITE NOISE IN MULTISTABLE DYNAMICS

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Abstract. We study the invariant measure of multistable dynamics under the influence of white noise. We show that the invariant measure exists and in the limit of vanishing white noise, the invariant measure approaches a Dirac type measure concentrated at the most stable equilibria if fluctuations are uniform; however, a lesser stable equilibrium may be selected by the fluctuation if its ability to fluctuate is sufficiently smaller than other stable equilibria. Certain related mathematical issues are also addressed.

1. **Introduction.** In the theory of ordinary differential equations, if an initial data is in the attraction basin of a stable equilibrium, the corresponding trajectory approaches the equilibrium. However, under the introduction of white noise, no matter how small it is, almost all trajectories oscillate among the stable equilibria infinite many times. As a practical application, this can be used to describe patterns in physics, chemistry, and biology. Indeed, it is commonplace that fluctuations give rise to many interesting phenomena; see for example, a recent experimental work of Chang, Oh, Ingber, and Huang [5] analyzing a multistep dynamics of mammalian cell differentiation with multistability, molecular collisions in gasses and liquids [13] and electronic fluctuations in solids [15]. It has been long observed that external noise can induce a phase transition; see for example, Bulsara, Schieve, and Gragg [3], Porrá, Masoliver and Lindenberg [19, 20], Schimansky-Geier, Hesse, and Zülick [23, 24], Xiao, Yan, and Zhang [31], Zhang, Cao, and Wu [32], de Rueda, Izús and Borzi [26], Weidlich and Grabert [29], and references therein. It is well-known that white noise best models the fluctuations generated by microscopic effects in a homogeneous physical system. In the case of Allen–Cahn equation [1] for bistable dynamics describing phase transition in binary alloys, interesting phenomena induced by white noise have been studied by Brassesco, De Masi, and

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Presutti [4], Erbar [8], Funaki [9, 10], Da Prato and J. Zabczyk [7], Fatkullin and Vanden-Eijnden [11], Katsoulakis, Kossioris, and Lakkis [14], Liu [16], Reznikoff and G. Vanden-Eijnden [21], Weber [27, 28], and others.

Mathematically a prototype bistable dynamics is

$$\frac{du}{dt} = 4(u-\alpha)(1-u^2) \quad \text{or} \quad du = 4(u-\alpha)(1-u^2)dt \tag{1}$$

where $\alpha \in [0,1)$ is a constant and $u \equiv 1$ and $u \equiv -1$ are two stable equilibria. In the context of phase transition, the stability of an equilibrium is described by the smallness of the potential $P(u) = 4\int(u-\alpha)(u^2-1)du$. In particular, when $\alpha \in (0,1)$, the equilibrium $u \equiv -1$ is regarded as more stable than $u \equiv 1$ since $P(-1) < P(1)$. When $\alpha = 0$, $P(u) = (u^2-1)^2$ is called a balanced double-well potential.

In this paper we investigate the effect of white noise on multistable dynamics. Replacing the bistable forcing $4(u-\alpha)(1-u^2)$ by a generic multistable forcing (or drift) $b(u)$, we consider the dynamics with white noise modeled by the stochastic differential equation (sde)

$$du_t = b(u_t)dt + \varepsilon\,\sigma(u_t)dW_t \tag{2}$$

where $\{W_t\}_{t\geqslant 0}$ is the Wiener process (Brownian motion), ε is a positive constant measuring the size of fluctuation, and $\sigma(\cdot)$ is a non-negative function modeling the dependence of fluctuation on system state. By a standard theory of sde (e.g. [25]), under certain technical conditions on σ and b, (2) subject to an initial condition admits a unique solution and the probability density $\rho = \rho(u,t)$ of the random variable u_t is the solution of the following Fokker-Planck or forward Kolmogorov equation

$$\begin{cases} \partial_t\rho = \partial_u\big\{\partial_u\big[\tfrac{1}{2}\,\varepsilon^2\sigma^2\rho\big] - b\rho\big\} & \text{on } \mathbb{R}\times(0,\infty), \\ \rho(\cdot,0) = \rho_0(\cdot) & \text{on } \mathbb{R}\times\{0\} \end{cases} \tag{3}$$

where ∂_t and ∂_u are partial derivatives and ρ_0 is the probability density function (pdf) of u_0. Ryter [22] studies exact solutions to the Fokker-Planck equation for the special case of exponential decreasing distributions. Chow et. al. [6] show that there is a connection (but also substantial differences) between the system of ordinary differential equations and the classical Fokker-Planck equation on Euclidean spaces. Furthermore, each of these systems of ordinary differential equations is a gradient flow for the free energy functional on a Riemanian manifold.

We expect that the system eventually achieve a stochastic equilibrium in the sense that

$$\lim_{t\to\infty}\rho(\cdot,t) = \rho^*(\cdot). \tag{4}$$

When ρ^* is non-trivial, it must be a pdf and is well-known as the **invariant measure** which describes the observed distribution density after a certain amount of initiation time that is used to get rid of the initial effect. If a non-trivial ρ^* exists then it must be a solution of the following linear second order ordinary differential equation (ode) subject to the pdf conditions:

$$\begin{cases} \big\{\big[\tfrac{1}{2}\,\varepsilon^2\sigma^2\rho^*\big]' - b\rho^*\big\}' = 0 \quad \text{on } \mathbb{R}, \\ \rho^* \geqslant 0 \text{ on } \mathbb{R}, \quad \int_{\mathbb{R}}\rho^*(u)du = 1. \end{cases} \tag{5}$$

Suppose there exists a unique solution ρ^* and that (4) holds. Then we conclude that fluctuation eventually drives a multistable system into an invariant state that does not depend on any initial setup. This is totally different from the ode dynamics (1) whose long time behavior of the solution crucially depends on the initial condition. This is indeed one of the main reason that multistable systems are studied with fluctuations.

In this paper, we establish the existence of a unique invariant measure and analyze its asymptotic limit as the white noise vanishes (i.e. as $\varepsilon \searrow 0$). The limit is shown to be Dirac measures. When σ is a constant, the Dirac measure is concentrated on the most stable equilibria of the potential

$$P(u) := -\int b(u)du.$$

When σ is biased, the measure may support on those lesser stable equilibria but with smaller σ. The overall selection is the most stable equilibria of the effective potential Q defined by

$$Q(u) = -\int \frac{b(u)}{\sigma^2(u)}du.$$

Observe that Q shares the same set of equilibria and set of stable equilibria as P, but with different stability. The smaller the σ, the more stable the stable equilibrium of Q. We shall also address a few fundamental mathematical issues related to the degenerate case when σ is not everywhere positive and to the well-posedness of (3).

The connection between the stochastic differential equations in high space dimension and the Fokker-Planck equation has been studied by Arnold, Markowich, Toscani and Unterreiter [2] and Markowich and Villiani [18] for the case b defined as the gradient of a confining potential, V, and subject to initial conditions that are proportional to (L^2 function multiplied by) the unique equilibrium state which is the Gibbs measure whose density function is proportional to e^{-V}. In particular, they studied the rate of convergence to equilibrium. Our new contribution to this direction of research is that we considered the most general initial condition: the initial data is a probability measure. Unlike earlier work, our proof of the convergence does not rely on the classical method of energy estimates together with Sobolev's imbedding.

White noise has been studies in the context of the Allen-Cahn equation which is a diffusion equation augmented by a cubic reaction term. This describes an interface between two phases with the same energy. Funaki [9] (in two dimensional space) and later Weber [27] (in arbitrary dimension) have studied the Allen-Cahn equation with noise for short times. The key result is that in the sharp interface limit, the solutions move according to motion by mean curvature with an additional stochastic forcing. See also Lions and Souganides [17]. The Allen-Cahn equation has a sharp interface limit that is described by the motion by mean curvature. Hence, the related problem involves noise injected directly into the geometric motion by mean curvature equation. This has been studied by Yip [30]. Recent work by Erbar [8] studies the low noise limit for the invariant measure of a multi-dimensional stochastic Allen-Cahn equation.

The rest of the paper is organized as follows. In §2 we establish the existence of a unique solution of (5) under the following regularity, non-degeneracy, and stability conditions:

$$\sigma, b \in C(\mathbb{R}), \varepsilon \in \mathbb{R}; \quad \varepsilon > 0,\ \sigma(\cdot) > 0; \quad \overline{\lim_{u\to\infty}}\ b(u) < 0 < \underline{\lim}_{u\to-\infty}\ b(u). \tag{6}$$

In §3 we convert (3) to its classically equivalent version for the cumulative distribution function (cdf) to establish its well-posedness under the conditions only listed in (6). In §4 we show that every solution of (3) with ρ_0 being a pdf approaches ρ^* as $t \to \infty$. In §5 we study the asymptotic limit of the invariant measure as the size ε of the white noise approaches zero.

Finally, in §6 we consider the case when σ is not everywhere positive. We regard the invariant measure as the limit of invariant measures with σ replaced by an approximation sequence of positive functions. We show that if σ vanishes at a set $\mathcal{S}$ of (non-degenerate) stable equilibria, then for any probability measure μ supported on $\mathcal{S}$, there exists a sequence of positive functions approximating σ uniformly such that the corresponding invariant measures approach μ. On the other hand, if σ vanishes only outside of all stable equilibria, the limit, although highly non-trivial, does not depend on the approximation in general, so an invariant measure can be uniquely defined.

In a subsequent paper, we shall investigate the asymptotic behavior, as $\varepsilon \searrow 0$, of the expected escape time, i.e., the expected length of time interval during which the solution of the sde stays in the attraction basin (in ode sense) of a stable state.

2. The invariant measure.

In this section we prove the following:

Theorem 2.1. *Assume that the regularity, non-degeneracy, and stability conditions listed in (6) hold. Then (5) admits a unique solution and the solution is given by*

$$\rho^*(u) = \frac{1}{Z}\,\frac{e^{q(u)}}{\sigma^2(u)}, \qquad q(u) := \int_0^u \frac{2b(x)}{\varepsilon^2\sigma^2(x)}dx, \qquad Z := \int_{\mathbb{R}} \frac{e^{q(x)}}{\sigma^2(x)}dx. \tag{7}$$

Proof. **Existence.** We show that ρ^* defined in (7) is a solution. By the positivity assumption $\varepsilon, \sigma > 0$ and the regularity assumption $\sigma, b \in C(\mathbb{R})$, the function q is well-defined. To see that ρ^* is a pdf, we need only show that Z is finite. For this we use the stability assumption $\overline{\lim}_{u\to\infty} b(u) < 0$ and $\underline{\lim}_{u\to-\infty} b(u) > 0$. There exist positive constants η and M such that $b(u) \leqslant -\eta$ for every $u \geqslant M$ and $b(u) \geqslant \eta$ for every $u \leqslant -M$. It then follows that

$$\int_M^\infty \frac{e^{q(x)}}{\sigma^2(x)}dx \leqslant \int_M^\infty \frac{-2b(x)}{2\eta}\,\frac{e^{q(x)}}{\sigma^2(x)}dx = \int_M^\infty \frac{\varepsilon^2 q'(x)e^{q(x)}}{-2\eta}dx \leqslant \frac{\varepsilon^2 e^{q(M)}}{2\eta}.$$

Similarly, $\int_{-\infty}^{-M} \sigma^{-2}(x)e^{q(x)}dx$ is also finite. Thus, ρ^* defined in (7) is a pdf. By differentiation, one can verify that ρ^* satisfies (5) and therefore is a solution of (5).
Uniqueness. Let ρ^* be a generic solution of (5). The ode in (5) can be solved by an integration followed by another integration with integrating factor e^{-q}, giving the general solution

$$\rho^*(u) = \frac{e^{q(u)}}{\sigma^2(u)}\Big(C_1 \int_0^u e^{-q(x)}\,dx + C_2\Big)$$

with integration constants C_1 and C_2. For such ρ^* to be non-negative, C_1 must be 0. Indeed,

$$\lim_{u\to\pm\infty} \int_0^u e^{-q(x)}dx = \int_0^{\pm M} e^{-q(x)}dx + \lim_{u\to\pm\infty} \int_{\pm M}^u e^{-q(x)}dx = \pm\infty$$

since $q(x) \leqslant q(M)$ for $x \geqslant M$ (as $b < 0$ in $[M, \infty)$) and $q(x) \leqslant q(-M)$ for $x \leqslant -M$. Thus, $C_1 = 0$. The condition $\int_{\mathbb{R}} \rho^*(x)dx = 1$ then gives $C_2 = 1/Z$. This completes the proof. □

Remark 1. In high space dimension, the equation for ρ^* is

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big(\frac{1}{2}\sum_{j=1}^{n}\frac{\partial(\sigma_{ij}\rho^*)}{\partial x_j} - b_i\rho^*\Big) = 0.$$

In the case $\sigma_{ij} = \delta_{ij}$ and $b_i = -\partial_{x_i} V$, the physically relevant solution is the Gibbs measure given by $\rho^* = e^{-V}/\int_{\mathbb{R}^n} e^{-V}dx$. See, for example, Markowich and Villani [18] for related analysis.

3. **Well-posedness of (3).** Under conditions only listed in (6), problem (3) may not admit a classical solution, i.e., a solution such that all derivatives appeared in (3) exist in the classical sense and the differential equation in (3) is satisfied. Hence, a certain notion of weak solutions is needed. From the probabilistic point of view and also to treat measure type initial values, it is natural to convert (3) to the cumulative distribution functions (cdf) defined by

$$f(u,t) := \int_{-\infty}^{u} \rho(x,t)dx, \quad f_0(u) := \mathbb{P}(\{u_0 < u\}), \quad f^*(u) := \int_{-\infty}^{u} \rho^*(x)dx.$$

Here $\mathbb{P}$ stands for the probability. Setting $a = \varepsilon^2\sigma^2/2$, one can derive formally that f satisfies

$$\begin{cases} \mathcal{L}f := \partial_t f - \partial_u(a\partial_u f) + b\,\partial_u f = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ \lim_{t\searrow 0} f(u,t) = f_0(u) & \text{a.e. } u\in\mathbb{R}, \\ \lim_{u\to-\infty} f(u,t) = 0,\ \lim_{u\to\infty} f(u,t) = 1 & \text{locally uniformly in } t\in[0,\infty). \end{cases} \tag{8}$$

We define a solution of (8) as follows: f IS CALLED A CLASSICAL SOLUTION OF (8) if

$$f,\ \partial_u f,\ \partial_t f,\ \partial_u(a\partial_u f) \in C(\mathbb{R}\times(0,\infty)), \qquad \partial_u f \geqslant 0 \tag{9}$$

and each equation in (8) is satisfied. Also, when f_0 is continuous, we require $f \in C(\mathbb{R}\times[0,\infty))$. We say that ρ IS A WEAK SOLUTION OF (3) if $\rho = \partial_u f$ and f is a classical solution of (8).

Theorem 3.1. *Assume that ε, σ, and b satisfy (6) and $a = \varepsilon^2\sigma^2/2$. Then for each given cumulative distribution function f_0, (8) admits a unique classical solution.*

The classical theory of linear parabolic pde of divergence form (e.g. [12]) involves weak solutions in Sobolev spaces and does not seem to be enough to provide the stated assertion of the theorem. There are two concerns: (i) the regularity (9) is in general not true in high space dimensions, and (ii) the asymptotic behavior of the solution as $u \to \pm\infty$ depends on behavior of b; that is, the stability property of b is crucial in the analysis since without this property, drifted Brownian particles could escape from $\pm\infty$ in finite time, so the solution may not be a cdf.

Proof of Theorem 3.1. We divide the proof into four steps. In the first step we establish the existence of a weak solution of $\mathcal{L}f = 0$ in $\mathbb{R}\times(0,\infty)$ with initial condition $f = f_0$ on $\mathbb{R}\times\{0\}$. In the second step we show that the weak solution satisfies the boundary condition at $u = \pm\infty$. In the third step we establish the regularity (9). Finally, we prove uniqueness.

1. The existence of a weak solution is standard so we keep it brief. Let $\{(a^n, b^n)\}_{n=1}^{\infty}$, where $a^n > 0$, be smooth functions that approach (a, b) locally uniformly in $\mathbb{R}$ as $n \to \infty$. Let $\{f_0^n\}_{n=1}^{\infty}$ be smooth increasing functions such that

$f_0^n(-n)=0$, $f_0^n(n)=1$, and $f_0^n \to f_0$ a.e. in $\mathbb{R}$ as $n\to\infty$. Let f^n be the smooth solution of the initial boundary value problem

$$\begin{cases} \mathcal{L}^n f^n := \partial_t f^n - \partial_u(a^n\partial_u f^n) + b^n\partial_u f^n = 0 & \text{in } (-n,n)\times(0,\infty),\\ f^n(-n,t)=0, \quad f^n(n,t)=1 & \forall\, t>0,\\ f^n(u,0)=f_0^n(u) & \forall\, u\in[-n,n]. \end{cases}$$

The existence of a unique smooth solution f^n follows from the classical theory of parabolic pde [12]. By the maximum principle, $0\leqslant f^n\leqslant 1$ on $[-n,n]\times[0,\infty)$, so $f^n(-n,t)=0$ is the global minimum, $f^n(n,t)=1$ is the global maximum, and $\partial_u f^n(\pm n,t)>0$ for every $t>0$. By the maximum principle again, $\partial_u f^n>0$ in $[-n,n]\times(0,\infty)$. Standard local regularity estimates (c.f. Step **3** below) allows us to pass to a limit to obtain a weak (in the sense of distribution) solution of $\mathcal{L}f=0$ in $\mathbb{R}\times(0,\infty)$ with initial condition $f(\cdot,0)=f_0$. The solution has the property

$$\partial_u f\in L^2_{\rm loc}(\mathbb{R}\times[0,\infty)),\qquad \partial_u f\geqslant 0,\qquad 0\leqslant f\leqslant 1.$$

2. We show that the solution obtained from Step **1** has the desired limit as $u\to\pm\infty$. It is in this step that the stability assumption on b is needed, by means of using the invariant measure.

Let η be any small positive constant. Since f_0 is a cdf, there exist $x_\eta<0$ and $y_\eta>0$ such that

$$f_0(u)\leqslant\eta \ \ \forall\, u\leqslant x_\eta,\qquad\qquad f_0(u)\geqslant 1-\eta \ \ \forall\, u\geqslant y_\eta.$$

Also, since $0\leqslant f_0\leqslant 1$, for every $u\in\mathbb{R}$ we have

$$\eta+\frac{f_n^*(u)}{f_n^*(x_\eta)} =: f_1^\eta(u)\geqslant f_0(u)\geqslant f_2^\eta(u) := 1-\eta-\frac{1-f_n^*(u)}{1-f_n^*(y_\eta)}$$

where f_n^* is the analog of f^*, the cdf of the invariant measure ρ^* defined in the previous section with (a,b) replaced by (a^n,b^n). Note that $\mathcal{L}^n f_1^\eta=0$ and $\mathcal{L}^n f_2^\eta=0$ on $\mathbb{R}\times[0,\infty)$. Also, when $n\geqslant\max\{|x_\eta|,|y_\eta|\}$, $f_1^\eta\geqslant f^n\geqslant f_2^\eta$ on the parabolic boundary of $(-n,n)\times[0,\infty)$. Hence, applying the comparison principle on $[-n,n]\times[0,\infty)$ we derive that

$$f_2^\eta(u)\leqslant f^n(u,t)\leqslant f_1^\eta(u)\quad \forall\, u\in[-n,n],\ t>0.$$

Passing to the limit, we see that

$$1-\eta-\frac{1-f^*(u)}{1-f^*(y_\eta)}\leqslant f(u,t)\leqslant \eta+\frac{f^*(u)}{f^*(x_\eta)}\quad \forall\, u\in\mathbb{R}, t\in[0,\infty).$$

Since $0\leqslant f\leqslant 1$, first sending $u\to\pm\infty$ and then $\eta\searrow 0$ we conclude that

$$\lim_{u\to-\infty}\sup_{t\in[0,\infty)}|f(u,t)|=0,\qquad \lim_{u\to\infty}\sup_{t\in[0,\infty)}|f(u,t)-1|=0. \tag{10}$$

3. Next we establish the needed regularity (9) for the solution obtained in Step **1.** We use energy estimates. For notational simplicity, we omit the superscript n so in the following energy estimates, (f,a,b) is indeed meant to be (f^n,a^n,b^n) with $n>R+2$.

Let $R\geqslant 1$ be any constant and let ζ be a cut-off function:

$$\zeta\in C^\infty(\mathbb{R}),\quad 0\leqslant\zeta,|\zeta'|\leqslant 1 \text{ on } \mathbb{R},\quad \zeta=1 \text{ on } [-R,R],\quad \zeta(u)=0 \text{ if } |u|>R+2.$$

For any non-negative integer k, integrating $\zeta^{4k+2}(u)[\partial_t^k f][\partial_t^k \mathcal{L}^n f] = 0$ over $\mathbb{R}\times\{t\}$ and using integration by parts we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \zeta^{4k+2}|\partial_t^k f|^2 + \int_{\mathbb{R}} a\zeta^{4k+2}|\partial_t^k\partial_u f|^2 \\ = \ & -\int_{\mathbb{R}}\Big[(4k+2)a\zeta^{4k+1}\zeta' + b\zeta^{4k+2}\Big](\partial_t^k f)(\partial_t^k\partial_u f) \\ \leqslant \ & \frac{1}{2}\int_{\mathbb{R}} a\zeta^{4k+2}|\partial_t^k\partial_u f|^2 + \int_{\mathbb{R}}\Big[(4k+2)^2 a + \frac{b^2}{a}\Big]\zeta^{4k}|\partial_t^k f|^2 \end{aligned}$$

by Cauchy inequality. Multiplying the resulting inequality by $2t^{2k}$ we obtain

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} t^{2k}\zeta^{4k+2}|\partial_t^k f|^2 + \int_{\mathbb{R}} at^{2k}\zeta^{4k+2}|\partial_t^k\partial_u f|^2 \\ \leqslant \ & \int_{\mathbb{R}}\Big[2(4k+2)^2 at + \frac{2b^2 t}{a} + 2k\Big]t^{2k-1}\zeta^{4k}|\partial_t^k f|^2. \end{aligned}$$

Similarly, integrating $2t^{2k+1}\zeta^{4k+4}(\partial_t^{k+1} f)(\partial_t^k\mathcal{L}^n f) = 0$ over $\mathbb{R}\times\{t\}$ we obtain

$$\begin{aligned} &\int_{\mathbb{R}} t^{2k+1}\zeta^{4k+4}|\partial_t^{k+1} f|^2 + \frac{d}{dt}\int_{\mathbb{R}} at^{2k+1}\zeta^{4k+4}|\partial_t^k\partial_u f|^2 \\ \leqslant \ & \int_{\mathbb{R}}\Big[2(4k+4)^2 at + \frac{2b^2 t}{a} + 2k+1\Big]at^{2k}\zeta^{4k+2}|\partial_t^k\partial_u f|^2. \end{aligned}$$

Alternately integrating these two inequalities with $k = 0, 1$ over $[0, T]$ we obtain

$$\int_{\mathbb{R}} \zeta^2 f^2(u,T)du + \int_0^T\int_{\mathbb{R}} a\zeta^2|\partial_u f|^2 dudt \leqslant \int_0^T\int_{\mathbb{R}}\Big(8a + \frac{2b^2}{a}\Big)f^2 dudt + \int_{\mathbb{R}}\zeta^2 f_0^2 du,$$

$$T\int_{\mathbb{R}} a\zeta^4|\partial_u f|^2 du + \int_0^T\int_{\mathbb{R}} t\zeta^4|\partial_t f|^2 dudt \leqslant \int_0^T\int_{\mathbb{R}}\Big(32at + \frac{2b^2 t}{a} + 1\Big)a\zeta^2|\partial_u f|^2 dudt,$$

$$T^2\int_{\mathbb{R}} \zeta^6|\partial_t f|^2 du + \int_0^T\int_{\mathbb{R}} at^2\zeta^6|\partial_t\partial_u f|^2 dudt \leqslant \int_0^T\int_{\mathbb{R}}\Big(72at + \frac{2b^2 t}{a} + 2\Big)t\zeta^4|\partial_t f|^2 dudt,$$

and

$$\begin{aligned} &T^3\int_{\mathbb{R}} a\zeta^8|\partial_t\partial_u f|^2 du + \int_0^T\int_{\mathbb{R}} t^3\zeta^8|\partial_t^2 f|^2 dudt \\ \leqslant \ & \int_0^T\int_{\mathbb{R}}\Big(128a^2 t + \frac{2b^2 t}{a} + 3\Big)at^2\zeta^6|\partial_t\partial_u f|^2 dudt. \end{aligned}$$

These inequalities imply that

$$\int_0^T\int_{-R}^R t^3|\partial_t(\partial_t f)|^2 dudt + \sup_{t\in[0,T]}\int_{-R}^R t^3|\partial_u(\partial_t f(u,t))|^2 du \leqslant C(R,T)$$

where $C(R,T)$ depends only on T and on $\|a + b^2/a + 1/a\|_{L^\infty([-R-2,R+2])}$. Thus, by Sobolev imbedding, changing notation f back to f^n which is what f meant to be here, we obtain

$$\|\partial_t f^n\|_{C^{1/4}([-R,R]\times[T,1/T])} \leqslant \hat{C}(R,T) \quad \forall\, R > 1, T > 1, n > R+2.$$

Passing this estimate to the limit, we see that $\partial_t f \in C^{1/4}(\mathbb{R}\times(0,\infty))$. Regarding $\partial_t f$ as a known function, writing $\mathcal{L}f = 0$ as $\partial_u(ae^{-q}\partial_u f) = e^{-q}\partial_t f$, and using $\partial_t(ae^{-q}\partial_u f) \in L^2_{\text{loc}}(\mathbb{R}\times(0,\infty))$ we derive that $ae^{-q}\partial_u f \in C^{1/4}(\mathbb{R}\times(0,\infty))$. Then, $\partial_u(a\partial_u f) = \partial_t f + b\partial_u f \in C(\mathbb{R}\times(0,\infty))$. Thus (8) admits a classical solution.

4. Finally we prove the uniqueness of the solution. The argument relies on the required behavior of the solution near $u = \pm\infty$. Suppose f_1 and f_2 are two solutions of (8). Let $T > 0$ be any positive constant. Fixing $\eta > 0$, we derive from the boundary condition $\lim_{u\to\pm\infty} f_i(u,t) = (1\pm 1)/2$ uniformly on $[0,T]$ that $|f_1(u,t) - f_2(u,t)| \leqslant \eta$ for all $|u| \geqslant y_\eta, t \in [0,T]$, for some large enough y_η. On the domain $[-y_\eta, y_\eta]\times[0,T]$ we can apply the maximum principle for weak solutions [12] to conclude that $|f_1 - f_2| \leqslant \eta$ in $[-y_\eta, y_\eta]\times[0,T]$. Sending $\eta \to 0$ we obtain $f_1 = f_2$ in $\mathbb{R}\times(0,T]$. As T is arbitrary, we see that $f_1 \equiv f_2$, and therefore obtain the uniqueness of the solution. This completes the proof of Theorem 3.1. □

Remark 2. One can further show that $\partial_t(a\partial_u f) \in C(\mathbb{R}\times(0,\infty))$. Then $\rho_t = \partial_t\partial_u f = \partial_u j$ with $j = \partial_u[a\rho] - b\rho$ is also continuous, so ρ is indeed a classical solution too, though it may not be even differentiable in u. The delicacy here is that the combinations $a\rho$ and j are differentiable in u. These properties in general hold only in one space dimension.

4. **Long time behavior.** In this section we prove (4).

Theorem 4.1. *Assume that ε, σ, b satisfy (6) and that ρ_0 is a probability density function. Let ρ^* be defined as in (7) and ρ be the weak solution of (3) defined in the previous section. Then*

$$\lim_{t\to\infty} \rho(\cdot,t) = \rho^*(\cdot) \quad \textit{locally uniformly in } \mathbb{R} \quad \textit{and in} \quad L^1(\mathbb{R}).$$

Consequently, for any Borel set A in $\mathbb{R}$, the solution $\{u_t\}_{t\geqslant 0}$ of the sde (2) satisfies

$$\lim_{t\to\infty} \mathbb{P}(\{u_t \in A\}) = \lim_{t\to\infty} \int_A \rho(x,t)dx = \int_A \rho^*(x)dx.$$

Note that ρ^* defined in (7) depends only on ε, σ, and b; in particular it does not depend on the initial distribution of u_0. Theorem 4.1 proclaims that regardless of initial setup, any solution of the std (2) approaches the same statistical equilibrium distribution as ρ^* in the long run.

Proof of Theorem 4.1. Let $f(\cdot,t)$ and $f^*(\cdot)$ be the cdfs of $\rho(\cdot,t)$ and $\rho^*(\cdot)$ respectively. Set $g = f - f^*$. Then

$$\mathcal{L}g = 0, \qquad g(\cdot,0) = f_0(\cdot) - f^*(\cdot), \qquad \lim_{|u|\to\infty} \sup_{t\in[0,\infty)} |g(u,t)| = 0$$

where the last equation follows from (10) (as the solution is unique).

Let $\eta > 0$ be any small positive constant. We shall show that $\|g(\cdot,t)\|_{L^\infty(\mathbb{R})} \leqslant 2\eta$ when t is large enough, by using comparison principle. The asymptotic behavior in (10) implies that there exists $y_\eta > 0$ such that

$$|g(u,t)| \leqslant \eta \quad \forall\, u \in (-\infty, -y_\eta] \cup [y_\eta, \infty),\ t \in [0,\infty).$$

Next we focus the solution on the spatially bounded domain $[-y_\eta, y_\eta]\times[0,\infty)$. Let

$$\varphi(u,t) := e^{-\lambda t}\sin(\pi f^*(u)), \qquad \lambda := \min_{u\in[-y_\eta,y_\eta]} \Big\{\pi^2 a(u)\rho^*(u)^2\Big\}.$$

For $u \in [-y_\eta, y_\eta]$ and $t > 0$, direct computation gives

$$\mathcal{L}\varphi = [-\lambda + a\pi^2(\partial_u\, f^*)^2]\varphi + \pi e^{-\lambda t}\cos(\pi f^*)\mathcal{L}f^* = [-\lambda + \pi^2 a\rho^{*2}]\varphi \geqslant 0.$$

On the set $[-y_\eta, y_\eta] \times (0, \infty)$ let

$$\Psi(u,t) := \eta + \frac{e^{-\lambda t}\sin(\pi f^*(u))}{\min\{\sin(\pi f^*(-y_\eta)), \sin(\pi f^*(y_\eta))\}}.$$

It is easy to see that $\mathcal{L}\Psi \geqslant 0$ on $[-y_\eta, y_\eta] \times [0, \infty)$, $\Psi(u, 0) > 1 \geqslant |g(u, 0)|$ for all $u \in [-y_\eta, y_\eta]$ and $\Psi(\pm y_\eta, t) > \eta \geqslant |g(\pm y_\eta, t)|$ for all $t > 0$. Hence, comparing Ψ with $\pm g$ on $[-y_\eta, y_\eta] \times [0, \infty)$ we see that $\pm g < \Psi$ on $[-y_\eta, y_\eta] \times [0, \infty)$. Since this estimate is already known to be true when $u \notin [-y_\eta, y_\eta]$, we hence conclude that $|g(u,t)| \leqslant \Psi(u,t)$ for every $u \in \mathbb{R}, t \in [0, \infty)$. Therefore, there exists $T_\eta > 0$ such that $|g| \leqslant 2\eta$ on $\mathbb{R} \times [T_\eta, \infty)$. Since η is arbitrary, we conclude that

$$\lim_{t\to\infty} \|g(\cdot, t)\|_{L^\infty(\mathbb{R})} = 0.$$

Finally, we apply the energy estimates in Step **3** of the regularity proof in the previous section to the function $g(\cdot, h + \cdot)$ in the domain $\mathbb{R} \times [0, 2]$ for any $h > 0$ to conclude that for every $R > 1$,

$$\lim_{t\to\infty} \|\rho(\cdot, t) - \rho^*(\cdot)\|_{L^\infty((-R,R))} = \lim_{t\to\infty} \|\partial_u g(\cdot, t)\|_{L^\infty((-R,R))} = 0.$$

This completes the proof. □

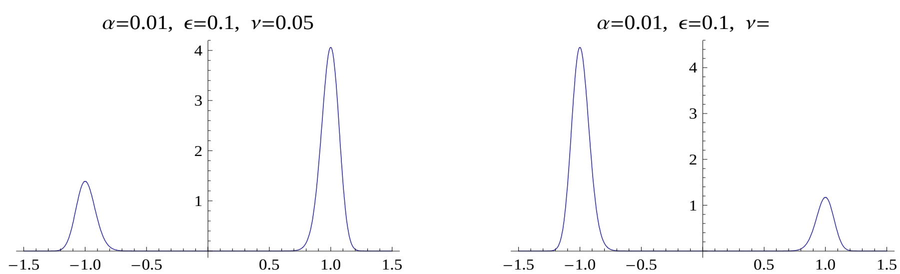

FIGURE 1. Graphs of $\rho = \rho^*(u)$ with $b(u) = (u - \alpha)(1 - u^2)$ and $\sigma(u) = 1 + \nu(u-1)^2$. When $\nu = 0.05$, the lesser stable equilibrium $u \equiv 1$ is selected by fluctuation.

5. **Limits of the invariant measure when white noise is vanishing.** In this section, we consider the asymptotic behavior of ρ^* as $\varepsilon \searrow 0$. We write it as

$$\rho^*(u) = \frac{\sigma^{-2}(u)\, e^{q(u)}}{\int_{\mathbb{R}} \sigma^{-2}(y) e^{q(y)} dy}, \qquad q(u) := -\frac{2Q(u)}{\varepsilon^2}, \qquad Q(u) = -\int_0^u \frac{b(x)}{\sigma^2(x)} dx.$$

Figure 1 displays the probability density function ρ^* for the case $b(u) = (u-\alpha)(1-u^2)$ and $\sigma(u) = 1 + \nu(u-1)^2$ with $(\alpha, \varepsilon, \nu) = (0.01, 0.1, 0.05)$ and $(0.01, 0.1, 0)$ respectively. Basic observation indicates that as $\varepsilon \searrow 0$, ρ^* approaches a Dirac type mass concentrated on the most stable equilibria of the effective potential Q. We confirm this observation into two Theorems, beginning with the model problem with constant σ.

Theorem 5.1. *Let* $\sigma(u) \equiv 1$, $b(u) = (u - \alpha)(1 - u^2)$, $0 < \varepsilon \ll 1$, *and* $\{u_t\}_{t \geqslant 0}$ *be any solution of* (2).

1. *If* $\alpha = 0$, *then* $\int_{|u\pm 1|\leqslant\sqrt{\varepsilon}} \rho^*(u)du = \frac{1}{2} + O(e^{-1/\varepsilon})$. *Consequently, for all* t *sufficiently large,*
$$\mathbb{P}(\{|u_t \pm 1| \leqslant \sqrt{\varepsilon}\}) = \tfrac{1}{2} + O(e^{-1/\varepsilon}).$$
2. *If* $\alpha \in (0,1)$, *then* $\int_{|u+1|>\sqrt{\varepsilon}} \rho^*(u)du = O(e^{-1/\varepsilon})$ *and for each sufficiently large* t,
$$\mathbb{P}(\{|u_t + 1| \geqslant \sqrt{\varepsilon}\}) = O(e^{-1/\varepsilon}).$$

The proof follows by a straightforward evaluation of the potential ρ^* and therefore is omitted.

The above theorem is indeed well-known; see a recent result of Weber [28] and the references therein.

Remark 3. Notice that without the white noise, i.e., for the ode solution of (1), we have
$$\lim_{t\to\infty} u(t) = \begin{cases} 1 & \text{if } u(0) > \alpha, \\ \alpha & \text{if } u(0) = \alpha, \\ -1 & \text{if } u(0) < \alpha. \end{cases}$$
Thus, adding an arbitrary small white noise totally eliminates the initial effect in the long run.

Remark 4. Statistically, it is important to find the mean time needed for the system to switch from one stable state to another by fluctuations; see [19, 20, 23, 24, 26, 31, 32] and references therein.

Next we consider the general case. We denote by $\delta(\cdot - x)$ the Dirac mass concentrated at x.

Theorem 5.2. *Let* σ *and* b *be in* $C^1(\mathbb{R})$ *and satisfy* (6), $Q(u) = -\int_0^u b(x)\sigma^{-2}(x)dx$, *and* $\mathcal{S}$ *be the set of global minimizers of* Q. *Assume that* $b'(x) \neq 0$ *for each* $x \in \mathcal{S}$. *Then in the sense of measure*
$$\lim_{\varepsilon\searrow 0} \rho^*(\cdot) = \frac{1}{A}\sum_{x\in\mathcal{S}} \frac{1}{\sqrt{|b'(x)|}\,\sigma(x)}\,\delta(\cdot - x), \qquad \textit{where} \quad A = \sum_{x\in\mathcal{S}} \frac{1}{\sqrt{|b'(x)|}\,\sigma(x)}.$$

Remark 5. Note that $\mathcal{S}$ is a subset of the stable steady states $\{u \mid P'(u) = 0 \leqslant P''(u)\}$ since $Q' = -b/\sigma^2 = P'/\sigma^2$ and on $\mathcal{S}$, $Q'' = -b'/\sigma^2 = P''/\sigma^2$.

The assumption $b'(x) \neq 0$ on $\mathcal{S}$ is critical for the validity of our theorem since it implies that the set $\mathcal{S}$ consists of finitely many points.

Proof of Theorem 5.2. Assume, without loss of generality, that $0 \in \mathcal{S}$. Then
$$\min_{u\in\mathbb{R}} Q(u) = Q(0) = 0.$$
Denote by M and η the positive constants such that $b(u) > \eta$ in $(-\infty, -M + \eta]$ and $b(u) < -\eta$ in $[M - \eta, \infty)$. Then $Q'(u) = -b(u)/\sigma^2(u) > 0$ in $[M - \eta, \infty)$ and $Q'(u) < 0$ in $(-\infty, -M+\eta]$, so that $\mathcal{S} \subset (-M+\eta, M-\eta)$. As $Q''(x) = -b'(x)/\sigma^2(x)$ is non-zero and therefore must be positive for every $x \in \mathcal{S}$, $\mathcal{S}$ is non-empty and has finitely many points. By taking smaller η if necessary we can assume that $Q''(y) > \frac{1}{2}Q''(x)$ if $|y - x| < \eta$ and $x \in \mathcal{S}$. We introduce
$$\mathrm{d}(y, \mathcal{S}) = \min_{x\in\mathcal{S}} |x - y|, \qquad c_1 = \min_{\mathrm{d}(y,\mathcal{S})\geqslant\eta} Q(y), \qquad c_2 = \min_{x\in\mathcal{S}} Q''(x).$$

We now calculate the contributions of the integrand towards the integral

$$Z := \frac{1}{\varepsilon}\int_{\mathbb{R}} \frac{e^{q(y)}}{\sigma^2(y)}dy \qquad \text{with } \ q(u) := -\frac{2Q(y)}{\varepsilon^2}.$$

We shall show that the contributions from the set $\{y \mid \mathrm{d}(y,\mathcal{S}) > 2\sqrt{\varepsilon}\}$ is negligible, whereas the contribution from each ball $\{y \mid \mathrm{d}(x,y) < 2\sqrt{\varepsilon}\}$ is proportional to $1/\sqrt{|b'(x)|\sigma^2(x)}$, if $x \in \mathcal{S}$.

(i) For the set $(-\infty,-M]\cup[M,\infty)$, we use the estimate in the proof of Theorem 2.1 to derive

$$\frac{1}{\varepsilon}\int_{|y|>M}\frac{e^{q(y)}}{\sigma^2(y)}dy \leqslant \frac{\varepsilon\big[e^{q(-M)}+e^{q(M)}\big]}{2\eta} \leqslant \frac{\varepsilon e^{-2c_1/\varepsilon^2}}{2\eta}.$$

(ii) In the bounded set $\{y \in [-M,M] \mid \mathrm{d}(y,\mathcal{S}) \geqslant \eta\}$ we have $Q \geqslant c_1$ so that

$$\frac{1}{\varepsilon}\int_{|y|<M,\mathrm{d}(y,\mathcal{S})>\eta}\frac{e^{q(y)}}{\sigma^2(y)}dy \leqslant \frac{2Me^{-2c_1/\varepsilon^2}}{\varepsilon\ \min\{\sigma^2(u)\mid u\in[-M,M]\}}.$$

(iii) Suppose $2\sqrt{\varepsilon} < |y-x| < \eta$ for some $x \in \mathcal{S}$. Then $Q(x) = 0$ and $Q'(x) = 0 < Q''(x)$. Using $Q''(\tilde{y}) \geqslant \frac{1}{2}Q''(x)$ for every $\tilde{y} \in [x-\eta, x+\eta]$ and Taylor expansion, we derive that

$$Q(y) = \tfrac{1}{2}Q''(\tilde{y})|y-x|^2 \geqslant Q''(x)\varepsilon \geqslant c_2\varepsilon,$$
$$\frac{1}{\varepsilon}\int_{2\sqrt{\varepsilon}<\mathrm{d}(y,\mathcal{S})\leqslant\eta}\frac{e^{q(y)}}{\sigma^2(y)}dy \leqslant \frac{2Me^{-2c_2/\varepsilon}}{\varepsilon\min\{\sigma^2(u)\mid u\in[-M,m]\}}.$$

(iv) Now for each $x \in \mathcal{S}$ and $|y-x| \leqslant 2\sqrt{\varepsilon}$, we have

$$q(y) = -\frac{2Q(y)}{\varepsilon^2} = -\Big[Q''(x)+o(1)\Big]\frac{(x-y)^2}{\varepsilon^2}, \quad \sigma^2(y) = \sigma^2(x)+o(1)$$

where $o(1) \searrow 0$ as $\varepsilon \searrow 0$. The change of variable $y = x+\varepsilon z$ gives

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\int_{|y-x|\leqslant 2\sqrt{\varepsilon}}\frac{e^{q(y)}}{\sigma^2(y)}dy &= \lim_{\varepsilon\searrow 0}\int_{-2/\sqrt{\varepsilon}}^{2/\sqrt{\varepsilon}}\frac{e^{-[Q''(x)+o(1)]z^2}}{\sigma^2(x)+o(1)}dz \\ &= \int_{\mathbb{R}}\frac{e^{-Q''(x)z^2}}{\sigma^2(x)}dz = \frac{\sqrt{\pi}}{\sqrt{Q''(x)}\sigma^2(x)} = \frac{\sqrt{\pi}}{\sqrt{|b'(x)|}\,\sigma(x)}.\end{aligned}$$

The above estimates can be summarized as follows:

$$\begin{aligned}\lim_{\varepsilon\searrow 0} Z &= \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}\int_{\mathbb{R}}\frac{e^{q(y)}}{\sigma^2(y)}dx = \sum_{x\in\mathcal{S}}\frac{\sqrt{\pi}}{\sqrt{|b'(x)|}\,\sigma(x)} = \sqrt{\pi}A, \\ \lim_{\varepsilon\searrow 0}\int_{|y-x|\leqslant 2\sqrt{\varepsilon}}\rho^*(y)dy &= \lim_{\varepsilon\searrow 0}\frac{1}{Z\,\varepsilon}\int_{|y-x|<2\sqrt{\varepsilon}}\frac{e^{q(y)}}{\sigma^2(y)}dy \\ &= \frac{1}{A\sqrt{|b'(x)|}\sigma(x)} \qquad \forall\, x\in\mathcal{S}, \\ \lim_{\varepsilon\searrow 0}\int_{\mathrm{d}(y,\mathcal{S})>2\sqrt{\varepsilon}}\rho^*(y)dy &= \lim_{\varepsilon\searrow 0}\frac{1}{Z\,\varepsilon}\int_{\mathrm{d}(y,\mathcal{S})>2\sqrt{\varepsilon}}\frac{e^{q(y)}}{\sigma^2(y)}dy = 0.\end{aligned}$$

The assertion of Theorem 5.2 thus follows. □

6. **The degenerate case Where σ is not everywhere positive.** Here we consider the degenerate case where σ is not everywhere positive. Our prototype is

$$\begin{aligned} b(u) &= (u-\alpha)(1-u^2) \qquad &(\alpha\in[0,1)),\\ \sigma(u) &= (\max\{\ell^2-u^2,0\})^\nu \qquad &(\nu>0, \ell>0). \end{aligned}$$

For the invariant measure ρ^* in (7) to be well-defined, we go through a regularization process defining ρ^* as the $\eta\searrow 0$ limit of those regular ρ_η obtained by replacing σ by positive σ_η:

$$\rho_\eta(x) := \frac{1}{Z_\eta}\,\frac{e^{q_\eta(x)}}{\sigma_\eta^2(x)}, \quad q_\eta(x) := \int_0^x \frac{b(s)}{\sigma_\eta^2(s)}ds, \quad Z_\eta := \int_{\mathbb{R}} \frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)}du. \tag{11}$$

Here by absorbing ε into σ, we assume for simplicity that $\varepsilon=\sqrt{2}$. It turns out that when $\ell\in(0,1]$, the limit depends sensitively on the regularization process, whereas when $\ell>1$, the limit does not depend on the regularization process and therefore ρ^* is well-defined.

We shall consider a special regularization for the case $\ell=1$ in the first subsection. Then consider a general regularization covering the case $\ell\in(0,1]$ in second subsection, showing that

$$\lim_{\eta\searrow 0}\rho_\eta(\cdot) \;=\; \theta\,\delta(\cdot+1)+[1-\theta]\,\delta(\cdot-1) \tag{12}$$

where $\theta\in[0,1]$ is a constant that depends on the regularization. Indeed, for any constant $\theta\in[0,1]$, there is a regularization such that (12) is true. The conclusion is then generalized to a much broad situation where σ vanishes at certain stable equilibria of a quite general potential. In the last subsection we consider a general regularization covering the case $\ell>1$, proving that

$$\lim_{\eta\searrow 0}\rho_\eta(\cdot) = \frac{m_1\,\delta(\cdot+\ell)\;+\;m_2\,\delta(\cdot-\ell)\;+\;\hat\rho(\cdot)\,\chi_{(-\ell,\ell)}(\cdot)}{m_1\;+\;m_2\;+\;m}$$

where χ_A is the characteristic function of the set A, $\hat\rho=e^q/\sigma^2$, m is a positive constant, and m_1,m_2 are non-negative constants depending only on α,ℓ, and ν; $m_1=m_2=0$ if and only if $\nu\geqslant 1/2$.

6.1. **The model case with special regularization.** We begin with a special regularization for a special case where σ vanishes at the stable equilibria.

Theorem 6.1. *Let $\alpha\in[0,1)$ and $\nu>0$ be constants and $b(u)=(u-\alpha)(1-u^2)$ and $\sigma=(\max\{1-u^2,0\})^\nu$ be functions of $u\in\mathbb{R}$. For $\eta\in(0,1]$ define ρ_η by (11) with $\sigma_\eta=\max\{\eta,\sigma\}$. Then (12) holds in the sense of measure with*

$$\theta=\frac{1}{1+m},\quad m=\begin{cases} 1 & \text{if }\ \alpha=0,\\ \sqrt{\frac{b'(1)}{b'(-1)}}\exp\left(\int_{-1}^1\frac{b(u)}{\sigma^2(u)}du\right) & \text{if }\ \alpha\in(0,1),\nu\in(0,1),\\ 0 & \text{if }\ \alpha\in(0,1),\nu\geqslant 1.\end{cases}$$

To prove the theorem, we first prove a lemma. For this we introduce

$$\rho_\eta^+(x) := \frac{e^{q_\eta(x)}}{\sigma_\eta^2(x)}\,\eta e^{-q_\eta(1)}\chi_{[\alpha,\infty)}(x), \qquad \rho_\eta^-(x) := \frac{e^{q_\eta(x)}}{\sigma_\eta^2(x)}\,\eta e^{-q_\eta(-1)}\chi_{(-\infty,\alpha)}(x).$$

Lemma 6.2. *As $\eta \searrow 0$, $\rho_\eta^\pm(\cdot) \longrightarrow M^\pm\, \delta(\cdot \mp 1)$ where, with $k = |b'(\pm 1)|/2$,*

$$M^\pm = \begin{cases} \int_{-\infty}^{\infty} e^{-kz^2}\, dz & \text{if } \nu > 1, \\ \int_{-1/2}^{\infty} e^{-kz^2}\, dz + \int_{-\infty}^{-1/2} \frac{e^{-k/4}\, dz}{|2z|^{k/2+2}} & \text{if } \nu = 1, \\ \int_0^\infty e^{-kz^2}\, dz & \text{if } \nu \in (0,1). \end{cases}$$

Proof of Theorem 6.1. Setting $M_\eta^\pm = \int_{\mathbb{R}} \rho_\eta^\pm(x)dx$ and $m_\eta = M_\eta^+ e^{q_\eta(1)}/[M_\eta^- e^{q_\eta(-1)}]$. Then

$$\rho_\eta(x) = \frac{m_\eta}{1+m_\eta}\, \frac{\rho_\eta^+(x)}{M_\eta^+} + \frac{1}{1+m_\eta}\, \frac{\rho_\eta^-(x)}{M_\eta^-}.$$

The assertion of Theorem 6.1 thus follows from Lemma 6.2 and the computation

$$\lim_{\eta\searrow 0} m_\eta = \lim_{\eta\searrow 0} \frac{M_\eta^+}{M_\eta^-} e^{q_\eta(1)-q_\eta(-1)} = \frac{M^+}{M^-}\, \lim_{\eta\searrow 0} \exp\Big(\int_{-1}^1 \frac{b(u)}{\sigma_\eta^2(u)} du\Big) = m.$$

□

Proof of Lemma 6.2. First consider ρ_η^+. We shall show that the major contribution toward the total mass M_η^+ of ρ_η^+ comes from the interval $(1-\eta|\ln\eta|, 1+\eta|\ln\eta|)$ for the case $\nu \geqslant 1$, and from the interval $[1, 1+\eta|\ln\eta|)$ for the case $\nu \in (0,1)$.

Set $\eta_\nu = 1 - \sqrt{1-\eta^{1/\nu}}$. Then $\sigma_\eta(u) = \eta$ when $|u| \geqslant 1-\eta_\nu$ and $\sigma_\eta(u) = (1-u^2)^\nu$ when $|u| \leqslant 1-\eta_\nu$.

Denote $k = -b'(1)/2$. Then for $x \geqslant 1-\eta_\nu$,

$$q_\eta(x) - q_\eta(1) = \int_1^x \frac{b(u)}{\eta^2} du \quad = \quad -k\Big(\frac{x-1}{\eta}\Big)^2 \Big(1 + \frac{b''(\hat{x})}{3b'(1)}(x-1)\Big)$$

by Taylor expansion, where $\hat{x}$ lies between 1 and x. Since $b'(1) < 0$ and $b'' < 0$ in $[1,\infty)$,

$$\overline{\lim_{\eta\searrow 0}} \int_{1+\eta|\ln\eta|}^\infty \rho_\eta^+(x)dx = \overline{\lim_{\eta\searrow 0}} \int_{1+\eta|\ln\eta|}^\infty \frac{e^{q_\eta(x)-q_\eta(1)}}{\eta} dx \leqslant \lim_{\eta\searrow 0} \int_{|\ln\eta|}^\infty e^{-kz^2} dz = 0$$

by the change of variable $x = 1+\eta z$. Also, by Lebesgue's Dominated Convergence Theorem,

$$\begin{aligned} &\lim_{\eta\searrow 0} \int_{1-\eta_\nu}^{1+\eta|\ln\eta|} \rho_\eta^+(x)dx \\ = \;& \lim_{\eta\searrow 0} \int_{-\eta_\nu/\eta}^{|\ln\eta|} e^{-kz^2(1+O(z\eta))} dz \\ = \;& \begin{cases} \int_{-\infty}^\infty e^{-kz^2}\, dz & \text{if } \nu > 1, \\ \int_{-1/2}^\infty e^{-kz^2}\, dz & \text{if } \nu = 1, \\ \int_0^\infty e^{-kz^2}\, dz & \text{if } \nu \in (0,1). \end{cases} \end{aligned}$$

Finally consider the integral on $(\alpha, 1-\eta_\nu]$. We shall use $\eta_\nu = \frac{1}{2}\eta^{\frac{1}{\nu}}[1+O(\eta^{\frac{1}{\nu}})]$ and

$$q_\eta(1-\eta_\nu) - q_\eta(1) = -k\Big(\frac{\eta_\nu}{\eta}\Big)^2\Big(1 - \frac{b''(\hat{x})}{3b'(1)}\eta_\nu\Big) = -\eta^{2/\nu-2}\Big[\frac{k}{4} + O(\eta_\nu)\Big].$$

(a) When $\nu > 1$, using $q_\eta(x) \leqslant q_\eta(1-\eta_\nu)$ and $\sigma_\eta(x) \geqslant (1+\alpha)^\nu(1-x)^\nu$ for $x \in (\alpha, 1-\eta_\nu]$, we obtain

$$\int_\alpha^{1-\eta_\nu} \rho_\eta^+(x)dx \leqslant \int_\alpha^{1-\eta_\nu} \frac{\eta e^{q_\eta(1-\eta_\nu)-q_\eta(1)}}{(1+\alpha)^{2\nu}(1-x)^{2\nu}}dx \leqslant \frac{\eta e^{-\eta^{2/\nu-2}[k/4+O(\eta_\nu)]}}{(2\nu-1)(1+\alpha)^{2\nu}\eta_\nu^{2\nu-1}} \longrightarrow 0,$$

as $\eta \searrow 0$.

(b) When $\nu \in (0,1)$, considering separately cases $\nu \in (0,1/2), \nu = 1/2$, and $\nu \in (1/2,1)$ we derive that

$$\int_\alpha^{1-\eta_\nu} \rho_\eta^+(x)dx \leqslant \int_\alpha^{1-\eta_\nu} \frac{\eta\, dx}{(1-x^2)^{2\nu}} \leqslant O(1)\eta[1+|\ln\eta_\nu| + \eta_\nu^{1-2\nu}] \longrightarrow 0 \quad \text{as } \eta \searrow 0.$$

(c) When $\nu = 1$, we have $\eta_\nu = \frac{\eta}{2} + O(\eta^2)$ and for $x \in (\alpha, 1-\eta_\nu]$,

$$\rho_\eta^+(x) = \frac{\eta e^{q_\eta(1-\eta_\nu)-q_\eta(1)} e^{\int_{1-\eta_\nu}^x \frac{u-\alpha}{1-u^2}du}}{(1-x^2)^2} = \frac{e^{-k/4+O(\eta)}}{\eta}\Big(\frac{\eta_\nu}{1-x}\Big)^{\frac{k}{2}+2}\Big(\frac{2-\eta_\nu}{1+x}\Big)^{\frac{1+\alpha}{2}+2}.$$

Using the change of variable $x = 1+\eta z$ we obtain

$$\begin{aligned}
\lim_{\eta\searrow 0}\int_{1-\eta|\ln\eta|}^{1-\eta_\nu} \rho_\eta^+(x)dx &= \lim_{\eta\searrow 0}\int_{-|\ln\eta|}^{-\eta_\nu/\eta} \Big(\frac{\eta_\nu}{-\eta z}\Big)^{\frac{k}{2}+2}\Big[e^{-k/4}+O(\eta\ln\eta)\Big]dz \\
&= \int_{-\infty}^{-1/2} \frac{e^{-k/4}\, dz}{|2z|^{k/2+2}}, \\
\overline{\lim_{\eta\searrow 0}}\int_\alpha^{1-\eta|\ln\eta|} \rho_\eta^+(x)dx &\leqslant \overline{\lim_{\eta\searrow 0}}\int_{-\infty}^{-|\ln\eta|} \frac{O(1)}{|z|^{k/2+2}}dz = 0.
\end{aligned}$$

In conclusion, defining $f_\eta(x) = \int_{-\infty}^x \rho_\eta^+(u)du$, we find that $\lim_{\eta\searrow 0} f_\eta(x) = 0$ if $x < 1$ and $= M^+$ if $x > 1$. Thus, $\rho_\eta^+(\cdot) \to M^+\delta(\cdot - 1)$. Similarly, by considering $\tilde\rho_\eta(x) = \rho_\eta(-x)$ with $k = -b'(-1)/2$, we can show that $\rho_\eta^-(\cdot) \to M^-\delta(\cdot+1)$. This completes the proof of Lemma 6.2. ☐

Remark 6. From the proof, one sees that $\rho^\pm$ has the following limiting profile:

$$\lim_{\eta\searrow 0} \eta\rho_\eta^\pm(\pm 1 \pm z\eta) = \begin{cases} e^{-kz^2} & \text{if } \nu > 1, \\ e^{-kz^2}\chi_{[0,\infty)} & \text{if } \nu \in (0,1), \\ e^{-kz^2}\chi_{[-\frac{1}{2},\infty)} + \frac{e^{-k/4}}{|2z|^{k/2+2}}\chi_{(-\infty,-\frac{1}{2})} & \text{if } \nu = 1. \end{cases}$$

6.2. **General regularization when σ vanishes at stable states.** Here we extend Theorem 6.1 to a general regularization for the general case when σ vanishes at stable equilibria. We assume that σ vanishes near ± 1 where -1 is the smallest and 1 is the largest zeros of b. Also σ is positive in $[x_1, x_2] \subset (-1,1)$ where (x_1, x_2) is an interval contains all zeros of b except ± 1.

Theorem 6.3. *Let b be a continuous function that satisfies, for some $x_2 \in (0,1)$ and $x_1 \in (-1,0)$,*

$$b < 0 \ \text{in } (-1, x_1] \cup (1,\infty), \quad b(u) > 0 \ \text{in } (-\infty,-1) \cup [x_2, 1), \quad \underline{\lim}_{|u|\to\infty} |b(u)| > 0.$$

Let σ be a non-negative continuous function such that, for some $\epsilon > 0$,

$$\sigma(u) > 0 \ \text{in} \ [x_1, x_2], \qquad \sigma(u) = 0 \ \text{in} \ [-1-\epsilon, -1] \cup [1, 1+\epsilon]. \tag{13}$$

For σ_η to be specified below, let ρ_η be defined as in (11). The following holds:

1. *Suppose* $\{\sigma_\eta\}_{\eta\in(0,1]}$ *is a family of continuous positive functions satisfying* $\lim_{\eta\searrow 0}\sigma_\eta(\cdot)=\sigma(\cdot)$ *uniformly on* $[-1-\epsilon,1+\epsilon]$. *Then for some* $\theta\in[0,1]$, (12) *holds along a sequence* $\eta\searrow 0$;
2. *For every* $\theta\in[0,1]$, *there exists a family* $\{\sigma_\eta\}_{\eta\in(0,1]}$ *of continuous positive functions satisfying* $\lim_{\eta\searrow 0}\sigma_\eta=\sigma$ *uniformly on* $\mathbb{R}$ *such that* (12) *holds.*

Proof of Theorem 6.3. Similar to the proof of Theorem 6.1, we introduce

$$\rho_\eta^{\pm}(x)=\frac{e^{q_\eta(x)-q_\eta(\pm1)}}{\sigma_\eta^2(x)}\chi_{[0,\infty)}(\pm x),\qquad M_\eta^{\pm}=\int_{\mathbb{R}}\rho_\eta^{\pm}(x)dx.$$

1. We first show that $\lim_{\eta\searrow 0}M_\eta^+=\infty$. For each $x>1$ setting $b^*(x)=\max_{u\in[1,x]}|b(u)|$ and using $q_\eta'=b/\sigma_\eta^2$, $b<0$ in $(1,x)$, $\lim_{\eta\searrow 0}\sigma_\eta=0$ on $[1,1+\epsilon]$, and $b(1)=0$, we derive that

$$\begin{aligned}M_\eta^+ &\geqslant \int_1^x\frac{|b(u)|}{b^*(x)}\rho_\eta^+(u)du=\frac{1-\exp(\int_1^x\frac{b(u)}{\sigma_\eta^2(u)}du)}{b^*(x)},\\ \underline{\lim}_{\eta\searrow 0}M_\eta^+ &\geqslant \lim_{x\searrow 1}\lim_{\eta\searrow 0}\frac{1-\exp(\int_1^x\frac{b(u)}{\sigma_\eta^2(u)}du)}{b^*(x)}=\lim_{x\searrow 1}\frac{1}{b^*(x)}=\infty.\end{aligned}$$

2. Next we estimate the mass on $[x,\infty)$ for each $x>1$. Setting $b_*(x)=\inf_{u\geqslant x}|b(u)|$ we have

$$\begin{aligned}\overline{\lim_{\eta\searrow 0}}\int_x^\infty\rho_\eta^+(u)du &\leqslant \overline{\lim_{\eta\searrow 0}}\int_x^\infty\frac{|b(u)|}{b_*(x)}\rho^+(u)du\\ &\leqslant \overline{\lim_{\eta\searrow 0}}\frac{e^{q_\eta(x)-q_\eta(1)}}{b_*(x)}=\lim_{\eta\searrow 0}\frac{1}{b_*(x)}\exp\Big(\int_1^x\frac{b(u)}{\sigma_\eta^2}du\Big)=0.\end{aligned}$$

3. For any $x\in(x_2,1)$, setting $B_*(x)=\min_{u\in[x^+,x]}b(u)$ we derive that

$$\begin{aligned}\int_{x^+}^x\rho_\eta^+(u)du &\leqslant \int_{x^+}^x\frac{b(u)}{B_*(x)}\rho_\eta^+(u)du=\frac{e^{-q_\eta(1)}}{B_*(x)}\int_{x^+}^x q'(u)e^{q_\eta(u)}du\leqslant\frac{1}{B_*(x)},\\ \overline{\lim_{\eta\searrow 0}}\frac{1}{M_\eta^+}\int_{-\infty}^x\rho_\eta^+(u)du &\leqslant \overline{\lim_{\eta\searrow 0}}\frac{1}{M_\eta^+}\Big(\int_0^{x_2}\frac{e^{q_\eta(u)-q_\eta(x_2)}}{\sigma_\eta^2(u)}du+\frac{1}{B^*(x)}\Big)=0.\end{aligned}$$

4. The above three estimates shows that as $\eta\searrow 0$, $\rho_\eta^+(x)/M_\eta^+\longrightarrow\delta(x-1)$. Similarly, as $\eta\searrow 0$, $\rho_\eta^-(x)/M_\eta^-\longrightarrow\delta(x+1)$. Finally, note that

$$\rho_\eta(x)=\theta_\eta\frac{\rho_\eta^-(x)}{M_\eta^-}+[1-\theta_\eta]\frac{\rho_\eta^+(x)}{M_\eta^+},\qquad\theta_\eta:=\frac{M_\eta^-e^{q_\eta(-1)}}{M_\eta^+e^{q_\eta(1)}+M_\eta^-e^{q_\eta(-1)}}.\tag{14}$$

As $\{\theta_\eta\}_{\eta\in(0,1]}$ is a family in $(0,1)$, there exists a sequence $\{\eta_i\}_{i=1}^\infty$ in $(0,1)$ such that as $i\to\infty$, $\eta_i\searrow 0$ and $\theta_{\eta_i}\to\theta$ for some $\theta\in[0,1]$. This implies that $\rho_{\eta_i}(\cdot)\to\theta\delta(\cdot+1)+[1-\theta]\delta(\cdot-1)$ as $i\to\infty$. The first assertion of the theorem thus follows. As the second assertion is a special case of the theorem to be presented next, this completes the proof. □

Remark 7. Following the same proof, one sees that the first assertion of Theorem **6.3** remains true if condition (13) is replaced by the following more applicable conditions:

$$\int_{x_1}^{x_2}\frac{du}{\sigma^2(u)}\leqslant\overline{\lim_{\eta\searrow 0}}\int_{x_1}^{x_2}\frac{du}{\sigma_\eta^2(u)}<\infty=\int_{-1-\varepsilon}^{-1}\frac{du}{\sigma^2(u)}=\int_1^{1+\varepsilon}\frac{du}{\sigma^2(u)}\quad\forall\,\varepsilon>0.$$

For example, $\sigma(u) = \sqrt{|u|}\ |\ln u^2|$.

Since the effective potential is $Q(u) = \int b(u)/\sigma^2(u)du$, the second assertion of Theorem 6.3 can be extended to a much more general case where σ vanishes at multiple stable equilibria.

Theorem 6.4. *Let σ be a non-negative continuous function and b be a $C^1(\mathbb{R})$ function satisfying*

$$\overline{\lim_{u\to\infty}}\, b(u) < 0 < \underline{\lim_{u\to-\infty}}\, b(u).$$

Assume that the set $\mathcal{S} := \{u \in \mathbb{R} \mid b(u) = 0, b'(u) < 0, \sigma(u) = 0\}$ is non-empty. Then for every probability measure μ supported on $\mathcal{S}$, these exists a family of continuous positive functions $\{\sigma_\eta\}_{0<\eta<1}$ such that $\lim_{\eta\searrow 0}\sigma_\eta = \sigma$ uniformly on $\mathbb{R}$ and the function ρ_η defined in (11) satisfies

$$\lim_{\eta\searrow 0}\rho_\eta = \mu \qquad \text{in the sense of measure.}$$

Proof. **1.** Let $\eta \in (0, 1]$ be any constant.

Observe that $\mathcal{S}$ is bounded and has only finitely many points. As $\sigma = 0$ and $b' < 0$ on $\mathcal{S}$, there exists $\epsilon > 0$ such that $\sigma(y) \leqslant \eta$ and $|b'(y)/b'(x) - 1| \leqslant 1/2$ when $y \in [x-\epsilon, x+\epsilon]$ and $x \in \mathcal{S}$. The distance between different points in $\mathcal{S}$ is $> 2\epsilon$ since $b(x-\epsilon) > 0 > b(x+\epsilon)$ for each $x \in \mathcal{S}$. Denote

$$A = \cup_{x\in\mathcal{S}}[x-\epsilon, x+\epsilon], \qquad A^c := \mathbb{R}\setminus A, \qquad \hat\sigma(u) = \max\{\sigma(u), \eta\},$$
$$\hat q(u) = \int_0^u \frac{b(y)}{\hat\sigma^2(y)}dy, \qquad \hat Z = \int_{\mathbb{R}} \frac{e^{\hat q(y)}}{\hat\sigma^2(y)}dy, \qquad \hat Z_c = \int_{A^c} \frac{e^{\hat q(y)}}{\hat\sigma^2(y)}dy.$$

Here without loss of generality, we assume that $0 \in A^c$. Recall that the stability assumption on b implies that $\hat Z$ is finite. For non-negative functions $s(x)$ and $L(x, u)$ to be chosen later, we define

$$\sigma_\eta(u) = \hat\sigma(u)\chi_{A^c}(u) + \sum_{x\in\mathcal{S}} \frac{\hat\sigma(u)}{\sqrt{1+\hat\sigma^2(u)s(x)L(x,u)}}\chi_{[x-\varepsilon,x+\varepsilon]}(u)$$

and the corresponding $\rho_\eta, q_\eta, Z_\eta$ as in (11). The central idea here is to choose an appropriate L such that $q_\eta = \hat q$ on A^c. Then the integral of ρ_η on A^c is still $\hat Z_c$, whereas in each interval $[x-\epsilon, x+\epsilon]$ with $x \in \mathcal{S}$, we still have the freedom to choose $s(x)$. For this we notice the following

$$\frac{1}{\sigma_\eta^2(u)} = \frac{1}{\hat\sigma^2(u)} + \sum_{x\in\mathcal{S}} s(x)L(x,u)\chi_{[x-\epsilon,x+\epsilon]}(u), \tag{15}$$

$$q_\eta(u) = \hat q(u) + \sum_{x\in\mathcal{S}} s(x)\int_0^u b(y)L(x,y)\chi_{[x-\epsilon,x+\epsilon]}(y)dy. \tag{16}$$

Hence, for $q_\eta \equiv \hat q$ on A^c, we need only $\int_{x-\epsilon}^{x+\epsilon} b(y)L(x,y)dy = 0$ for each $x \in \mathcal{S}$. More details will be given later. Here we point out that if $L(x, u)$ is continuous in u and $L(x, x \pm \varepsilon) = 0$ for each $x \in \mathcal{S}$, then σ_η is positive and continuous. In addition, since $\sigma \leqslant \eta$ on A,

$$\|\sigma - \sigma_\eta\|_{L^\infty(\mathbb{R})} \leqslant \eta.$$

2. Now we define $L(x, \cdot)$ for each $x \in \mathcal{S}$ as follows: $L(x, \cdot)$ is zero in $(-\infty, x-\epsilon/2]$, linear with slop 2 in $[x-\epsilon/2, x]$, linear with slop $-k$ in $[x, x+\epsilon/k]$ and zero in

$[x+\epsilon/k,\infty)$; more precisely,

$$L(x,u) = [2(u-x)+\epsilon]\,\chi_{[x-\epsilon/2,x]}(u) + [\epsilon+(x-u)k]\chi_{(x,x+\epsilon/k]}(u).$$

Here $k = k(x) \geqslant 1$ is the root of the equation $j(k)=0$ where

$$\begin{aligned} j(k): &= \int_{\mathbb{R}} L(x,u)b(u)du \\ &= \int_{x-\epsilon/2}^{x} [2(u-x)+\epsilon]b(u)du + \epsilon\int_x^{x+\epsilon/k} b(u)[\epsilon+(x-u)k]du \\ &= \epsilon^2\int_0^1 (1-s)\Big[\frac{b(x-\epsilon s/2)}{2} + \frac{b(x+\epsilon s/k)}{k}\Big]ds \\ &= \epsilon^3\int_0^1(1-s)s\int_0^1\Big[\frac{b'(x+\epsilon\theta s/k)}{k^2} - \frac{b'(x-\epsilon\theta s/2)}{4}\Big]d\theta ds. \end{aligned}$$

There exists a (unique) $k\in[1,4]$ such that $j(k)=0$ since $|b'(y)/b'(x)-1|\leqslant 1/2$ for $y\in[x-\epsilon,x+\epsilon]$.

Since $0\notin A$, we see from (16) that $q_\eta=\hat q$ on A^c. In addition, since $\int_{x-\varepsilon}^u b(y)L(x,y)dy$ is increasing in $(-\infty,x]$ and decreasing in $[x,\infty)$, it is non-negative in $\mathbb{R}$.

3. Now we define $s(x)=s$ for $x\in\mathcal{S}$ as the root of

$$\begin{aligned} &\int_{x-\epsilon}^{x+\epsilon}\Big(\frac{1}{\hat\sigma^2(u)} + sL(x,u)\Big)\exp\Big(\hat q(u) + s\int_{x-\epsilon}^u b(y)L(x,y)dy\Big)du \\ &= \frac{\hat Z\,[\mu(\{x\})+\eta]}{\eta}. \end{aligned} \tag{17}$$

There exists a unique solution $s>0$ since the left-hand side is an increasing function of $s\in[0,\infty)$ (as $\int_{x-\varepsilon}^u b(y)L(x,y)\geqslant 0$ for each $u\in[x-\epsilon,x+\epsilon]$), approaches ∞ as $s\to\infty$, and is less than the right-hand side when $s=0$.

4. Now we are ready to complete the proof. Since $q_\eta=\hat q$ on A^c, we have

$$\int_{A^c}\frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)}du = \int_{A^c}\frac{e^{\hat q(u)}}{\hat\sigma^2(u)}du = \hat Z_c \leqslant \hat Z.$$

Also, for each $x\in\mathcal{S}$ using (15), (16) and the definition of $s=s(x)$ in (17) we find that

$$\int_{x-\epsilon}^{x+\epsilon}\frac{e^{q_\eta(y)}}{\sigma_\eta^2(y)}dy = \frac{\hat Z\,[\mu(\{x\})+\eta]}{\eta}.$$

Consequently,

$$\begin{aligned} Z_\eta &= \int_{A^c}\frac{e^{q_\eta(y)}}{\sigma_\eta^2(y)}dy + \sum_{x\in\mathcal{S}}\int_{x-\epsilon}^{x+\epsilon}\frac{e^{q_\eta(y)}}{\sigma_\eta^2(y)}dy \\ &= \hat Z_c + \sum_{x\in\mathcal{S}}\frac{\hat Z\,[\mu(\{x\})+\eta]}{\eta} = \hat Z_c + |\mathcal{S}|\hat Z + \frac{\hat Z}{\eta} \end{aligned}$$

since $\sum_{x\in\mathcal{S}}\mu(\{x\}) = \mu(\mathbb{R}) = 1$, where $|\mathcal{S}|$ is number of points of $\mathcal{S}$. Hence,

$$\begin{aligned} \int_{A^c}\rho_\eta(y)dy &= \frac{1}{Z_\eta}\int_{A^c}\frac{e^{q_\eta}}{\sigma_\eta^2}dy = \frac{\eta\hat Z_c}{(1+\eta|\mathcal{S}|)\hat Z + \eta\hat Z_c}, \\ \int_{x-\epsilon}^{x+\epsilon}\rho_\eta(y)dy &= \frac{1}{Z_\eta}\int_{x-2\epsilon}^{x+2\epsilon}\frac{e^{q_\eta}}{\sigma_\eta^2}dy = \frac{\mu\{x\}+\eta}{1+\eta|\mathcal{S}|+\eta\hat Z_c/\hat Z} \quad \forall x\in\mathcal{S}. \end{aligned}$$

Sending $\eta \searrow 0$ we then conclude that $\rho_\eta \longrightarrow \mu$. This completes the proof. □

6.3. **The case when σ vanishes only beyond all critical points.** Here we consider a case based on the prototype $b(u) = (u-\alpha)(1-u^2)$ and $\sigma(u) = (\max\{\ell^2 - u^2, 0\})^\nu$ with $\ell > 1$. Let $[-1,1]$ be an interval that contains all critical points of the potential, i.e., the set $\{x \in \mathbb{R} \mid b(x) = 0\}$. We consider a degenerate case where σ is positive in an open interval (x_1, x_2) that contains $[-1,1]$ and vanishes outside (x_1, x_2). It turns out that the limit of the regularized invariant measure does not depend on the modification. Therefore, the limit can be defined as the invariant measure.

Theorem 6.5. *Let b be a continuous function on $\mathbb{R}$ that satisfies*

$$b > 0 \text{ in } (-\infty,-1), \quad b < 0 \text{ in } (1,\infty), \quad \varliminf_{|u|\to\infty} |b(u)| > 0.$$

Let σ be a continuous function on $\mathbb{R}$ such that, for some $x_1 < -1$, $x_2 > 1$ and $\epsilon > 0$,

$$\sigma > 0 \text{ in } (x_1, x_2), \qquad \sigma = 0 \text{ in } [x_1 - \epsilon, x_1] \cup [x_2, x_2 + \epsilon]. \tag{18}$$

Let $\rho_\eta(\cdot)$ be defined as in (11) where $\{\sigma_\eta\}_{\eta\in(0,1]}$ is any family of continuous positive functions in $\mathbb{R}$ that satisfies $\lim_{\eta\searrow 0} \sigma_\eta(\cdot) = \sigma(\cdot)$ uniformly on $[x_1 - \epsilon, x_2 + \epsilon]$. Then in measure

$$\lim_{\eta\searrow 0} \rho_\eta(x) = \frac{1}{m_1 + m_2 + m}\left(m_1\delta(x - x_1) + m_2\delta(x - x_2) + \frac{e^{q(x)}}{\sigma^2(x)}\,\chi_{(x_1,x_2)}(x)\right) \tag{19}$$

where

$$q(x) := \int_0^x \frac{b(s)}{\sigma^2(s)}\,ds \;\forall\, x \in [x_1, x_2], \quad m_i := \frac{e^{q(x_i)}}{|b(x_i)|}, \quad m := \int_{x_1}^{x_2} \frac{e^{q(x)}}{\sigma^2(x)}dx.$$

Before the proof, we remark on the sizes of m, m_1 and m_2.

Remark 8. Clearly, m_1, m_2, and m are finite positive constants if $\int_{x_1}^{x_2} \sigma^{-2}(x)dx < \infty$.

Suppose $\int_0^{x_2} \sigma^{-2}(x)dx = \infty$. Then $q(x_2) = -\infty$ since $b(x_2) < 0$. This implies that $m_2 = 0$. In addition, setting

$$\epsilon_1 := \frac{1}{2}\min\{x_2 - 1, -1 - x_1\}, \qquad b_* := \inf_{x\in(-\infty,x_1+\epsilon_1]\cup[x_2-\epsilon_1,\infty)} |b(x)|, \tag{20}$$

we have $b_* > 0$ and

$$\int_{x_2-\epsilon_1}^{x_2} \frac{e^{q(x)}}{\sigma^2(x)}dx \leqslant -\frac{1}{b_*}\int_{x_2-\varepsilon}^{\infty} q'(x)e^{q(x)}dx = \frac{e^{q(x_2-\epsilon_1)}}{b_*} < \infty.$$

Similarly, if $\int_{x_1}^0 \sigma^{-2}(x)dx = \infty$, then $q(x_1) = -\infty, m_1 = 0$, and $\int_{x_1}^{x_1+\varepsilon_1} \sigma^{-2}e^{q(x)}dx < \infty$.

Thus, m *is a finite positive constant whereas* m_1 *and* m_2 *are non-negative finite constants.*

Remark 9. The limit does not depend on regularization, so the right-hand side of (19) can well be defined as the invariant measure ρ^*. Here the contribution of the point masses at x_1 and x_2 is highly non-trivial and has to be obtained through a regularization process.

Proof of Theorem 6.5. **1.** Introduce

$$F_\eta(x) := \int_0^x \frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)}du \quad \forall\, x \in \mathbb{R}, \qquad F(x) := \int_0^x \frac{e^{q(u)}}{\sigma^2(u)}du \quad \forall\, x \in [x_1, x_2].$$

By Remark 8, F is continuous on $[x_1, x_2]$. Since $\sigma > 0$ in (x_1, x_2) and $\lim_{\eta\searrow 0}\sigma_\eta = \sigma$ uniformly,

$$\lim_{\eta\searrow 0} q_\eta(x) = q(x), \quad \lim_{\eta\to 0} F_\eta(x) = F(x) \quad \forall\, x \in (x_1, x_2). \tag{21}$$

2. Next we consider $F_\eta(\infty) - F_\eta(x)$ for $x \in (x_2, \infty)$. Let b_* be as in (20). For $\beta > 0$,

$$\begin{aligned}\int_{x_2+\beta}^\infty \frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)}du &\leqslant \int_{x_2+\beta}^\infty \frac{|b(u)|}{b_*\sigma_\eta^2(u)}e^{q_\eta(x)}du \\ &= -\frac{1}{b^*}\int_{x_2+\beta}^\infty q_\eta'(u)e^{q_\eta(u)}du \leqslant \frac{e^{q_\eta(x_2+\beta)}}{b^*}.\end{aligned}$$

In addition, since $\lim_{\eta\searrow 0}\sigma_\eta = 0$ in $[x_2, x_2+\epsilon]$ and $b < 0$ in $(1, \infty)$,

$$\lim_{\eta\searrow 0} q_\eta(x_2+\beta) = q(1) + \lim_{\eta\searrow 0}\int_1^x \frac{b(s)}{\sigma_\eta^2(s)}ds = -\infty. \tag{22}$$

It then follows that

$$\lim_{\eta\searrow 0}\int_{x_2+\beta}^\infty \frac{e^{q_\eta(u)}}{\sigma_\eta^2(u)}du = 0 \quad \forall\, \beta > 0. \tag{23}$$

3. Now we consider the jump of F_η across x_2. For every $\beta \in (0, \varepsilon_1)$, by the mean value theorem, there exists $x_{\eta,\beta} \in [x_2-\beta, x_2+\beta]$ such that

$$\begin{aligned}e^{q_\eta(x_2-\beta)} - e^{q_\eta(x_2+\beta)} &= -\int_{x_2-\beta}^{x_2+\beta} q_\eta'(x)e^{q_\eta(x)}dx = -\int_{x_2-\beta}^{x_1+\beta}\frac{b(x)}{\sigma_\eta^2(x)}e^{q_\eta(x)}dx \\ &= -b(x_{\eta,\beta})\int_{x_2-\beta}^{x_2+\beta}\frac{e^{q_\eta(x)}}{\sigma_\eta^2(x)}dx \\ &= |b(x_{\eta,\beta})|[F_\eta(x_2+\beta) - F_\eta(x_2-\beta)].\end{aligned}$$

Thus, for any fixed $x > x_2$, in view of (23), (22) and (21), we have

$$\begin{aligned}\overline{\lim_{\eta\searrow 0}} F_\eta(x) &= \lim_{\beta\searrow 0}\overline{\lim_{\eta\searrow 0}} F_\eta(x_2+\beta) \\ &= \lim_{\beta\searrow 0}\overline{\lim_{\eta\searrow 0}}\left(F_\eta(x_2-\beta) + \frac{e^{q_\eta(x_2-\beta)} - e^{q_\eta(x_2+\beta)}}{|b(x_{\eta,\beta})|}\right) \\ &= \lim_{\beta\searrow 0}\left(F(x_2-\beta) + \frac{e^{q(x_2-\beta)}}{\underline{\lim}_{\eta\searrow 0}|b(x_{\eta,\beta})|}\right) \\ &= F(x_2) + \frac{e^{q(x_2)}}{|b(x_2)|} \\ &= F(x_2) + m_2.\end{aligned}$$

Similarly, we derive $\underline{\lim}_{\eta\searrow 0} F_\eta(x) = F(x_2)+m_2$. Hence, $\lim_{\eta\searrow 0} F_\eta(x) = F(x_2)+m_2$ for each $x > x_2$.

In summary, we have

$$\lim_{\eta\searrow 0} F_\eta(x) = \begin{cases} F(x) & \text{if } x \in (x_1, x_2), \\ m_2 + F(x_2) & \text{if } x > x_2. \end{cases}$$

After a similar analysis for $F_\eta(x)$ for $x \leqslant x_1$, we then obtain the assertion of the Theorem. □

Remark 10. Following the same proof, one sees that Theorem 6.5 remains true if condition (18) is replaced by the following more applicable conditions:

$$\begin{aligned}\lim_{\eta\searrow 0}\int_{x_1+\epsilon}^{x_2-\epsilon}\frac{du}{\sigma_\eta^2(u)} &= \int_{x_1+\epsilon}^{x_2-\epsilon}\frac{du}{\sigma^2(u)} < \infty = \int_{x_1-\epsilon}^{x_1}\frac{du}{\sigma^2(u)}\\ &= \int_{x_2}^{x_2+\varepsilon}\frac{du}{\sigma^2(u)}\forall\,\epsilon\in(0,\tfrac{x_2-x_1}{2}).\end{aligned}$$

For example, $\sigma(u) = \sqrt{|(u-x_1)(x_2-u)|}\,|\sin(2\pi u)|^{1/4}$ with $x_1 < -1$ and $x_2 > 1$.

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