

THE MILNOR CONJECTURE AND THE MOTIVIC BLOCH-KATO CONJECTURE

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The following document consists of augmented notes for Talk 9 of the London number theory study group on Motives, which I gave on 7 March 2018.¹ The main sources I consulted to put together these talks are [Dug04, Kah97, MVW06]. Of course, errors I introduce are my own responsibility and I welcome emails suggesting corrections.

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We are now turning from developments in the theory of motives to its applications. Principal among these are the Milnor conjecture and the motivic Bloch-Kato conjecture (henceforth “Bloch-Kato conjecture”). The main goal of this talk is to outline how motivic cohomology is applied in order to prove these conjectures.

Outline. In §1, we state the motivic Bloch-Kato conjecture. In §2, we outline

- (1) how motivic cohomology is related to the objects of the conjecture and
- (2) the main property of motivic cohomology needed in order to prove the conjecture.

The goal of §3 is to give explicit examples of the kinds of calculations that are involved in proving (1). In §4 we give an outline of the proof of (2). In §5, we give an outline of the full proof of (1). However, §5 is incomplete; I have decided to post these notes anyway, since it is unclear whether I will finish §5.

1. THE CONJECTURES

The choices necessary to set up the conjectures are the following. Let $m \geq 1$ be an integer. Let F be a field.

Crucial running assumption. We will always assume that $\text{char}(F)$ does not divide m , unless otherwise stated. When m is replaced by a prime ℓ , we maintain this assumption.

1.1. The actors. We recall the elements involved in the statements of the Milnor and Bloch-Kato conjecture, which were developed in the previous talk (Algebraic K -theory, by Adam Morgan).

Definition 1.1.1. The *Milnor K -theory* of F is the graded commutative ring

$$K^M(F) = \bigoplus_{n \geq 0} K_n^M(F),$$

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¹Study group webpage: <https://nms.kcl.ac.uk/alexander.betts/motives.html>

generated over $K_0^M = \mathbb{Z}$ by the abelian group F^\times in degree 1, and subject to the *Steinberg relations*

$$\{a, 1 - a\} = 0 \quad \text{for all } a \in F \setminus \{0, 1\},$$

i.e. the ideal of relations is generated in degree 2. Here $\{a_1, \dots, a_n\}$ denotes the element $a_1 \otimes \dots \otimes a_n \in K_n^M(F)$ (or in $\text{Sym}_{\mathbb{Z}}^2(F^\times)$, when we define the Steinberg relation).

Remark 1.1.2. As usual, by “graded commutative” we mean “skew-commutative” in that the product of a degree i and j element is intertwined by $(-1)^{ij}$. In fact, this relation follows from the Steinberg relations on the free tensor algebra generated in degree 1 by F^\times .

Definition 1.1.3. Let μ_m denote the m -th roots of unity, which may be thought of as an étale sheaf in finite abelian groups (on the big étale site of $\text{Spec } \mathbb{Z}$). Let G_F be a choice of absolute Galois group of F , arising from a separable closure \overline{F} . For today, by “the Galois cohomology of F (modulo m)” we will mean the graded ring

$$H(F, m) = \bigoplus_{n \geq 0} H^n(F, m) := \bigoplus_{n \geq 0} H^n(G_F, \mu_m^{\otimes n}) \cong \bigoplus_{n \geq 0} H_{\text{ét}}^n(\text{Spec } F, \mu_m^{\otimes n}),$$

where $H^n(G_F, -)$ and $H_{\text{ét}}^n(\text{Spec } F, -)$ denote continuous group cohomology and étale cohomology, respectively; and we use the usual identification between the two, depending on the usual choices.

Here the multiplication operation arises from the cup product in cohomology, written \cup . The cup product \cup is well-known to be graded commutative. For a reference, see e.g. [Sha16].

1.2. The action. There is a natural candidate for a map $K^m(F)/m \rightarrow H(F, m)$.

Proposition 1.2.1 (Kummer theory). *There is a canonical isomorphism*

$$\chi : (F^\times)/m(F^\times) \xrightarrow{\sim} H^1(G_F, \mu_m)$$

induced by $F^\times \ni a \mapsto (\sigma \mapsto \sigma(a^{1/m})/a^{1/m})$ for all $\sigma \in G_F$ and some choice of $a^{1/m}$ in \overline{F} .

Proof. See e.g. [Sha16, Prop. 2.4.12]. □

We call χ the *Kummer map* or Kummer isomorphism. The proof uses Hilbert Theorem 90.

In order to draw a map $K^m(F)/m \rightarrow H(F, m)$, we need to know that the Steinberg relations on the Kummer map vanish in Galois cohomology.

Proposition 1.2.2 (Tate). *For all $a \in F \setminus \{0, 1\}$ and $b = 1 - a$, the cup product $\chi_a \cup \chi_b \in H^2(G_F, \mu_m^{\otimes 2})$ vanishes.*

Proof. Let $E := F[t]/(t^m - a)$ and let α be a the choice $t = a^{1/m}$, producing χ_a . One can check that the norm map for E/F satisfies $N_{E/F}(1 - \alpha) = 1 - a$.

Next we use the restriction and corestriction maps in Galois cohomology

$$\text{res}_{E/F} : H^*(G_F, -) \rightarrow H^*(G_E, -), \quad \text{cores}_{E/F} : H^*(G_E, -) \rightarrow H^*(G_F, -),$$

the fact that $\text{cores}_{E/F} \circ \text{res}_{E/F}$ amounts to multiplication by $[E : F]$, and the compatibility with cup products

$$\text{cores}(\text{res}(\alpha) \cup \beta) = \alpha \cup \text{cores}(\beta).$$

For a reference, see [Sha16, Props. 1.8.22 and 1.9.8]. We also need to know that the corestriction map $\text{cores}_{E/F} : H^1(G_E, \mu_m) \rightarrow H^1(G_F, \mu_m)$ amounts to the norm map $N_{E/F}$.

Applying this to $\chi_a \cup \chi_b$ and dropping “ E/F ” from the notation, we find

$$\chi_a \cup \chi_b = \chi_a \cup \text{cores}(\chi_{1-\alpha}) = \text{cores}(\text{res}(\chi_a) \cup \chi_{1-\alpha}) = \text{cores}(\chi_{\alpha^m} \cup \chi_{1-\alpha}),$$

which is zero because $\chi_{\alpha^m} \in H^1(G_E, \mu_m) \cong E^\times/m(E^\times)$ is zero. \square

As a result, we have the desired map.

Corollary 1.2.3. *There exists a homomorphism of graded rings*

$$\eta = \eta_m : K^m(F)/m \longrightarrow H(F, m)$$

distinguished by sending $a \in F^\times$ to its image $\chi_a \in H^1(G_F, \mu_m)$ under the Kummer isomorphism.

We also write $\eta_{m,n} : K_n^m(F)/m \longrightarrow H^n(F, m)$ for the degree n part of η_m .

1.3. The question. With the map η_m in place, we can state the main conjectures.

Conjecture 1.3.1 (Milnor, Bloch-Kato). *For any $m \geq 1, n \geq 0$ and field F (satisfying the assumption $\text{char}(F) \nmid m$, as always), the statement*

$$K(n, m, F) : \quad \eta_{m,n} : K_n^m(F)/m \longrightarrow H^n(F, m) \text{ is an isomorphism}$$

is true. Equivalently, for any $m \geq 1$ and field F , η_m is an isomorphism of graded rings.

Likewise, we write $K(n, m)$ for the statement that $K(n, m, F)$ holds for all fields F (such that $\text{char}(F) \nmid m$).

In this form and generality, the conjecture is due to Kato, and is known as the *motivic Bloch-Kato conjecture*. The conjecture in the case case $m = 2$ is due to Milnor, so this case is known as the *Milnor conjecture*.

1.4. Cases of the conjecture that are already clear. Let’s immediately observe that there are some cases of the conjecture that we already know are true, before we get close to using input from the theory of motives. This observations are drawn from [Kah97, §1.1].

$K(0, m, F)$ Firstly, for the case $n = 0$ and any (F, m) , we know that $K(0, m, F)$ is true because we have $\mathbb{Z}/m \cong K_0^M(F)/m \xrightarrow{\sim} H^0(G_F, \mathbb{Z}/m) \cong \mathbb{Z}/m$.

$K(1, m, F)$ For $n = 1$, the map $K_1^M(F)/m \xrightarrow{\sim} H^1(G_F, \mu_p)$ is precisely the Kummer isomorphism χ .

Then there are cases that were known before Voevodsky’s application of motives to the conjecture.

$K(2, m, F)$ Tate proved this in the case of global fields F , using *class field theory*. This proof also applies to local fields.

More generally, using the isomorphism $K_2^M(F) \xrightarrow{\sim} K_2(F)$ with algebraic K -theory (the *theorem of Matsumoto*; see the previous talk), $K(2, 2)$ was proved by Merkurjev and $K(2, m)$ for all $m \geq 1$ was proved by Merkurjev-Suslin.

$n \geq 3$ For such n and for global fields F , Bass and Tate proved $K(n, m, F)$ for all $m \geq 1$. In fact, $K_n^M(F) \simeq \mathbb{Z}/2\mathbb{Z}^{\oplus r_1(F)}$, where $r_1(F)$ is the number of real places of F .

For henselian discrete valuation fields F such that $\text{char}(F) = 0$ and its residue characteristic is $p > 0$, $K(n, p, F)$ is a theorem of Bloch-Kato-Gabber.

2. MOTIVIC COHOMOLOGY

Our goal in this section is to outline, as concisely as possible, the relationship between motivic cohomology and the actors in the Bloch-Kato conjecture $K(n, m)$. To this end, we first recall what has been introduced in previous talks. Most of what we recall is in talk 7, two weeks ago (on the triangulated category of motives, by Giada Grossi). Then we relate motivic cohomology to the Milnor K -groups and Galois cohomology groups present in the Bloch-Kato conjecture. However, we find it necessary, in order to be as concrete as possible, to introduce motivic cohomology in a more lowbrow manner than in talk 7: it will be convenient to express it as Zariski hypercohomology of certain complexes.

2.1. Recalling talk 7: Voevodsky’s triangulated category of motives. In talk 7 we were introduced to the following categories.

- A category $\text{SmCor}(F)$ of smooth F -schemes with finite correspondences.
- The homotopy category $\mathcal{H}^b(\text{SmCor}(F))$ of bounded complexes (of abelian groups) over $\text{SmCor}(F)$.
- The category of “geometric motives” $\text{DM}_{gm}(F)$, which is the pseudo-abelian envelope of the localization of $\mathcal{H}^b(\text{SmCor}(F))$ at its thick subcategory generated by \mathbb{A}^1 -homotopy invariance and Mayer-Vietoris.
- The category of sheaves with transfers for the Nisnevich topology, denoted $\text{Sh}_{\text{tr}}^{\text{Nis}}(F)$. We write $D^-(\text{Sh}_{\text{tr}}^{\text{Nis}}(F))$ for its bounded-above derived category.
- Finally, defining the *triangulated tensor category of motives* $\text{DM}_-(F)$ to be the full subcategory of $D^-(\text{Sh}_{\text{tr}}^{\text{Nis}}(F))$ consisting of complexes with homotopy-invariant cohomology sheaves.

There was some subtlety in defining a new (not the natural one)

$$\iota : \text{DM}_{gm}(F) \hookrightarrow \text{DM}_-(F)$$

respecting both the tensor and triangulated structure. The *Tate object*

$$\mathbb{Z}(1) := ([\mathbb{P}^1]_0 \rightarrow [\text{Spec } F]_1)[-2]$$

(where the subscripts denote the degree in the complex) exists in $\text{DM}_{gm}(F)$, but we will write $\mathbb{Z}(1)$ for $\iota(\mathbb{Z}(1))$ in $\text{DM}_-(F)$. As usual, we write $Z(n) := \mathbb{Z}(1)^{\otimes n}$ and $M(n) = M \otimes \mathbb{Z}(n)$ in $\text{DM}_-(F)$.

Remark 2.1.1. For the relationship between this notion of Tate object and the alternate definition we will use today (Definition 2.2.1), see [MVW06, Example 6.15].

Finally, we were introduced in talk 7 to *motivic cohomology* $H_{\text{mot}}^p(X, \mathbb{Z}(q))$ for $p, q \in \mathbb{Z}$, defined as

$$\text{Hom}_{\text{DM}_{gm}(F)}(X, \mathbb{Z}(k)[i]).$$

As is common, we will write this in a bigraded form as

$$H_{\text{mot}}^{p,q}(X, \mathbb{Z}) := H_{\text{mot}}^p(X, \mathbb{Z}(p)),$$

which we will use from now on.

2.2. A most basic construction of motivic cohomology. Because our goal today is to illustrate as much of the proof of the Bloch-Kato conjecture as possible, it will be convenient to use a more basic definition of motivic cohomology, amenable to our desired computations. It also happens to be useful for merely stating the relationship between motivic cohomology and the actors in the Bloch-Kato conjecture, which is why we discuss this basic definition first.

This motivic cohomology arises in the Zariski topology. Of course this motivic cohomology can be found to be representable in the richer categories above (see e.g. [MVW06, Lecture 13]); we will not discuss this further.

We follow [MVW06, Lecture 2-3].

Recall that a presheaf with transfers (over F) is a contravariant additive functor F from $\text{SmCor}(F)$ to abelian groups. There is a usual Yoneda embedding, from $\text{SmCor}(F)$ into such presheaves, and we write such a presheaf arising from $X \in \text{Sm}_F$ as $\mathbb{Z}_{\text{tr}}(X)$.² (Here we are implicitly using the usual faithful functor $\text{Sm}_F \rightarrow \text{SmCor}(F)$.)

In order to define the Tate object in this context, we require a few more constructions.³

- The presheaf with transfers (valued in abelian groups) associated to a pointed object of Sm_F . Given a pointed F -scheme (X, x) , let $\mathbb{Z}_{\text{tr}}(X, x)$ be the cokernel of $x_* : \mathbb{Z} \rightarrow \mathbb{Z}_{\text{tr}}(X)$ associated to $x : \text{Spec } F \rightarrow X$. We will apply this in the case of $\mathbb{G}_m := (\mathbb{A}^1 \setminus \{0\}, 1)$.
- The cosimplicial scheme

$$\Delta^n := \text{Spec } F[t_0, \dots, t_n] / \left(1 - \sum_{i=0}^n t_i \right),$$

where the i -th face maps $\Delta^n \rightarrow \Delta^{n+1}$ is given by inserting $x_i = 0$.

- Given a presheaf \mathcal{F} on Sm_F , we have simplicial abelian groups $\mathcal{F}(U \times_F \Delta^\bullet)$ for any $U \in \text{Sm}_F$. This presheaf of simplicial abelian groups will be written $C_\bullet \mathcal{F}$.
- Given a simplicial abelian group, there is a standard associated complex.⁴ It satisfies $(C_\bullet \mathcal{F})_n = (C_* \mathcal{F})_n$ for all $n \in \mathbb{Z}$. We will write the complex associated to $C_\bullet \mathcal{F}$ as $C_* \mathcal{F}$.
 - As an example, when \mathcal{F} is a constant presheaf taking the value A on all $U \in \text{Sm}_F$, then $C_* \mathcal{F}$ is the complex

$$\cdots \xrightarrow{\text{id}} A \xrightarrow{0} A \xrightarrow{\text{id}} A \xrightarrow{0} A \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

It is quasi-isomorphic to the complex with A concentrated in degree 0.

- Note, however, that $\mathbb{Z}(1)$ below is not constant. We will work on relating $\mathbb{Z}(1)$ to something “more” constant in §3.

We apply these constructions in the definition of $\mathbb{Z}(1)$.

²It was written as $L(X)$ in talk 7. See also [MVW06, p. 15].

³Those who attended the talk should take note of these items, as I bungled these in the talk.

⁴Apparently it is sometimes called the “alternating face map complex.”

Definition 2.2.1. The *Tate object* or *motivic 1-sphere* is the complex of presheaves with transfers

$$\mathbb{Z}(1) := (C_*(\mathbb{Z}_{\text{tr}}(\mathbb{G}_m)))[-1].$$

Likewise, we define the q -th *motivic complex* or *motivic q -sphere* to be

$$\mathbb{Z}(q) := C_*(\wedge^q(\mathbb{Z}_{\text{tr}}(\mathbb{G}_m)))[-q].$$

Here \wedge^q denotes the cokernel of the map to $\mathbb{Z}_{\text{tr}}((\mathbb{A}^1 \setminus \{0\})^{\times q})$ from

$$\bigoplus_{i=1}^q \mathbb{Z}_{\text{tr}}((\mathbb{A}^1 \setminus \{0\})^{\times q-1})$$

sending (x_1, \dots, x_{q-1}) to $(x_1, \dots, x_{i-1}, 1, x_i, \dots, x_{q-1})$ on the i -th summand.

We will attempt to give some motivating comments explaining what is going on here, using an analogy with topological homotopy theory.

Remark 2.2.2. We attempt to explain why $\mathbb{Z}(1)$ can be called a “motivic 1-sphere” and $\mathbb{Z}(q)$ a “motivic q -sphere.” The starting point is to aim for a (pointed) \mathbb{G}_m to play the role of the pointed 1-sphere $(S^1, *)$ in usual topological homotopy theory. The problem is that “algebraic geometry has too few maps” for \mathbb{G}_m to be able to calculate homotopy groups. Therefore we move to the category of presheaves with transfers, which has more maps, and has the flexibility to keep track of smash products (see below) of pointed objects. The object $\mathbb{Z}(1)$ is simply the complex associated to the simplicial object of maps from Δ^\bullet into $(\mathbb{G}_m, 1)$.

More specifically, the simplicial set $C_\bullet \mathbb{Z}_{\text{tr}}(\mathbb{G}_m)$ is analogous to the simplicial set of singular chains of S^1 . And $C_* \mathbb{Z}_{\text{tr}}(\mathbb{G}_m)$ is analogous to the singular chain complex of S^1 . So we expect to be able to use $C_* \mathbb{Z}_{\text{tr}}(\mathbb{G}_m)$ to calculate “ π_1 .” We will apply it to construct cohomology.

And while a smash product does not exist in the category of pointed schemes, it does exist in the category of presheaves with transfers. Namely, the definition of $\wedge^q \mathbb{Z}_{\text{tr}}(\mathbb{G}_m)$ represents the q -th smash power of $(\mathbb{G}_m, 1)$, in analogy with the fact that $(S^q, *)$ is the q -th smash power of $(S^1, *)$.

Putting this altogether, the sequence $\mathbb{Z}(q)$ behaves like a spectrum (a “motivic sphere spectrum,” I suppose), as a spectrum in topology is a sequence of topological spaces X_i equipped with maps from the smash product $X_i \wedge S^1 \rightarrow X_{i+1}$. Evaluating $\mathbb{Z}(q)$ on $X \in \text{Sm}_F$ amounts to taking certain correspondences from X into $\mathbb{G}_m^{\times q}$, it makes sense that we are relating $\mathbb{Z}(q)(X)$ to cohomology. (As to any spectrum, there is an associated cohomology theory given by taking maps into it.)

We also observe that this grading according to the q in $\mathbb{Z}(q)$ makes sense as a cohomological grading, but then there is another grading by the degree p in sheaf cohomology that will appear shortly.

Remark 2.2.3. As noted in talk 7, and in light of the previous remark, we want the cohomology of motivic objects like $\mathbb{Z}(1)$ to be homotopy invariant. This is verified in [MVW06, Cor. 2.19].

Of course the motivic complexes $\mathbb{Z}(q)$ have many extra properties and structures. For example, there is a product $\mathbb{Z}(q) \otimes \mathbb{Z}(q') \rightarrow \mathbb{Z}(q+q')$ that is commutative and associative up to homotopy (see e.g. [MVW06, Constr. 3.11]). But for today’s purposes we mainly need the following.

Lemma 2.2.4. *Let $X \in \text{Sm}_F$. The restriction $\mathbb{Z}(q)_X$ of $\mathbb{Z}(q)$ to the (small) Zariski site over X is a complex of sheaves. Likewise, $\mathbb{Z}(q)$ is sheafy for the étale topology on Sm_F .*

Proof. This follows directly from $\mathbb{Z}_{\text{tr}}(T)$ being a sheaf for these topologies for any F -scheme T . See [MVW06, Lemmas 3.2 and 6.2], and also §3. \square

Now we can give our (relatively!) lowbrow definition of motivic cohomology.

Definition 2.2.5. Let $X \in \text{Sm}_F$. The *motivic cohomology* $H^{p,q}(X, \mathbb{Z})$ is defined to be the hypercohomology in the Zariski topology on X of $\mathbb{Z}(q)$, i.e.

$$H_{\text{mot}}^{p,q}(X, \mathbb{Z}) := \mathbb{H}_{\text{zar}}^p(X, \mathbb{Z}(q)_X).$$

Remark 2.2.6. As eluded to above, it is a non-trivial theorem that the incarnation of $\mathbb{Z}(q)$ presented in talk 7 represents these motivic cohomology groups. For examples of motivic cohomology groups that can be computed without a great deal of difficulty, see [MVW06, Lecture 4]. We will discuss some aspects of this when we investigate $\mathbb{Z}(1)$ in §3.

2.3. Relationship with Milnor K -theory. One of the main desiderata for motivic cohomology is the following relationship with Milnor K -theory, which is our first indication of how one hopes to prove the Bloch-Kato conjecture using motivic cohomology.

Theorem 2.3.1. *For any field F and $n \geq 0$, we have*

$$H_{\text{mot}}^{n,n}(\text{Spec } F, \mathbb{Z}) \cong K_n^M(F).$$

We will aim to illustrate the proof later in this talk (initially in §3, and concluding in §5). The impatient reader can immediately turn to [MVW06, Lecture 5] as a reference.

2.4. Étale motivic cohomology. While motivic cohomology arises in the Zariski topology, so that Milnor K -theory is a Zariski hypercohomology group, we already know that the Galois cohomology appearing in the Bloch-Kato conjecture is an étale cohomology group. So we expect some étale realization of motivic cohomology. Because we have already discussed that the motivic complexes $\mathbb{Z}(q)$ are also sheafy for the étale topology on Sm_F , we can do this without delay.

We especially follow [MVW06, Lecture 10] here.

Definition 2.4.1. The *étale motivic cohomology* of $X \in \text{Sm}_F$ is the hypercohomology in the Zariski topology of $\mathbb{Z}(q)$, i.e.

$$H_L^{p,q}(X, \mathbb{Z}) := \mathbb{H}_{\text{ét}}^p(X, \mathbb{Z}(q)|_{X_{\text{ét}}}).$$

Here the “L” stands for “Lichtenbaum,” as he is credited with envisioning this construction.

The following result gives us a relationship with conventional étale cohomology.

Theorem 2.4.2. *For $m \geq 1$ such that $\text{char}(F) \nmid m$, and $X \in \text{Sm}_F$,*

$$H_L^{p,q}(X, \mathbb{Z}/m) \cong H_{\text{ét}}^p(X, \mu_m^{\otimes q})$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{Z}_{\geq 0}$.

The proof will be discussed in §5.1.

Remark 2.4.3. This theorem is to-be-expected given the discussion of realization functors on motivic cohomology discussed at the end of talk 7. Indeed, ℓ -adic étale cohomology is built out of $H_{\text{ét}}^p(X, \mu_m^{\otimes q})$ for $m = \ell^a$.

The following particular case is relevant for the Bloch-Kato conjecture.

Corollary 2.4.4. *For $\text{char}(F) \nmid m$ and some $n \geq 0$,*

$$H_L^{n,n}(\text{Spec } F, \mathbb{Z}/m) \cong H^p(G_F, \mu_m^{\otimes n}).$$

2.5. The Bloch-Kato conjecture restated in terms of motivic cohomology.

With Theorem 2.3.1 and Corollary 2.4.4 in hand, we have a new way to understand the map $\eta_{m,n}$ between the actors in the Bloch-Kato conjecture. Namely, we have the morphism of sites $\alpha_X : X_{\text{ét}} \rightarrow X_{\text{Zar}}$ inducing maps $H_{\text{mot}}^{p,q}(X, A) \rightarrow H_L^{p,q}(X, A)$ for an abelian group A – from motivic cohomology to étale motivic cohomology. Unsurprisingly, the resulting map in the case $p = q = n$ and $X = \text{Spec } F$ is compatible with the map η of the the motivic Bloch-Kato conjecture.

Lemma 2.5.1. *For $n \geq 0$ and $\text{char}(F) \nmid m \geq 1$, the square*

$$\begin{array}{ccc} H_{\text{mot}}^{n,n}(\text{Spec } F, \mathbb{Z}/m) & \xrightarrow{\alpha_F} & H_L^{n,n}(\text{Spec } F, \mathbb{Z}/m) \\ \downarrow & & \downarrow \\ K_n^M(F)/m & \xrightarrow{\eta_{m,n}} & H^p(G_F, \mu_m^{\otimes n}) \end{array}$$

commutes.

Proof. This lemma will follow from the discussion of the case $n = 1$ appearing in §3. \square

Therefore, assuming Theorem 2.3.1 and Corollary 2.4.4, we have the following equivalent formulation of the Bloch-Kato conjecture.

Theorem 2.5.2. *Let $m \geq 1$ such that $\text{char}(F) \nmid m$. The map $(\text{Spec } F)_{\text{ét}} \rightarrow (\text{Spec } F)_{\text{Zar}}$ induces an isomorphisms, for all $n \geq 0$,*

$$H_{\text{mot}}^{n,n}(\text{Spec } F, \mathbb{Z}/m) \xrightarrow{\sim} H_L^{n,n}(\text{Spec } F, \mathbb{Z}/m).$$

Therefore, the remaining goals of this talk are to illustrate as much as possible of the proofs of the following three statements.

- Motivic-Milnor** i.e. Theorem 2.3.1, that motivic cohomology $H_{\text{mot}}^{n,n}(\text{Spec } F, \mathbb{Z})$ realizes Milnor K -theory $K_n^M(F)$.
 - For this, see §5 for a reference.
- Motivic-Galois** i.e. Corollary 2.4.4, that étale motivic cohomology $H_L^{n,n}(\text{Spec } F, \mathbb{Z}/m)$ realizes Galois cohomology $H^n(G_F, \mu_m^{\otimes n})$ when $\text{char}(F) \nmid m$.
 - For this, see §5.1.
 - Theorem 2.5.2.
 - For this, see §4.

3. EXPLICATING $\mathbb{Z}(1)$: “MOTIVIC KUMMER THEORY”

Both Motivic-Milnor and Motivic-Galois relations have a common starting point, which is understanding the Tate object motivic complex, known as $\mathbb{Z}(1)$. We mainly follow [MVW06, Lecture 4]. This amounts to a motivic setting of Kummer theory,

and immediately gives rise to facts about motivic cohomology that are of interest independent of the Bloch-Kato conjecture.

This study, combined with a brief calculation, will result in the proof of the Motivic-Galois relation and Motivic-Milnor relations in degree $n = 1$ only. We will discuss general n in §5.

The reason that we are especially dwelling on $\mathbb{Z}(1)$ is that it is a fitting laboratory to see how motives and correspondences are related to objects we are used to approaching from other perspectives. The proof of Theorem 3.1.1 is the most detail we will give in these notes.

3.1. The main result on $\mathbb{Z}(1)$. Our goal is to illustrate this theorem. First we state it and observe some consequences. We follow [MVW06, Lecture 4] very closely, so here we give the main points and refer the reader to *loc. cit.* for further details.

Theorem 3.1.1. *There is a quasi-isomorphism of complexes of presheaves with transfers*

$$\mathbb{Z}(1) \xrightarrow{\sim} \mathcal{O}^\times[-1].$$

Firstly we should note that the transfers on \mathcal{O}^\times are not completely obvious from the start. Indeed, it is not representable as the presheaf with transfers $\mathbb{Z}_{\text{tr}}(\mathbb{G}_m)$ arising from \mathbb{G}_m , even though $\mathcal{O}^\times(X) = \mathbb{G}_m(X)$ as groups. When we write \mathcal{O}^\times , we are considering the transfers to arise from the norm map on functions fields [MVW06, Ex. 2.4].

3.2. Consequences of Theorem 3.1.1. Here are some immediate consequences of the theorem.

Corollary 3.2.1. *Let $X \in \text{Sm}_F$.*

- (1) $H_{\text{mot}}^{1,1}(X, \mathbb{Z}) \cong H_L^{1,1}(X, \mathbb{Z}) \cong \mathcal{O}^\times(X)$.
- (2) $H_{\text{mot}}^{2,1}(X, \mathbb{Z}) \cong \text{Pic}(X)$.
- (3) $H_{\text{mot}}^{p,1}(X, \mathbb{Z}) = 0$ for $p \neq 1, 2$.

We can also deduce some consequences toward the Bloch-Kato conjecture. Applying the theorem, we have

$$\mathbb{Z}/m(1) \simeq \mathcal{O}^\times[-1] \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Z}/m = [\mathcal{O}^\times \xrightarrow{m} \mathcal{O}^\times],$$

where the complex is concentrated in degrees 0 and 1. Computing with this, we have

Corollary 3.2.2. *There is a quasi-isomorphism of étale sheaves $\mathbb{Z}/m(1)_{\text{ét}} \cong \mu_m$, where $(-)_{\text{ét}}$ denotes étale sheafification.*

From this we can derive the Motivic-Galois relation (Corollary 2.4.4) in degree $n = 1$. Combining this with Corollary 3.2.1, we can deduce that the Zariski to étale motivic cohomology map in $H_{(-)}^{1,1}(\text{Spec } F, \mathbb{Z})$ realises the Bloch-Kato conjecture in the already-known case $n = 1$.

Because no aspect of the extension of these results to $n > 1$ is trivial, further discussion in this direction is delayed to §5.

3.3. Beginning the proof of Theorem 3.1.1: Constructions. The main constructions of the proof are a presheaf with transfers \mathcal{M} and a map λ . Throughout this discussion, we let t denote the standard coordinate on $\mathbb{P}^1 \supset \mathbb{A}^1 \supset \mathbb{A}^1 \setminus \{0\}$.

Definition 3.3.1. Let \mathcal{M} denote the functor (and, one readily observes, Zariski sheaf) $\text{Sm}_F \rightarrow \text{Ab}$ sending X to the group of rational functions on $X \times \mathbb{P}^1$ that are regular in an open neighborhood of $X \times \{0, \infty\}$ and equal to 1 on $X \times \{0, \infty\}$.

We want there to be a morphism of presheaves with transfers $\mathcal{M} \hookrightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}^1 \setminus \{0\})$ given by sending $f \in \mathcal{M}(X)$ to the Weil divisor of $D(f)$ on $X \times \mathbb{A}^1$ of $f|_{X \times \mathbb{A}^1}$. It will then be clear enough that this is a monomorphism. The least obvious part of this claim is that $D(f)$ is finite and surjective over X . The proof is straightforward and appears in [MVW06, Lem. 4.3]: reducing to the affine case $X = \text{Spec } A$, one can find that f is a ratio of elements of $A[t]$ and check that these two polynomials have the same degree with leading coefficients in A^\times .

The following basic lemma will be useful for understanding the image of $\mathcal{M} \hookrightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}^1 \setminus \{0\})$. I am elaborating on it because I found its very quick treatment in the proof of [MVW06, Lem. 4.4] confusing at first.

Lemma 3.3.2. *Given some $Z \in \text{Cor}(X, \mathbb{A}^1)$, there exists a unique rational function f on $X \times \mathbb{P}^1$ and an integer n such that the Weil divisor of its restriction to $X \times \mathbb{A}^1$ satisfies $D(f|_{X \times \mathbb{A}^1}) = Z$ and $(f/t^n)|_{X \times \infty} = 1$, where t is the coordinate on \mathbb{P}^1 .*

Proof. The main input to the proof of this lemma is that there is an isomorphism $\text{Pic}(X \times \mathbb{P}^1) \cong \text{Pic}(X) \times \text{Pic}(\mathbb{P}^1)$, the rightward map arising from restriction on the factors and the leftward map sending a line bundles $\mathcal{L}/X, \mathcal{L}'/\mathbb{P}^1$ to $\mathcal{L} \boxtimes \mathcal{L}'$. Consequently, the following a priori right exact sequence (see e.g. [Har77, Prop. II.6.5]⁵)

$$\text{Pic}(X \times \{\infty\}) \longrightarrow \text{Pic}(X \times \mathbb{P}^1) \longrightarrow \text{Pic}(X \times \mathbb{A}^1) \longrightarrow 0$$

is actually short exact. Therefore we have an induced isomorphism $\text{Pic}(X \times \mathbb{A}^1) \xrightarrow{\sim} \text{Pic}(\mathbb{P}^1)$. Applying this isomorphism to divisors, we find that any divisor in $X \times \mathbb{A}^1$, thought of as a divisor in $X \times \mathbb{P}^1$, is linearly equivalent to a divisor that is a multiple n of $X \times \{\infty\}$. Therefore, thinking of Z as a divisor of $X \times \mathbb{P}^1$, there exists $f \in F(X \times \mathbb{P}^1)$ and $n \in \mathbb{Z}$ such that $D(f) = Z + n(X \times \{\infty\})$. As any non-constant rational function on \mathbb{P}^1 has a non-trivial divisor, there is a unique unit $u \in \mathcal{O}^\times(X \times \mathbb{P}^1)$ such that $uf/t^n|_{X \times \{\infty\}} = 1$. \square

Definition 3.3.3. For each $X \in \text{Sm}_F$, we use the lemma to define a map of abelian groups

$$\lambda : \mathbb{Z}_{\text{tr}}(\mathbb{A}^1 \setminus \{0\}) \longrightarrow \mathbb{Z} \oplus \mathcal{O}^\times(X)$$

by sending $Z \in \text{Cor}(X, \mathbb{A}^1 \setminus \{0\})$ to

$$Z \mapsto (n, (-1)^n f(0)),$$

where f, n are associated uniquely to Z in the lemma and $f(0)$ refers to the restriction $f|_{X \times \{0\}}$. It is clear from the lemma that $f(0) \in \mathcal{O}^\times(X)$.

Lemma 3.3.4. *There is a short exact sequence of presheaves*

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\lambda} \mathbb{Z} \oplus \mathcal{O}^\times \longrightarrow 0.$$

⁵Check that this reference is appropriate.

Proof. We have discussed above that we have an injection $\mathcal{M} \hookrightarrow \mathbb{Z}_{\text{tr}}(\mathbb{A}^1 \setminus \{0\})$. The sequence is exact in the middle because those $Z \in \mathbb{Z}_{\text{tr}}(\mathbb{A}^1 \setminus \{0\})(X)$ sent to $(0, 1) \in \mathbb{Z} \times \mathcal{O}^\times(X)$ are exactly those Z where there exists (by the lemma) a unique $f \in F(X \times \mathbb{P}^1)$ such that $f(0) = 1$, $f(\infty) = 1$, and $D(f) = Z$.

The sequence is exact on the right because if one chooses $u \in \mathcal{O}^\times(X \times \{\infty\})$, u extends to $X \times \mathbb{P}^1$ constant along the fibres and then one observes that $\lambda(D(t-u)) = (1, u)$. Similarly, $\lambda(D(t-u) - D(t-1)) = u$, so λ is surjective. \square

3.4. Properties of λ and \mathcal{M} . The right two terms of the short exact sequence are endowed with transfers. We want to prove the following lemma in order to compatibly endow \mathcal{M} with transfers as well.

Lemma 3.4.1. *λ is compatible with transfers.*

Proof. The source $\mathbb{Z}_{\text{tr}}(\mathbb{A}^1 \setminus \{0\})$ is representable, as is the factor $\mathbb{Z} = \mathbb{Z}_{\text{tr}}(\text{Spec } F)$ of the target. The projection of λ onto \mathbb{Z} is compatible with transfers because it arises from the structure morphism $\mathbb{A}^1 \setminus \{0\} \rightarrow \text{Spec } F$. It remains to address the projection onto \mathcal{O}^\times .

Here one may first reduce to case that X is the spectrum of a finitely generated field over F . This is given in [MVW06, Exercise 1.13]: $X \in \text{Sm}_F$ can be replaced with its function field $F(X)$. Abusing notation by writing F for such a field, and then E/F for a finite extension, it remains to verify the compatibility with transfers

$$\begin{array}{ccc} \text{Cor}_F(\text{Spec } E, \mathbb{A}^1 \setminus \{0\}) & \xrightarrow{\lambda} & E^\times \\ \downarrow \alpha & & \downarrow N_{E/F} \\ \text{Cor}_F(\text{Spec } F, \mathbb{A}^1 \setminus \{0\}) & \xrightarrow{\lambda} & F^\times. \end{array}$$

As a special case of Lemma 3.3.2, we observe that $\text{Cor}_F(\text{Spec } E, \mathbb{A}^1 \setminus \{0\}) \cong \text{Cor}_E(\text{Spec } E, \mathbb{A}_E^1 \setminus \{0\})$ (where this equivalence comes from [MVW06, Exer. 1.12]) is in bijective correspondence with elements of $E(t)$ that can be written as a ratio of monic polynomials $f, f' \in E[t]$ with constant coefficients in E^\times . Then λ sends f/f' to $f(0)/f'(0)$. It will suffice to prove the commutativity of the square assuming $f' = 1$, i.e. we work only with $f \in E[t]$.

We see that α amounts to intersecting $D(f) = V(f) \subset \mathbb{A}_E^1$ with $\mathbb{A}_F^1 \subset \mathbb{A}_E^1$. Write this intersection as $\alpha(V(f)) = V(f_F)$, for some monic $f_F \in F[t]$. Choose a primitive element $s \in E$ so that $E = F[s]/(g(s))$, where $g \in F[s]$ is monic. We observe that $f_F(0) = \lambda(\alpha(V(f)))$ is the constant coefficient (as a polynomial in s) of $f(0) \in E^\times$. This is known to be $N_{E/F}(f(0)) = N_{E/F}(\lambda(V(f)))$, irrespective of the choice of s . \square

Now we have the desired consequence.

Corollary 3.4.2. *\mathcal{M} has a structure of a presheaf with transfers determined by the isomorphism $\mathcal{M} \xrightarrow{\sim} \ker(\lambda)$ of presheaves. Thus the short exact sequence of Lemma 3.3.4 is a short exact sequence in the category of presheaves with transfers.*

Applying the exact functor C_* this short exact sequence and excising the extra copy of $C_*\mathbb{Z}$ (changing $\mathbb{Z}_{\text{tr}}(\mathbb{A}^1 \setminus \{0\})$ to $\mathbb{Z}_{\text{tr}}(\mathbb{G}_m)$) gives rise to a short exact sequence of complexes of presheaves with transfers

$$(3.4.3) \quad 0 \longrightarrow C_*(\mathcal{M}) \longrightarrow \mathbb{Z}(1)[1] \longrightarrow C_*(\mathcal{O}^\times) \longrightarrow 0.$$

The main goal of this section, Theorem 3.1.1, now follows from this final lemma.

Lemma 3.4.4. $C_*(\mathcal{M})$ is an acyclic complex. That is, for every $X \in \text{Sm}_F$, $C_*(\mathcal{M})(X)$ is an acyclic complex of abelian groups.

For this proof, we use the normalized chain complex⁶ $C_*^{\text{DK}}(\mathcal{M})$ associated to the simplicial presheaf with transfers $C_\bullet(\mathcal{M})$. It is a subcomplex of $C_*(\mathcal{M})$ consisting of those elements of $C_n(\mathcal{M})$ that vanish under the face maps $d_i : C_n(\mathcal{M}) \rightarrow C_{n-1}(\mathcal{M})$ for $0 < i \leq n$. The differential on $C_n^{\text{DK}}(\mathcal{M})$ is the 0-th face map. Therefore a cycle in $C_n^{\text{DK}}(\mathcal{M})$ is an element of $C_n(\mathcal{M})$ vanishing under every face map.

A crucial property of the $C_*^{\text{DK}}(A_\bullet)$ of a simplicial abelian group A_\bullet is that $C_*^{\text{DK}}(A_\bullet) \hookrightarrow C_*(A_\bullet)$ is a quasi-isomorphism. That is why we can deal with $C_*^{\text{DK}}(\mathcal{M})$ in place of $C_*(\mathcal{M})$ in this proof.

Proof. Let $f \in C_n^{\text{DK}}(\mathcal{M})(X)$ be a cycle. This means that f is a regular function on some open neighborhood of

$$Z := X \times \Delta^n \times \{0, \infty\} \subset X \times \Delta^n \times \mathbb{P}^1$$

that vanishes on Z as well as on every face $\simeq X \times \Delta^{n-1} \times \mathbb{P}^1$. Now consider that $h_X(f) := 1 - t(1 - f)$ is an element of $C_*(\mathcal{M})(\mathbb{A}^1 \times X)$, where t is the coordinate on \mathbb{A}^1 . Moreover, it follows from our discussion of what a cycle is in $C_n^{\text{DK}}(\mathcal{M})(X)$ that $h_X(f)$ is a cycle in $C_n^{\text{DK}}(\mathcal{M})(\mathbb{A}^1 \times X)$. We claim that $h_X(f)|_{t=0} = 1$ and $h_X(f)|_{t=1} = f$ are chain homotopic in $C_n(\mathcal{M})(X)$. The lemma follows from the claim, because C_*^{DK} produces a subcomplex of C_* and all boundaries in C_* are also in C_*^{DK} .

It remains to prove the claim. In fact this is a general lemma that is useful for observing that certain presheaves with transfers are homotopy invariant, so we separate it off. \square

Lemma 3.4.5. Let \mathcal{F} be a presheaf and let $X \in \text{Sm}_F$. Then the maps

$$i_0^*, i_1^* : C_*(\mathcal{F})(\mathbb{A}^1 \times X) \longrightarrow C_*(\mathcal{F})(X)$$

are chain homotopic.

Proof. Omitted. \square

Remark 3.4.6. This lemma appears as [MVW06, Lem. 2.18], and the reader can see in that context the relation of this lemma to homotopy invariance of presheaves. We omit the proof as I feel that I have especially little to add on this point.

3.5. Conclusion of the proof of Theorem 3.1.1. One final observation is needed.

Proof. We have a short exact sequence of complexes of presheaves with transfers (3.4.3). Lemma 3.4.4 tells us that $C_*\lambda$ induces a quasi-isomorphism $\mathbb{Z}(1) \xrightarrow{\sim} C_*(\mathcal{O}^\times)[-1]$. It remains to observe that $C_*(\mathcal{O}^\times)$ is quasi-isomorphic to \mathcal{O}^\times . This follows from the fact that $\mathcal{O}^\times(X \times \mathbb{A}^n) \xrightarrow{\sim} \mathcal{O}^\times(X)$ for all $n \in \mathbb{N}$ and $X \in \text{Sm}_F$, along with the observation that $C_*(A_\bullet)$ of a constant simplicial abelian group $A_\bullet = A$ is quasi-isomorphic to the complex with A concentrated in degree 0. \square

⁶It is also known as the Moore complex.

4. OUTLINE OF THE PROOF OF THE CONJECTURE

Having completed the Motivic-Milnor relation and the Motivic-Galois relation, it remains to show Theorem 2.5.2, i.e. the statement that

$$\alpha_F : H_{\text{mot}}^{n,n}(\text{Spec } F, \mathbb{Z}/m) \rightarrow H_L^{n,n}(\text{Spec } F, \mathbb{Z}/m)$$

is an isomorphism. In this we follow [Kah97].

4.1. Reduction steps. First we give some reduction steps that do not require motivic cohomology, following [Kah97, §1.2]. The conclusion of the reduction will be that we can address only primes $m = \ell$ and fields F of characteristic zero.

Proposition 4.1.1. *Let $n \geq 0$.*

- (1) *Let $m_1, m_2 \geq 1$, not divisible by $\text{char}(F)$ and such that $(m_1, m_2) = 1$. Then $K(n, m_1 m_2, F)$ is equivalent to $(K(n, m_1, F) \text{ and } K(n, m_2, F))$.*
- (2) *For $\text{char}(F) \nmid m$ and E/F of degree prime to m , $K(n, m, E) \Rightarrow K(n, m, F)$.*
- (3) *Let ℓ be a prime number, $\ell \neq \text{char}(F)$. Then $K(n-1, \ell, F)$ and $K(n, \ell, F)$ imply $K(n, \ell^v, F)$ for all $v \geq 1$.*

Proof. See [Kah97, Prop. 1.1]. □

Proposition 4.1.2. *Let E be a complete discrete valuation field with residue field F . Then for all $m \geq 1$ such that $\text{char}(F) \nmid m$ and for all $n \geq 1$,*

$$K(n, m, E) \iff (K(n, m, F) \text{ and } K(n-1, m, F)).$$

Proof. We have a map $F^\times \rightarrow E^\times/m(E^\times)$ defined by choosing lifts from F^\times to \mathcal{O}_E^\times . It is unique because any two choices of a lift have ratio equal to a unit of \mathcal{O}_E^\times reducing to $1 \in F^\times$, and any such unit has an m -th root in \mathcal{O}_E^\times by Hensel's lemma and the fact that $\text{char}(F) \nmid m$. This map extends to a map of commutative graded rings

$$K^M(F) \longrightarrow K^M(E)/m.$$

Next we rely on this lemma of Milnor [?, Lem. 2.1]: given a choice of uniformizer $\pi \in E$, there exists a unique group homomorphism

$$K_n^M(E) \longrightarrow K_M(F)$$

mapping $\{\pi, u_2, \dots, u_n\}$ to $\{u_2, \dots, u_n\}$, where the u_i vary over elements of \mathcal{O}_E^\times . Then one checks that

$$0 \longrightarrow K_n^M(F)/m \longrightarrow K_n^M(E)/m \longrightarrow K_{n-1}^M(F) \longrightarrow 0$$

is a short exact sequence.

We now construct a corresponding short exact sequence in Galois cohomology, following [Ser02, p. 111]. We use the splittable short exact sequence of Galois groups

$$1 \longrightarrow I_E \longrightarrow G_E \longrightarrow G_F \longrightarrow 1.$$

It is clear enough that μ_m is an unramified G_E -module, and it follows that the Hochschild-Serre spectral sequence degenerates, resulting in short exact sequences

$$0 \longrightarrow H^n(G_F, \mu_m^{\otimes n}) \longrightarrow H^i(G_E, \mu_m^{\otimes n}) \longrightarrow H^{i-1}(G_F, \text{Hom}(I, \mu_m^{\otimes n})) \longrightarrow 0.$$

Finally, we note that a choice of uniformizer $\pi \in F$ results in an isomorphism $\text{Hom}(I, \mu_m^{\otimes n}) \xrightarrow{\sim} \mu_m^{\otimes n-1}$.

Now we consider the maps η between the elements of these short exact sequences. Each of these short exact sequences relied on a choice of a uniformizer $\pi \in F$, and

it can be checked that a common choice of uniformizer results in η defining a map of short exact sequences.

An isomorphism of the middle arrow amounts to $K(n, m, E)$, while $(K(n, m, F)$ and $K(n - 1, m, F))$ amount to isomorphisms of the outer two arrows. So the proposition follows from the five lemma. \square

Corollary 4.1.3. $K(n, m) \implies K(n - 1, m)$.

Proof. Apply the proposition to the discrete valuation field $F((t))$. \square

Corollary 4.1.4. $K(n, m)$ in characteristic zero implies $K(n, m)$ in all characteristics (prime to m).

Proof. Assume that F has positive characteristic. Upon a reduction step [Kah97, Prop. 1.1(b)], one may assume that F is perfect. Then one may take E to be the fraction field of the Witt ring of F and apply the propositions above. \square

4.2. Motivic Hilbert theorem 90. As a result of the reduction steps we have just discussed, it suffices to restrict to studying $K(n, \ell, F)$ for primes ℓ and F of characteristic zero. Also, we can work by induction on n . We are aiming towards the motivic translation of the Bloch-Kato conjecture, Theorem 2.5.2.

Here is a main claim toward proving Theorem 2.5.2.

Theorem 4.2.1 (Motivic Hilbert theory 90, i.e. HT90). *Let ℓ be a prime number and assume that $\text{char}(F) = 0$. Let $n \geq 0$. For all $p \leq n$,*

$$H_L^{p+1, p}(\text{Spec } F, \mathbb{Z}_{(\ell)}) = 0$$

We write $H90(n, \ell)$ for the veracity of this theorem for a particular n and ℓ .

Example 4.2.2. Using the Motivic-Galois relation, we record cases of motivic Hilbert theorem 90 that we already know.

$H90(0, \ell)$ amounts to $H^1(G_F, \mathbb{Z}_{(\ell)}) = 0$, which is clear because G_F is profinite.

$H90(1, \ell)$ amounts to $H_{\text{ét}}^1(\text{Spec } F, \mathbb{G}_m) \otimes \mathbb{Z}_{(\ell)} = 0$, which follows from the classical Hilbert theorem 90, $H^1(G_F, \bar{F}^\times) = 0$.

It turns out that $HT90(n, \ell)$ implies that motivic cohomology and étale motivic cohomology (of any smooth variety) are identical in many cases. We formulate this as follows.

Let α denote the morphism from the étale site to the Zariski site on Sm_F . Our maps $H_{\text{mot}} \rightarrow H_L$ arise from $\mathbb{Z}(q) \mapsto \alpha^* \mathbb{Z}(q)$.

Definition 4.2.3. Let $B(n)$ denote the Zariski sheaf $\tau_{\leq n+1} R\alpha_* \alpha^* \mathbb{Z}(n)$. We denote by B_n the natural morphism

$$B_n : \mathbb{Z}(n) \rightarrow B(n).$$

Theorem 4.2.4. *The following are equivalent.*

- (1) $HT90(n, \ell)$ is true.
- (2) For all $p \leq n$, $(B_p) \otimes \mathbb{Z}_{(\ell)}$ is a quasi-isomorphism.

Remark 4.2.5. Here is a more concrete interpretation of condition (2), demonstrating the broad impact of $HT90(n, \ell)$. It means that

$$H_{\text{mot}}^{p, q}(X, \mathbb{Z}/\ell) \xrightarrow{\alpha^*} H_L^{p, q}(X, \mathbb{Z}/\ell)$$

is an isomorphism with $q \geq 0$ and $p \leq q \leq n$; and it is a monomorphism for $p-1 = q \leq n$.⁷ The Milnor conjecture is merely the case $p = q = n$ and $X = \text{Spec } F$.

Proof. We have seen firsthand that $\mathbb{Z}(n)$ is concentrated in degree $\leq n$, so it is clear that (2) implies (1).

Of course (1) implies (2) is more interesting, but we simply refer to [Kah97, Thm. 2.3] for now. \square

Remark 4.2.6. The easy vanishing properties of $H_{\text{mot}}^{*,*}$ and $H_L^{*,*}$ are different. For example, for A an abelian group, $H_{\text{mot}}^{p,q}(X, A) = 0$ for $p > q + \dim X$ [MVW06, Thm. 3.6]; in particular, $H_{\text{mot}}^{p,q}(\text{Spec } F, A) = 0$ for $p > q$. In contrast, this is non-trivial for $H_L^{n+1,n}(\text{Spec } F, A)$: HT90 is an example.

4.3. Discussion of the strategy. With the role of motivic Hilbert theorem 90 in place, we can give an outline of the remainder of the proof, following [Dug04, §2.2].

- (1) Prove $HT90(n, \ell)$ for “big fields,” i.e. those with no extensions of degree prime to ℓ , and also satisfying $\ell K_n^M(F) = K_n^M(F)$.
- (2) If F were some other counterexample to $HT90(n, \ell)$ (but we have assumed $HT90(n-1, \ell)$), produce a “bigger” counterexample, eventually producing a counterexample for the “big fields” of step (1).

4.4. Proof of the conjecture for “big fields”. To be written. See [Kah97, §3]. Here is the main result.

Theorem 4.4.1. *Let $\text{char}(F) = 0$ and let ℓ be prime. We fix $n \geq 0$. Assume that F has no extensions of degree prime to ℓ and also that $\ell K_n^M(F) = K_n^M(F)$. Then $HT90(n-1, \ell) \Rightarrow HT90(n, \ell)$.*

Thus, for fields F that are “big” (i.e. they satisfy the conditions of the theorem), we can prove the Bloch-Kato conjecture at the prime ℓ .

The proof of the theorem is not trivial, but reduces to calculations in Galois cohomology.

4.5. Making counterexamples “bigger”. For concreteness, and due to following [Dug04, §2.6], we restrict to the case $\ell = 2$.

Suppose that F has characteristic zero and is a counterexample to $HT90(n, 2)$, despite satisfying $HT90(n', 2)$ for $n' \leq n-1$. Proposition 4.1.1(2) implies that any odd degree extension of F is also a counterexample, so we may assume that F has no such extensions. Thus Theorem 4.4.1 implies that $2K_n^M(F) \neq K_n^M(F)$.

Choose some $\underline{a} = \{a_1, \dots, a_n\} \in K_n^M(F) \setminus 2K_n^M(F)$. Next, we want to find a splitting variety for \underline{a} .

Definition 4.5.1. A splitting variety for $\underline{a} \in K_n^M(F)$ is an object of Sm_F such that $\underline{a} \in 2K_n^M(F(X))$.

The extension $F(X)/F$ is the “bigger” extension advertised above. In fact, the composite of $F(X)$ all \underline{a} and $X = X_{\underline{a}}$ provides exactly such an extension.

Here is the exact X that we will use.

⁷Check this statement of [Dug04, Prop. 2.4].

Definition 4.5.2. Let $\phi : \{0, \dots, 2^{n-1} - 1\} \xrightarrow{\sim} \mathcal{P}(\{1, \dots, n-1\})$ be a bijection, where \mathcal{P} denotes power set. Given $\underline{a} \in K_n^M(F)$, we let $Q_{\underline{a}}$ be the projective quadric in $\mathbb{P}^{2^{n-1}}$ given by

$$\left(\sum_{i=0}^{2^{n-1}-1} x_i^2 \prod_{j \in \phi(i)} (-a_j) \right) - a_n x_{2^{n-1}}^2 = 0.$$

Proposition 4.5.3. $Q_{\underline{a}}$ is a splitting variety for \underline{a} .

Proof. See [Kah97, Thm. 4.13]. \square

4.6. **The argument.** Now we summarise the remainder of the argument to prove $HT90(n, 2)$. What remains to prove is

Proposition 4.6.1. *If $HT90(n, 2, F)$ is false and $HT90(n-1, 2, F)$ is true, then $HT90(n, 2, F(Q_{\underline{a}}))$ is false. More specifically,*

$$H_L^{n+1, n}(\text{Spec } F, \mathbb{Z}_{(2)}) \longrightarrow H_L^{n+1, n}(\text{Spec } F(Q_{\underline{a}}), \mathbb{Z}_{(2)})$$

is injective.

We now summarise the proof of the proposition, basically repeating the summary given in [Dug04, §2].

- (1) Consider the Čech complex $\check{C}Q_{\underline{a}}$ of $Q_{\underline{a}}$, which is a simplicial F -scheme homotopy equivalent to $\text{Spec } F$. It turns out that this means that $\check{C}Q_{\underline{a}} \simeq \text{Spec } F$ in the étale world (i.e. $\check{C}Q_{\underline{a}}$ is “étale contractible”), but this is not necessarily so in the Zariski = motivic setting. Therefore we have that all of the étale motivic cohomology groups of $\text{Spec } F$ and $\check{C}Q_{\underline{a}}$ are identical, and, in particular,

$$H_L^{n+1, n}(\text{Spec } F, \mathbb{Z}_{(2)}) \xrightarrow{\sim} H_L^{n+1, n}(\check{C}Q_{\underline{a}}, \mathbb{Z}_{(2)}).$$

This is purely homotopy theory and étale descent, and has nothing to do with our special choice of $Q_{\underline{a}}$.

- (2) Let \tilde{C} be defined by the cofiber sequence

$$\check{C}Q_{\underline{a}} \longrightarrow \text{Spec } F \longrightarrow \tilde{C},$$

so the reduced motivic cohomology of \tilde{C} calculates the difference between the motivic cohomology of $\check{C}Q_{\underline{a}}$ vs. $\text{Spec } F$. Using a long exact sequence in the degree p of $H_{\text{mot}}^{p, q}$, and the fact that $H^{i, n}(\text{Spec } F, \mathbb{Z}) = 0$ for $i > n$,⁸ we have

$$H_{\text{mot}}^{n+1, n}(\check{C}Q_{\underline{a}}, \mathbb{Z}_{(2)}) \xrightarrow{\sim} \tilde{H}_{\text{mot}}^{n+2, n}(\tilde{C}, \mathbb{Z}_{(2)}).$$

The fact that $Q_{\underline{a}}$ has a point valued in a degree 2 extension of F , along with Proposition 4.1.1(2), implies that this group is 2-torsion. So it will suffice to show that the image of

$$\tilde{H}_{\text{mot}}^{n+2, n}(\tilde{C}, \mathbb{Z}_{(2)}) \longrightarrow \tilde{H}_{\text{mot}}^{n+2, n}(\tilde{C}, \mathbb{Z}/2)$$

is zero.

- (3) Now we prepare to apply *Steenrod operations* on motivic cohomology with coefficients in $\mathbb{Z}/2$. These consist of the Bockstein homomorphism with bi-degree $(1, 0)$ and maps Sq^{2^i} of bi-degree $(2^i, 2^{i-1})$ for $i \geq 1$. Milnor modified these to produce operators Q_i of bi-degree $(2^{i+1} - 1, 2^i - 1)$.

⁸As the complex defining $\mathbb{Z}(n)$ is concentrated in degrees $\leq n$, recall.

- (4) We use this input, which again is not specific to $Q_{\underline{a}}$, but simply applies to quadrics.

Proposition 4.6.2. *Let $X \subset \mathbb{P}^{2^n}$ be a smooth quadric and define $\tilde{C}X$ by the cofiber sequence*

$$\check{C}X \longrightarrow \mathrm{Spec} F \longrightarrow \tilde{C}X.$$

Then for $i \leq n$, every element of $H_{\mathrm{mot}}^{i,n}(\tilde{C}X, \mathbb{Z}/2)$ killed by Q_i is also in the image of Q_i .

Dugger tells us that the tools to prove this result are the Steenrod operations, along with basic properties of quadric and motivic cohomology.

- (5) Finally, we use this input, which is where the motivic approach finally pays off by using input from geometry about $Q_{\underline{a}}$.

Proposition 4.6.3.

$$\tilde{H}_{\mathrm{mot}}^{2^n, 2^{n-1}}(\tilde{C}, \mathbb{Z}_{(2)}) = 0.$$

Proof. This is a result of Voevodsky, using the a result of Rost that the motive of $Q_{\underline{a}}$ has a certain direct summand. \square

- (6) Now we complete the proof as follows. I appreciate Dugger's concrete example of the argument: for concreteness let $n = 4$, so we want to show that certain elements in $\tilde{H}_{\mathrm{mot}}^{6,4}(\tilde{C}, \mathbb{Z}/2)$ are zero.

Dugger observes that the induction hypothesis, applied as in Remark 4.2.5, implies that

$$\tilde{H}_{\mathrm{mot}}^{p,q}(\tilde{C}, \mathbb{Z}/2) = 0 \text{ for } p \leq q \leq n - 1.$$

Applying these vanishings in small degrees, along with Proposition 4.6.2, we get that both

$$Q_1 : H^{6,4} \longrightarrow H^{9,5} \quad \text{and} \quad Q_2 : H^{9,5} \longrightarrow H^{16,8}$$

are injective. Altogether we inject $Q_2 \circ Q_1 : \tilde{H}_{\mathrm{mot}}^{6,4}(\tilde{C}, \mathbb{Z}/2) \hookrightarrow \tilde{H}_{\mathrm{mot}}^{16,8}(\tilde{C}, \mathbb{Z}/2)$, but elements in the target coming from cohomology with $\mathbb{Z}_{(2)}$ -coefficients is zero, by Proposition 4.6.3.

5. THE MOTIVIC-MILNOR AND MOTIVIC-GALOIS RELATIONS

The goal of this section is to illustrate, to some extent, the proof of the Motivic-Milnor and Motivic-Galois relations: Theorem 2.3.1 and Corollary 2.4.4. We hoped to be relatively thorough in discussing Motivic-Milnor, following [MVW06, Lecture 5], but have decided to put this online as more time has passed without doing this. We will only give an outline of Motivic-Galois relation.

5.1. The Motivic-Galois relation: sketch of the proof. We will actually sketch the proof of the isomorphism

$$(5.1.1) \quad H_L^{p,q}(X, \mathbb{Z}/m) \cong H_{\acute{e}t}^p(X, \mu_m^{\otimes q})$$

(where $m \geq 1$ such that $\mathrm{char}(F) \nmid m$, and $X \in \mathrm{Sm}_F$) of Theorem 2.4.2, of which the Motivic-Galois relation needed for the Bloch-Kato conjecture is just the special case $X = \mathrm{Spec} F$.

All that is needed is to prove

Theorem 5.1.2. *For all $q \geq 0$, $m \geq 1$ such that $\text{char}(F) \nmid m$, and $X \in \text{Sm}_F$, we have an quasi-isomorphism of complexes of étale sheaves*

$$\mathbb{Z}/m(q)|_{X_{\text{ét}}} \cong \mu_m^{\otimes q}.$$

We have given a full proof of this theorem for $q = 1$, finishing in Corollary 3.2.2. Thus we already know (5.1.1) for $q = 1$ and arbitrary p .

One might suspect that Theorem 5.1.2 arises from simply taking tensor powers of this relation. This is indeed the case – the catch is that one has to make sense of this tensor product. So, here, we will simply list the steps involved, based on a cursory reading of [MVW06]. The interested reader should definitely go directly to this source. As in talk 7 in this series, dealing with the tensor product is non-trivial.

- Show that étale sheafification preserves transfers [MVW06, Lecture 6].
- Produce a tensor product on the derived category of étale sheaf with transfers [MVW06, Lecture 8], so that we can make sense of $\mathbb{Z}(1)_{\text{ét}}^{\otimes q}$.
- Set up \mathbb{A}^1 -weak equivalence in this category, appropriately setting up \mathbb{A}^1 -homotopy equivalence [MVW06, Lecture 9]. For objects known as \mathbb{A}^1 -local, Homs into them from weakly equivalent objects are identifiable.
- Prove that
 - Both $\mu_m^{\otimes q}$ and $\mathbb{Z}/m(q)$ are \mathbb{A}^1 -local;
 - Using the multiplication map $\mathbb{Z}(1)_{\text{ét}}^{\otimes q} \rightarrow \mathbb{Z}(q)$ (not discussed in these notes; see [MVW06, Const. 3.11]), produce $\mu_m^{\otimes q} \rightarrow (\mathbb{Z}/m)(1)_{\text{ét}}^{\otimes q} \rightarrow (\mathbb{Z}/m)(q)_{\text{ét}}$ and show that the composite is a \mathbb{A}^1 -weak equivalence, from which Theorem 5.1.2 follows.

5.2. Beginning the proof of the Motivic-Milnor relation. Unfortunately I have not yet written this. See [MVW06, Lecture 5] for the proof.

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