Prime numbers and the Riemann zeta function
Undergraduate Mathematics Seminar, University of Pittsburgh

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Prime numbers

A natural number is called *prime* when it cannot be properly divided into factors.

Example: $5 = a \cdot b$ means either a or b is 1, so it's prime!

We find some primes: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$
... $2^{82589933} - 1$ is the largest known one!

Let's ask some questions – this is doing *number theory*:

The big question

How are the primes distributed among *all* of the natural numbers?

People have been asking about this question for millennia.

The distribution of prime numbers

The big question

How are the primes distributed among all of the natural numbers?

This is a question about the interaction of addition and multiplication:

- Primes are the "atoms" under multiplication.

  Fundamental theorem of arithmetic. Each natural number admits a unique factorization into prime numbers.

  $15 = 3 \cdot 5$, $24 = 2^3 \cdot 3$, etc.

- 1 is the single atom for addition: $2 = 1 + 1$, $3 = 1 + 1 + 1$, etc.

First let's find out how many primes there are!

How many primes are there?

It would be really nice if we could argue just like we can with even numbers:

There are infinitely many even numbers because

- *the process "find the next even number by adding 2 to the last one" clearly never stops*, OR
- *the closed formula* 2*n* *gives us an even number for each number n.*

But primes don't seem to have a nice discernible pattern like

- even numbers $(2, 4, 6, 8, 10, 12, 14, \dots)$
- squares $(1, 4, 9, 25, 36, \dots)$
- exponentials $(1, 2, 4, 8, 16, 32, 64, \dots)$

Stare at it: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, . . .

There are infinitely many primes: Proof

Here is Euclid and friends' argument for the infinitude of the primes.

First, establish two background facts, which we will treat as assumptions.

1. If n is at least 2, then for any x, $(n \cdot x + 1)$ is not divisible by n.
2. If y is at least 2, then it is divisible by some prime number.

Let $\{p_1, \ldots, p_a\}$ be a finite list of the primes that you know so far. Then multiply all of them together and add 1:

$$y := p_1 \cdot p_2 \cdots p_a + 1.$$

Assumption (1) says that each of the primes p_i do not divide y. Assumption (2) says that some prime must divide y.

Putting this together, we get a new prime, p_{a+1}, that was not already on the list!

There are infinitely many primes

So we are done with the proof: any finite list of primes does not have all of the primes on it!

This means that we can't answer our *big question* by just computing enough...

and after much effort, we still don't know a closed expression "nth prime is $f(n)$ for some function f."

Upshot

It seems like we can't just analyze something finite to study the primes. We have to go out and use logical argument to study "all infinity" of the primes.

(Of course, if you could figure out how to do it in a finite way, do it!)

The distribution of the infinitude of primes

There are lots of aspects of "the distribution of the primes" that we could ask about. Lots are interesting!

- *Are there infinitely many primes that leave a remainder 1 when divided by 4?*

Yes! Same for those that leave a remainder of 3.

- *How big can a gap between primes be?*

We know this: there are arbitrarily big gaps between primes!

- *It looks like 2 can be the smallest gap between primes that could possibly occur infinitely often. Does it?*

The guess that the answer is "yes!" is known as the *twin prime conjecture*.

They begin: $(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), \ldots$

The distribution of the infinitude of primes

We choose one refinement of the big question to especially focus on today.

Our choice of refinement of the big question

For primes up to a real number A, what is the average size of the gap between primes?
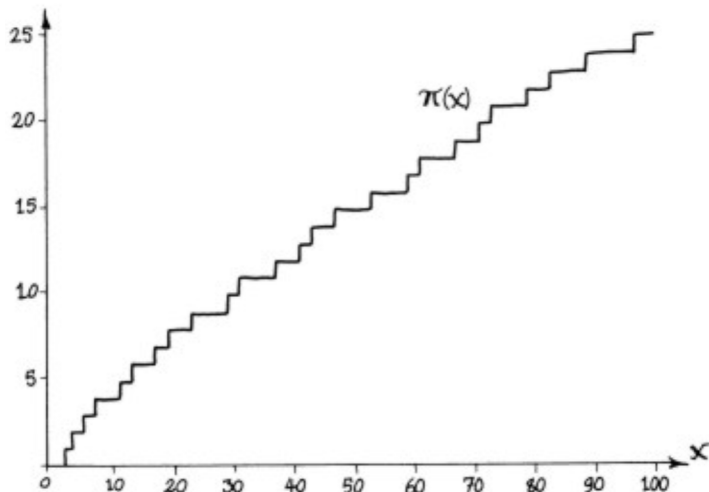
Stated differently: Of the $\sim A$ numbers up to a real number A, what proportion of them are prime?

To study our (refined) big question, let's define

$$\pi(X) := \text{ the number of primes } \leq X.$$

We call this the **prime counting function**.

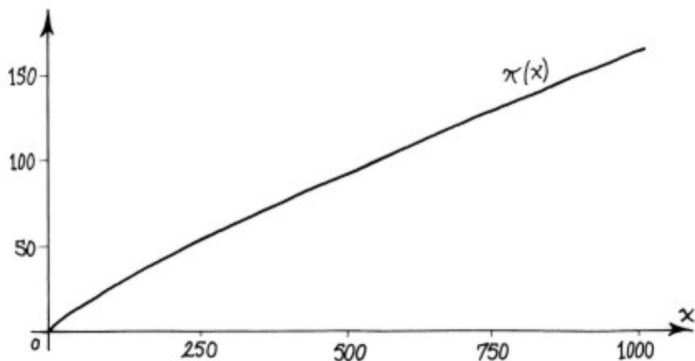# How are the primes distributed on average?

Picture credit: Matthew Watkins

How are the primes distributed on average?

We can rephrase our big question in terms of $\pi(X)$. Remember: $\pi(X)$ counts the primes up to X, e.g. $\pi(5.5) = \pi(5) = 3$, $\pi(15) = 6$.

- "Average size of gaps up to A" $= \dfrac{A}{\pi(A)}$.

- "Proportion of numbers up to A that are prime" $= \dfrac{\pi(A)}{A}$.

How are the primes distributed on average?

Picture credit: Matthew Watkins

How are the primes distributed on average?

To get an idea of what "distribution on average" means, let's try to compare the primes to other sets Σ of natural numbers that we understand very well.

Do the primes grow (or "thin out") like...

1. $\Sigma = $ *multiples* of a number a? $\Sigma = \{a, 2a, 3a, \dots\}$.

2. $\Sigma = b$th powers of numbers? $\Sigma = \{1, 2^b, 3^b, 4^b, 5^b, \dots\}$. (for $b > 1$)

3. $\Sigma = $ exponentials of a number c? $\Sigma = \{1, c^2, c^3, c^4, c^5, \dots\}$.

... or something in between them?

Using a computer, the distribution seems to be between "multiples" and "powers."

How are the primes distributed on average?

Computed values of $\pi(X)$: (Source: Wikipedia)

x	$\pi(x)$	$\pi(x) - \frac{x}{\log x}$	$\frac{\pi(x)}{x / \log x}$
10	4	−0.3	0.921
10^2	25	3.3	1.151
10^3	168	23	1.161
10^4	1 229	143	1.132
10^5	9 592	906	1.104
10^6	78 498	6 116	1.084
10^7	664 579	44 158	1.071
10^8	5 761 455	332 774	1.061
10^9	50 847 534	2 592 592	1.054
10^{10}	455 052 511	20 758 029	1.048
10^{11}	4 118 054 813	169 923 159	1.043

We find that the function $\frac{X}{\log X}$ is a good approximation for $\pi(X)$. (The base of "log" is e.)

How are the primes distributed on average?

We propose that $\pi(X)$ behaves like $X/\log X$ as X gets large. This means that:

- the average gap between primes up to A is about $\log A$.
- the proportion of numbers up to A that are prime is about $1/\log A$.

Example: Up to $e^7 \approx 1096.6$, about 1 in every 7 numbers is prime.

This proposal means that primes are

- more thinly distributed than $\Sigma = \{$multiples of $a\}$
  $\rightarrow$ because $\pi_\Sigma(X) = X/a$ overtakes $X/\log X$ for $X \gg 0$.
- more thickly distributed than powers $\Sigma = \{1, 2^b, 3^b, \dots\}$
  $\rightarrow$ because even for b just a tiny bit more than 1, $\pi_\Sigma(X) = X^{1/b}$ is overtaken by $X/\log X$ for $X \gg 0$.

How are the primes distributed on average?

Let's make this guess more precise.

A precise claim that "$\pi(X)$ behaves like $X/\log X$"

We propose that

$$\text{``}\pi(X) \sim \frac{X}{\log X}.\text{''}$$

The meaning of this "$\sim$" expression is

$$\lim_{X \to +\infty} \left(\frac{\pi(X)}{X/\log X} \right) = 1.$$

In plain language: the difference between $\pi(X)$ and $X/\log X$ becomes very small relative to the size of $\pi(X)$ and $X/\log X$ themselves.
(For example, $e^X \sim e^X + X^n$ for any number n.)

Can we prove the claim?

The prime number theorem

The claim was proven in 1896.

The prime number theorem (Hadamard, De la Vallée Poussin)

We have $\pi(X) \sim \dfrac{X}{\log X}$.

Remarkably, the theorem was proved by...

proving a fact about the location in the complex plane $\mathbb{C}$ of the complex numbers s that are sent to 0 by a particular complex meromorphic function

$$\zeta : \mathbb{C} \to \mathbb{C}$$

called the *Riemann zeta function!*   (What?)

The Riemann zeta function

The Riemann zeta function is defined by the following infinite series.

The Riemann zeta function

$$\zeta(s) = 1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{(for Re}(s) > 1\text{)}.$$

Let's understand what this means:

- Case of real s ($s \in \mathbb{R}$).
    - We know what n^{-s} means.
    - The series converges *absolutely* for $s > 1$.
    - The series at $s = 1$, $\sum_{n=1}^{\infty} n^{-1}$, famously diverges.
    - For even s, like $s = 2, 4, 6, \ldots$, we know equalities like

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

The Riemann zeta function

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{(for Re}(s) > 1).$$

- Case of real s ($s \in \mathbb{R}$).
  - We know what n^{-s} means.
  - The series converges absolutely for $s > 1$.

- The case of general $s \in \mathbb{C}$. Write it as "$s = \sigma + it$" where $\sigma, t \in \mathbb{R}$, so that Re$(s) = \sigma$ ("the real part of s is σ").
  - We can factor
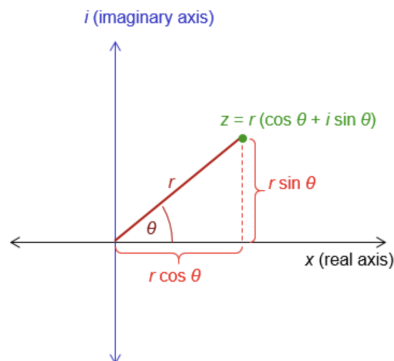 $$n^{-s} = n^{-\sigma - it} = n^{-\sigma} \cdot n^{-it}.$$
  - The $n^{-\sigma}$ part is positive real: it controls the *size* of n^{-s}.
  - The n^{-it} part controls the angle (or "argument") of n^{-s}.

The Riemann zeta function: the term n^{-s}

A picture of a complex number z in polar form: $z = r \cdot e^{i\theta}$.
Think: $z = (\text{size } r) \cdot (\text{angle } \theta)$.

Complex Numbers In Polar Form

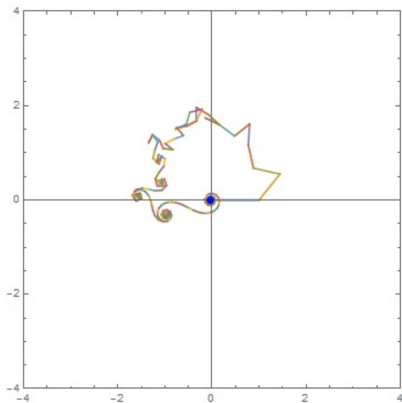Picture credit: onlinemathlearning.com.

For $z = n^{-s}$ and $s = \sigma + it$,

$$r = n^{-\sigma} \quad \text{and} \quad \theta = t \cdot \log n.$$

The Riemann zeta function: the "partial sums" $\sum_{n=1}^{N} n^{-s}$

Here is a picture of what the partial sums look like, for $s = \frac{1}{2} + i \cdot 986.764$.
Each line in the "connect the dots" is a single term n^{-s}.

Source: Mathematica StackExchange #133340.

Primes and the Riemann zeta function?

So what do primes have to do with the Riemann zeta function?

A lot!

The starting point is the *Euler product*:
Using the fundamental theorem of arithmetic, we have

$$\begin{aligned}
\zeta(s) &= 1 + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + 7^{-s} + \cdots \\
&= (1 + 2^{-s} + 4^{-s} + 8^{-s} + \cdots) \cdot (1 + 3^{-s} + 9^{-s} + \cdots) \\
&\quad \cdot (1 + 5^{-s} + 25^{-s} + \cdots) \cdot \cdots \\
&= \prod_{p:\text{ prime}} (1 + p^{-s} + p^{-2s} + p^{-3s} + \cdots) \\
&= \prod_{p:\text{ prime}} (1 - p^{-s})^{-1}.
\end{aligned}$$

Primes and the Riemann zeta function

The Euler product

$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ for Re$(s) > 1$.

The Euler product gives us new proof of the infinitude of the primes:

We know that $\lim_{s \to 1^+} \zeta(s) = +\infty$, because we know that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges to $+\infty$.

But each of the quantities $(1 - p^{-s})^{-1}$ stays bounded at $s = 1$. (It's $\frac{p}{p-1}$.) So the divergence of the harmonic series implies that there are infinitely many primes!

The Riemann zeta function

Let's mention things about ζ that we need in order to connect it with the distribution of prime numbers.

Facts in pictures:

- ζ has a unique(!) complex analytic extension to all of $\mathbb{C}$, except it has a pole at 1.
- ζ has a "functional equation" that gives $\zeta(1-s)$ in terms of $\zeta(s)$.

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s)$$

  where $\Gamma(s) := \int_0^\infty e^x x^{s-1}dx$. (Fact: $\Gamma(n) = (n-1)!$ for $n \in \mathbb{Z}_{\geq 1}$.)

- **Upshot:** We now understand $\zeta(s)$ except on the "critical strip," which is the region

$$\{s \in \mathbb{C} : 0 \leq \text{Re}(s) \leq 1\}.$$

Counting primes so that we can relate the count to ζ

Now let's count primes in a bit of a different way. (*What?*)

The Lambda-function Λ, and counting function ψ

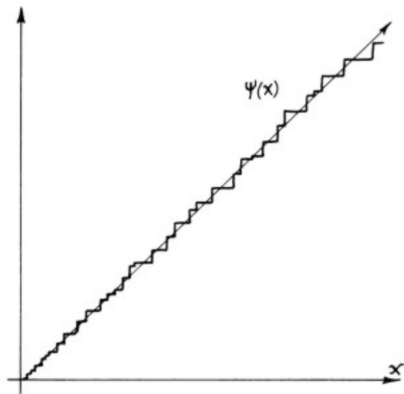Let $\psi(X) := \sum_{n \leq X} \Lambda(n)$, where Λ is the von Mangoldt function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^i \text{ (some } i) \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that it is not hard (in the scope of these efforts) to show that

$$\pi(X) \sim \frac{X}{\log X} \iff \psi(X) \sim X.$$

That is, the prime number theorem is the same as "$\psi(X) \sim X$."

Here is $\psi(X)$ superimposed on the function X.

Picture credit: Matthew Watkins

The "explicit formula"

We can very explicitly relate ψ to the Riemann zeta function!

Let ρ' vary over the complex numbers where $\zeta(\rho') = 0$, the "zeros of the zeta-function."
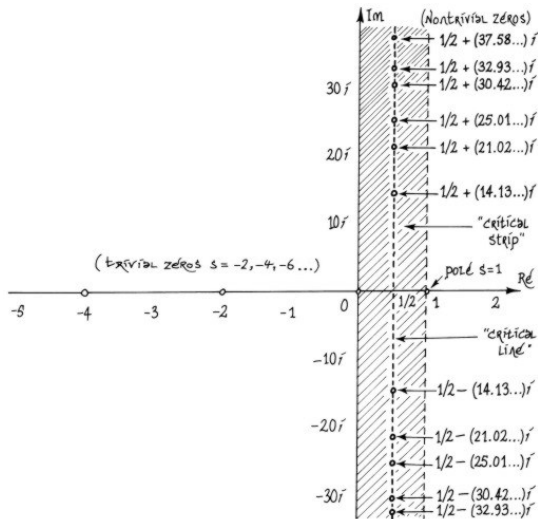
These "zeros" ρ' are:

- The negative even integers $-2, -4, -6, \ldots$. ("Trivial zeros.")
- The rest occur symmetrically the critical strip, and are denoted by ρ.

For example, $\rho = \frac{1}{2} + i \cdot 14.134725...$ is the zero closest to the real axis.

The explicit formula!

$$\psi(X) = X - \sum_{\rho'} \frac{X^{\rho'}}{\rho'} - \log(2\pi)$$

$$= X - \sum_{\rho} \frac{X^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2}).$$

A picture of a few zeros of the Riemann zeta function

Picture credit: Matthew Watkins

Applying the explicit formula

The explicit formula

$$\psi(X) = X - \sum_{\rho} \frac{X^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2}).$$

The explicit formula comes from standard methods in complex analysis! It's hard, but not super deep.

Observe: We have a "formula" for the primes!!

What does it really mean to us, if we want to show $\psi(X) \sim X$??

- The last two terms, $\log(2\pi)$ and $\frac{1}{2}\log(1 - X^{-2})$, are negligible relative to X. Good!
- So we just want to know that $\frac{X^{\rho}}{\rho}$ doesn't get too big relative to X!

Applying the explicit formula

The explicit formula

$$\psi(X) = X - \sum_\rho \frac{X^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2}).$$

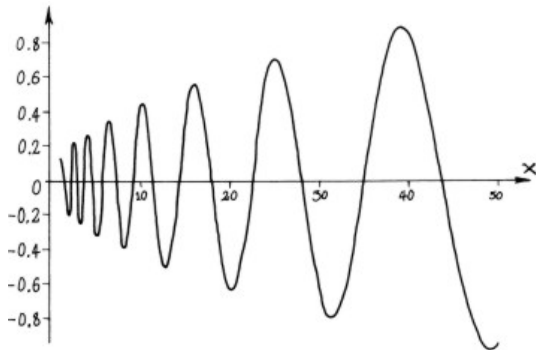What does the influence of $\sum_\rho X^\rho/\rho$ look like? Here's how we can break it down.

- Put together two symmetric zeros of ζ: call them $\rho = \beta + i\gamma$ and its complex conjugate $\bar\rho = \beta - i\gamma$.
- The two terms coming from ρ and $\bar\rho$ add up to

$$X^\rho/\rho + X^{\bar\rho}/\bar\rho \approx 2\frac{X^\beta}{|\rho|}\sin(\gamma \cdot \log X).$$

A "logarithmically-rescaled sinusoid"

The contribution of ρ and $\bar{\rho}$ looks like:

$$X^{\rho}/\rho + X^{\bar{\rho}}/\bar{\rho} \approx 2\frac{X^{\beta}}{|\rho|}\sin(\gamma \cdot \log X).$$

Picture credit: Matthew Watkins. This is the graph for $\rho = \frac{1}{2} + i \cdot 14.13....$

Using the explicit formula to prove the prime number theorem

The explicit formula

$$\psi(X) = X - \sum_{\rho} \frac{X^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2}).$$

We get that $\psi(X)$ differs from X by something of size X^{β}, where $\beta \leq 1$. !!

- As long as the infinite sum over the ρ behaves well, we can hope to prove the prime number theorem by showing that the real part β of ρ must be less than 1!

- This hope turns out to be valid!

- Hadamard and De la Vallée Poussin proved the prime number theorem by proving that $\beta = 1$ is impossible. That is, when $\zeta(\beta + i\gamma) = 0$, they showed that $\beta < 1$.

The Riemann Hypothesis, and primes

We have just seen that larger β in zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ results in larger deviation of $\psi(X)$ from X. How about if the deviation were as small as possible?

Because of the symmetries in the zeros ρ of $\zeta(s)$, the smallest possible maximum for β is $\frac{1}{2}$.

The Riemann Hypothesis

If $\rho \in \mathbb{C}$ such that $\zeta(\rho) = 0$, then either

$$\rho \in \{-2, -4, -6, \dots\} \quad \text{or} \quad \text{Re}(\rho) = \frac{1}{2}.$$

In words: *All of the non-trivial zeros of the Riemann zeta function lie on the critical line $\text{Re}(s) = \frac{1}{2}$.*

The Riemann Hypothesis, and primes

The Riemann Hypothesis

If $\rho \in \mathbb{C}$ such that $\zeta(\rho) = 0$, then either

$$\rho \in \{-2, -4, -6, \dots\} \quad \text{or} \quad \text{Re}(\rho) = \frac{1}{2}.$$

The Riemann Hypothesis is considered to be one of the most important open questions in mathematics.

Why?

- It says that the primes have really nice structure.
- It seems to point toward deeper ideas that should underlie the connection between ζ and primes.
- It has a lot of connections with generalizations of the ζ-function called "L-functions," which are connected with other objects of arithmetic (other than primes).

The Riemann Hypothesis, and primes

The mathematician and author Marcus du Sautoy has popularized the notion that the Riemann hypothesis means that there is "music in the primes." This idea comes from Fourier analysis: the "frequencies" of all of the logarithmically rescaled sinusoidal waves go hand-in-hand with the distribution of the prime numbers among the natural numbers.

Example. If there were just one exception to the Riemann Hypothesis, we would get a certain kind of wild variation in the distribution of the primes. (Picture!)

Let's watch a short video showing how the sum of all these waves gets closer and closer to ψ.

The Riemann Hypothesis, and primes

Takeaways:

- Mathematics is about making connections, less about manipulating formulas.
    - We have seen that studying primes has a lot to do with functions, limits, and calculus!
    - Formulas are useful when they help us understand a connection between ideas.
- By asking questions and thinking carefully, sometimes we learn about better questions to ask.
    - It seems like the ψ counting function is better than the π counting function for connecting with ζ.
    - A lot of learning mathematics is understanding why people ask the questions they do. In other words, it's about learning the culture of mathematics.
- The more we learn, the more we appreciate that there is much more to learn.

Attribution and additional information

- The images credited to Matthew Watkins in these slides come from his webpages, which form an accessible compendium of topics related to this talk. Relative to this talk, a good place to start accessing his webpages is.

  `http://empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/encoding2.htm`

- There are many books on the topic of this talk, such as *Prime numbers and the Riemann hypothesis*, by Barry Mazur and William Stein.