SUMMARY OF "COHOMOLOGICAL CONTROL OF DEFORMATION THEORY VIA A_∞-STRUCTURE"

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Abstract. This is a brief summary of one part of the forthcoming work "Cohomological control of deformation theory via A_∞-structure." Let G be a profinite group. The main result is that a natural A_∞-structure on cohomology groups induces presentations of universal deformation rings for G-representations, more general moduli spaces for G-representations, and universal deformation rings for Galois pseudorepresentations. Nothing in this summary is particular to the case that G is a Galois group. Remaining parts of the forthcoming paper (not described here) give applications to number theory.

1. Fine and coarse moduli of Galois representations

In this section, we give background for the result, quickly summarizing [WE15].

1.1. **Fine moduli of representations.** The most often-applied moduli theory of representations of a profinite group, due to Mazur [Maz89], proceeds as follows: fix a residual representation $\bar\rho : G \to \mathrm{GL}_d(\mathbb{F})$ and study its deformations, which is often represented by a universal deformation ring $R_{\bar\rho}$. In [WE15], I have studied the moduli of all representations, a space we will call "$\mathcal{R}$ep." Universal deformations rings $R_{\bar\rho}$ are complete local rings in $\mathcal{R}$ep. Because we must take account of the profinite topology on G, it is natural to restrict the coefficient rings (on which we evaluate $\mathcal{R}$ep) to quotients of completions of $\mathbb{Z}[x_1, \dots, x_n]$ at some ideal containing a rational prime p.

To understand $\mathcal{R}$ep, it is helpful to introduce pseudorepresentations, a notion due to Chenevier [Che14].[1] An A-valued *pseudorepresentation* $D : G \to A$ of dimension d is a collection of characteristic polynomial coefficient functions

$$D = (f_1 = \mathrm{Tr}, f_2, \dots, f_d = \det) : G \to A$$

satisfying conditions that would be expected if it came from an A-valued representation. We write PsR for the (fine) moduli scheme of pseudorepresentations. There is a natural map $\psi : \mathcal{R}\mathrm{ep} \to \mathrm{PsR}$ associating a representation to its characteristic polynomial. Although not every pseudorepresentation arises from a representation, it is critically important that pseudorepresentations valued in a field are in bijection with semi-simple representations [Che14, Thm. A]. Accordingly, we write $\bar D : G \to \mathbb{F}$ for a residual pseudorepresentation valued in a finite field $\mathbb{F}$, and write $\bar\rho^{ss}_{\bar D} : G \to \mathrm{GL}_d(\mathbb{F})$ for the associated semi-simple representation.

Chenevier has shown that each $\bar D$ has a universal deformation ring $R_{\bar D}$, which we call a **pseudodeformation ring**. Unlike the moduli of representations $\mathcal{R}$ep, PsR is the disjoint union of deformation spaces of residual pseudorepresentations [Che14, Thm. F]. Consequently, we study one connected component of $\mathcal{R}$ep at a time, written $\psi : \mathcal{R}\mathrm{ep}_{\bar D} \to \mathrm{Spec}\, R_{\bar D}$.

Date: 2015-11-02.

[1] Chenevier's definition develops notions due to Wiles [Wil88] and Taylor [Tay91].

The attention to $\mathcal{R}\mathrm{ep}_{\bar{D}}$ broadens the scope of the usual moduli theory of representations of G, initiated by Mazur [Maz89]. Mazur defined the universal deformation ring $R_{\bar{\rho}}$ of a single residual representation $\bar{\rho} : G \to \mathrm{GL}_d(\mathbb{F})$; $\mathbb{F}$ is a finite field. The deformation rings $R_{\bar{\rho}}$ are local rings in $\mathcal{R}\mathrm{ep}_{\bar{D}}$ when the semi-simplification $(\bar{\rho})^{ss}$ equals $\bar{\rho}^{ss}_{\bar{D}}$. When $\bar{\rho}^{ss}_{\bar{D}}$ is reducible, ψ is not an isomorphism and $\mathcal{R}\mathrm{ep}_{\bar{D}}$ is not local. For example, when $\bar{\rho}^{ss}_{\bar{D}}$ has two simple factors $\bar{\rho}_1, \bar{\rho}_2$, the special fiber of ψ consists of $\mathbb{F}$-valued representations of the forms

$$\begin{pmatrix} \bar{\rho}_1 & * \\ 0 & \bar{\rho}_2 \end{pmatrix}, \quad \begin{pmatrix} \bar{\rho}_1 & 0 \\ 0 & \bar{\rho}_2 \end{pmatrix}, \quad \text{and } \begin{pmatrix} \bar{\rho}_1 & 0 \\ * & \bar{\rho}_2 \end{pmatrix}.$$

However, when $\bar{\rho}^{ss}_{\bar{D}}$ is irreducible, ψ is an isomorphism [Che14, Thm. B], and $\mathcal{R}\mathrm{ep}_{\bar{D}} = \mathrm{Spec}\, R_{\bar{\rho}^{ss}_{\bar{D}}}$. The coarse moduli and fine moduli are identical. Consequently, the attention to coarse moduli is novel for *residually reducible* Galois representations.

1.2. **Coarse moduli of representations.** Let's make the definition of fine and coarse moduli spaces clear. *Fine* moduli spaces $\mathcal{X}$ parameterize precisely the objects one desires up to isomorphism, but are often not realizable as schemes. Rather, they are algebraic stacks that are often non-separated, making their geometry somewhat difficult. This is the case with $\mathcal{R}\mathrm{ep}_{\bar{D}}$.

Coarse moduli spaces, which in many cases are produced by Mumford's geometric invariant theory (GIT) [Mum65], are *schemes* that are the best possible scheme approximating $\mathcal{X}$. Namely, X receives a morphism $\phi : \mathcal{X} \to X$ that is universal for morphisms from $\mathcal{X}$ to affine schemes. However, coarse moduli spaces such as X do not automatically represent a known moduli functor. It was an achievement of GIT to describe the geometric points of X moduli-theoretically, but not much more was known.

I am interested in determining natural (fine) moduli functors that *coarse* moduli spaces represent. I will describe my past and continuing work along these lines for two particular moduli problems: the moduli of representations and the moduli of semi-linear modules.

By GIT arguments, it is known that the coarse moduli scheme of representations has the same set of geometric points as PsR (see e.g. [Ric88]), i.e. semi-simple representations. An isomorphism of points is not useful for deformation theory, and my work provides a refinement (see also [Che13, Prop. 2.3]).

Theorem 1.2.1 ([WE15, §3])**.** *$\psi : \mathcal{R}\mathrm{ep}_{\bar{D}} \to \mathrm{Spec}\, R_{\bar{D}}$ is a map from the fine moduli space to the coarse moduli space of representations, perhaps with a p-torsion, nilpotent defect when the simple factors of $\rho^{ss}_{\bar{D}}$ are not distinct. In particular, ψ is universally closed.*

Assume that the defect vanishes. When $\mathcal{R}\mathrm{ep}_{\bar{D}}$ is presented as a quotient stack of a reducible group $\mathcal{G}$ acting on an affine scheme $\mathrm{Spec}\, S$, then the statement that $\mathrm{Spec}\, R_{\bar{D}}$ is the coarse moduli space associated to $\mathcal{R}\mathrm{ep}_{\bar{D}}$ means that $R_{\bar{D}} = S^{\mathcal{G}}$. We will use this equality to determine $R_{\bar{D}}$, below.

2. Cohomological control of deformation theory via A_∞-structure.

We will now describe how an A_∞-structure on cohomology influences the moduli spaces/rings described above. See [Kel06] for an introduction to A_∞-algebras appropriate to the applications described in this note.

2.1. **Overview.** In this paragraph, we overview the main results under discussion, determining equi-characteristic moduli spaces $R_{\bar\rho}$, $\mathcal{R}\mathrm{ep}_{\bar D}$, and $R_{\bar D}$ in terms of A_∞-structure on cohomology.[2] These results are motivated by the following questions.

(1) Can one find information about a deformation ring $R_{\bar\rho}$ in cohomology making the standard tangent and obstruction theory explicit, e.g. a presentation for the ring $R_{\bar\rho}$?
(2) Generalizing the first question, can one determine the global structure of the fine moduli spaces $\mathrm{Rep}_{\bar D}$ in terms of cohomology?
(3) Can the structure of the coarse moduli space of representations, i.e. the moduli of pseudorepresentations represented by the pseudodeformation ring $R_{\bar D}$, be determined in terms of cohomology?

We will answer these questions positively, using an A_∞-structure on cohomology. This structure extends the usual cup product in cohomology to "higher cup products." The A_∞-structure offers a language with which to describe how an obstruction theory works. This language helps significantly with understanding $\mathrm{Rep}_{\bar D}$ and $R_{\bar D}$. Before stating precise theorems and introducing notation necessary to state them, we summarize the results in the following

Theorem 2.1.1 ([WE])**.** *Let $\bar\rho$ be a semi-simple representation of G over a field k. Let $\bar D$ denote the induced pseudorepresentation of G. Then there is an A_∞-algebra structure on $\bigoplus_{i\geq 0}\mathrm{Ext}^i_G(\bar\rho,\bar\rho)$ such that the restriction of this structure to $i = 0, 1, 2$ gives presentations for the following rings (or spaces):*

(1) When $\bar\rho$ is irreducible, the deformation ring $R_{\bar\rho}$.
(2) The moduli stack $\mathrm{Rep}_{\bar D}$ of deformations of representations of G with residual pseudorepresentation $\bar D$.
(3) When the simple factors of $\bar\rho$ are distinct, the presentation of $\mathrm{Rep}_{\bar D}$ induces a presentation for the pseudodeformation ring $R_{\bar D}$ via the invariant-theoretic relationship between $\mathrm{Rep}_{\bar D}$ and $R_{\bar D}$ established in Theorem 1.2.1.

Answering (3) especially interesting, because not even the tangent space dimension for $R_{\bar D}$ had been worked out in the generality I achieve, much less an obstruction theory. And the tangent dimension is especially critical for number-theoretic applications. The tangent space is determined in Corollary 2.3.5. The best past result along the lines of (3) is work of Bellaïche [Bel12], who determines the tangent space of $R_{\bar D}$ when there are two simple factors.

The complication in determining the tangent space of $R_{\bar D}$ is that one *needs* the explicit obstruction theory for representations produced in part (2). Indeed, obstructions to representations influence the tangent space for pseudorepresentations – this basically reflects the fact that characteristic polynomial coefficients, other than the trace, are of degree ≥ 2 in matrix coefficients, and that obstructions appear only degree ≥ 2.

Remark 2.1.2. I am optimistic that workable formulas will come of the case with multiplicity, in future work. This will demand avoiding using invariant theory to determine $R_{\bar D}$. In contrast, we heavily use invariant theory to calculate $R_{\bar D}$ in the multiplicity-free case discussed in the paper.

[2]We use $R_{\bar\rho}$, $\mathcal{R}\mathrm{ep}_{\bar D}$, and $R_{\bar D}$ to denote equi-characteristic moduli spaces/rings for the rest of this note.

2.2. **A_∞-algebras and presentations for $R_{\bar\rho}$, $\mathcal{R}\mathrm{ep}_{\bar D}$.** It is well known that there is a graded multiplicative cup product structure m_2 on $H^*_G(\mathrm{ad}\bar\rho)$ arising from the usual cup product in cohomology and the multiplication map $\mathrm{ad}\bar\rho \otimes \mathrm{ad}\bar\rho \to \mathrm{ad}\bar\rho$. In fact, due to a theorem of Kadeishvili [Kad82], there are "higher cup products"

$$m_n : H^*_G(\mathrm{ad}\bar\rho)^{\otimes n} \to H^*_G(\mathrm{ad}\bar\rho), \quad \text{of graded degree } 2-n, \quad n \geq 2$$

extending m_2. This structure $(H^*_G(\mathrm{ad}\bar\rho), (m_n)_{n\geq 2})$ is known as an A_∞-*algebra*. While the choice of (m_n) is not unique, the theorems below do not depend on the choices.

Remark 2.2.1. See the appendix §3 for a concrete explanation of how the m_n influence the moduli of representations. Theorems 2.2.2 and 2.2.3 can be deduced, in principle, from the examples explained in §3.

Consider the dual maps

$$m_n^* : H^2_G(\mathrm{ad}\bar\rho)^* \longrightarrow H^1_G(\mathrm{ad}\bar\rho)^{*\otimes n}, \qquad m^* : H^2_G(\mathrm{ad}\bar\rho)^* \xrightarrow{\Pi m_n^*} \prod_{n\geq 2} H^1_G(\mathrm{ad}\bar\rho)^{*\otimes n}.$$

These give a presentation for $R_{\bar\rho}$ when $\bar\rho$ is absolutely irreducible. We write $k[\![V]\!]$ for the completed symmetric algebra of the k-vector space V, i.e. $k[\![V]\!] := \prod_{n\geq 0} \mathrm{Sym}^n_k V$.

Theorem 2.2.2. *Let $\bar\rho$ be absolutely irreducible. Then there is a canonical isomorphism*

$$\frac{k[\![\,\mathrm{Ext}^1_G(\bar\rho,\bar\rho)^*]\!]}{(m^*\,\mathrm{Ext}^2_G(\bar\rho,\bar\rho)^*)} \xrightarrow{\sim} R_{\bar\rho}.$$

Notice that the surjection to $R_{\bar\rho}$ from $k[\![\,\mathrm{Ext}^1_G(\bar\rho,\bar\rho)^*]\!]$ follows from the standard result that the tangent space of $R_{\bar\rho}$ is canonically isomorphic to $\mathrm{Ext}^1_G(\bar\rho,\bar\rho)$. The higher cup products determine the kernel.

The generalization to the description of $\mathrm{Rep}_{\bar D}$ when $\bar D$ is not irreducible includes Theorem 2.2.2 as a special case. We set up some notation in order to state it:

- Let $\bar\rho = \bigoplus_{1\leq i\leq r} \bar\rho_i$ be a semi-simple representation with no multiplicity (i.e. $\bar\rho_i \simeq \bar\rho_j \Leftrightarrow i = j$), and let $\bar D$ be the induced pseudorepresentation.
- Write $\mathbf{r}$ for $\{1, 2, \dots, r\}$, and $\mathbf{l}$ for $\{0, 1, \dots, l\}$.
- Now $\mathrm{Ext}^i_G(\bar\rho,\bar\rho)$ has a "matrix-coordinate" decomposition $\mathrm{Ext}^k_G(\bar\rho_j,\bar\rho_i) = \bigoplus_{1\leq i,j\leq r} \mathrm{Ext}^k_G(\bar\rho_j,\bar\rho_i)$, and the higher cup products m_n respect this decomposition.
- Let $\mathcal{C} \subset \mathrm{Sym}^*_k \mathrm{Ext}^1_G(\bar\rho,\bar\rho)^*$ be the ideal generated by cyclic tensors, where a *cyclic tensor* is an element of

$$\mathrm{Ext}^1_G(\gamma)^* := \bigotimes_{0\leq s\leq l(\gamma)-1} \mathrm{Ext}^1_G(\bar\rho_{\gamma(s)}, \bar\rho_{\gamma(s+1)})^*,$$

  where $\gamma : \mathbf{l} \to \mathbf{r}$ is a *closed path* of length l, i.e. $\gamma(0) = \gamma(l)$.
- We write $S^\wedge_I$ for the completion of a ring S at an ideal I.

Theorem 2.2.3. *There is a map*

$$\mathrm{Spf}\, \frac{(\mathrm{Sym}^*_k \mathrm{Ext}^1_G(\bar\rho,\bar\rho)^*)^\wedge_{\mathcal{C}}}{(m^*\,\mathrm{Ext}^2_G(\bar\rho,\bar\rho)^*)} \longrightarrow \widehat{\mathrm{Rep}}_{\bar D}$$

presenting the stack $\widehat{\mathrm{Rep}}_{\bar D}$ as a quotient by the natural action of the torus of units in $\mathrm{End}_G(\bar\rho,\bar\rho)$.

Remark 2.2.4. Consider that the quotient $\mathrm{Sym}^*_k \mathrm{Ext}^1_G(\bar\rho, \bar\rho)^*/(\mathcal{C}, m^* \mathrm{Ext}^2(\bar\rho, \bar\rho)^*)$ parameterizes k-valued representations whose pseudorepresentation is $\bar D$ (i.e. whose semi-simplification is $\bar\rho$), i.e. this is the special fiber over $\mathrm{Spf}\, R_{\bar D}$.

Remark 2.2.5. Theorem 2.2.3 may be thought of as an "abelianization" of the results of Segal [Seg08], with attention to the profinite topology of G.

2.3. **Invariant theory.** We have seen that $\mathrm{Spec}\, R_{\bar D}$ is the GIT quotient of the quotient stack $\mathrm{Rep}_{\bar D}$ (Theorem 1.2.1), where we remind the reader of the assumption that the simple factors of $\bar\rho^{ss}_{\bar D}$ are distinct. Working with the presentation of this stack in Theorem 2.2.3 (and the comments afterward), we can determine $R_{\bar D}$ explicitly.

First we consider the case $\mathrm{Ext}^2_G(\bar\rho, \bar\rho) = 0$, which we call the *representation-unobstructed* case. In this case, is is quite easy to describe the tangent space to $R_{\bar D}$. The obstructions may be non-trivial, and have been determined in the literature in a combinatorial way. Indeed, $R_{\bar D}$ is simply the invariant subring for the natural torus action on $(\mathrm{Sym}^*_k \mathrm{Ext}^1_G(\bar\rho, \bar\rho))^\wedge_{\mathcal{C}}$. Here is some notation and the result.

- A closed path in $\gamma : \mathbf{l} \to \mathbf{r}$ is called *simple* if $\gamma(i) = \gamma(j) \Rightarrow \{i, j\} = \{0, l\}$.
- A *cycle* in $\mathbf{r}$ is an equivalence class of closed paths of length l under the equivalence relation $\gamma \sim \gamma'$ iff there exists $i \in \mathbf{l}$ such that $\gamma(j) = \gamma(i + j \pmod l)$ for $j \in \mathbf{l}$.
- Write $SC(\mathbf{r})$ for the set of equivalence classes of simple cycles in $\mathbf{r}$.
- Say that $\mathrm{Ext}^1_G(\bar\rho, \bar\rho)$ (or $\bar\rho$) is *strongly connected* if for any $i, j \in \mathbf{r}$, there exists a path γ from i to j and $\mathrm{Ext}^1_G(\gamma)$ is non-trivial.

Theorem 2.3.1 ([BLBVdW03]). *Assume that $\bar\rho = \rho^{ss}_{\bar D}$ is representation-unobstructed. Then $R_{\bar D}$ is isomorphic to the image of the natural map*

$$k[\![\bigoplus_{\gamma \in SC(\mathbf{r})} \mathrm{Ext}^1_G(\gamma)^*]\!] \longrightarrow k[\![\mathrm{Ext}^1_G(\bar\rho, \bar\rho)^*]\!].$$

In particular, the tangent dimension of $R_{\bar D}$ is equal to the dimension of $\bigoplus_{\gamma \in SC(\mathbf{r})} \mathrm{Ext}^1_G(\gamma)$, and, if $\bar\rho$ is strongly connected, the Krull dimension of $R_{\bar D}$ is given by $\dim_k \mathrm{Ext}^1(\bar\rho, \bar\rho) - r + 1$.

There is also a combinatorial expression for $R_{\bar D}$ in terms of the simplicial homology of the quiver associated to $\bar\rho$.

Now we continue to the general case, which may not be representation-unobstructed. We will write $R^1_{\bar D}$ for the $R_{\bar D}$ of Theorem 2.3.1, i.e. ignoring the presence of Ext^2. We will present $R_{\bar D}$ as a quotient of $R^1_{\bar D}$, which is possible given the invariant theory involved. We require a bit more notation.

- Write $SCC(i, j)$ for the set of "simple closed complements" of the length one path from j to i; $i = j$ is allowed, but $SCC(i, i) = \varnothing$.

Theorem 2.3.2. *There is an isomorphism*

$$R_{\bar D} \xrightarrow{\sim} \frac{R^1_{\bar D}}{\left(\bigoplus_{i,j \in \mathbf{r}} m^* \mathrm{Ext}^2_G(\bar\rho_j, \bar\rho_i)^* \otimes \left(\bigoplus_{\gamma \in SCC(i,j)} \mathrm{Ext}^1_G(\gamma)^* \right) \right)}$$

Remark 2.3.3. In view of Theorem 2.3.1, Theorem 2.3.2 gives a formula for an upper bound on the tangent dimension and Krull dimension of $R_{\bar{D}}$ in terms of dimensions of cohomology groups.

Remark 2.3.4. The presentation of Theorem 2.3.2 differs from a usual presentation of a deformation problem in that the relations can kill tangent vectors.

In particular, one can readily find an expression for the tangent space $\mathfrak{t}$ of $R_{\bar{D}}$ in terms of the expression given in Theorem 2.3.2, generalizing the result of Bellaïche [Bel12]. Bellaïche's result determines the tangent space when $r \leq 2$. The ith cup products for $2 \leq i \leq r$ are needed to determine $\mathfrak{t}$, for general r, which explains the limitation on the techniques of [Bel12].

Corollary 2.3.5. *There is a non-canonical isomorphism*

$$\mathfrak{t} \xrightarrow{\sim} \ker\left(\bigoplus_{\gamma \in SC(\mathbf{r})} \mathrm{Ext}^1_G(\gamma) \longrightarrow \bigoplus_{i,j \in \mathbf{r}} \mathrm{Ext}^2_G(\rho_j, \rho_i) \otimes \Big(\bigoplus_{\gamma' \in SCC(i,j)} \mathrm{Ext}^1_G(\gamma') \Big) \right),$$

where for a triple $(\gamma, (i,j), \gamma')$, the corresponding factor of the map is non-zero exactly when γ contains a length n path γ'' from j to i with complementary path γ', in which case the map is

$$\mathrm{Ext}^1_G(\gamma) = \mathrm{Ext}^1_G(\gamma'') \otimes \mathrm{Ext}^1_G(\gamma') \xrightarrow{b_n \otimes \mathrm{id}} \mathrm{Ext}^2_G(\rho_j, \rho_i) \otimes \mathrm{Ext}^1_G(\gamma').$$

Remark 2.3.6. Bellaïche determines a "complexity filtration" on $\mathfrak{t}$ [Bel12]. The fact that we express $\mathfrak{t}$ as a direct sum, instead of determining graded pieces of this filtration, reflects the non-canonical choices of A_∞-structure (m_n).

3. Appendix: The representation-theoretic significance of A_∞-structure

In this appendix, we illustrate, in concrete, representation-theoretic terms, what question the A_∞-structure on cohomology answers, and to explain the notion of A_∞-algebras so that its usefulness to answer this question is clear.

We will consider representations ρ_i of G on finite-dimensional k-vector spaces. We know that $\mathrm{Ext}^1_G(\rho_2, \rho_1)$ describes extensions up to natural equivalence. Given such an extension $e_{12} \in \mathrm{Ext}^1_G(\rho_2, \rho_1)$, we will represent it as

$$\begin{pmatrix} \rho_1 & e_{12} \\ & \rho_2 \end{pmatrix}.$$

Given another extension $e_{23} \in \mathrm{Ext}^1_G(\rho_3, \rho_2)$, we ask whether there is a representation of the form

$$\begin{pmatrix} \rho_1 & e_{12} & ? \\ & \rho_2 & e_{23} \\ & & \rho_3 \end{pmatrix}.$$

There will be such a representation precisely when the cup product $m_2(e_{12}, e_{23}) \in \mathrm{Ext}^2_G(\rho_3, \rho_1)$ vanishes. Then, the ways to "fill in" the "?" with f_{13} are a principal homogenous space under $\mathrm{Ext}^1_G(\rho_3, \rho_1)$. Continuing on, given that there are two representations of length three,

$$\begin{pmatrix} \rho_1 & e_{12} & f_{13} \\ & \rho_2 & e_{23} \\ & & \rho_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \rho_2 & e_{23} & f_{24} \\ & \rho_3 & e_{34} \\ & & \rho_4 \end{pmatrix},$$

there is a representation of length 4 inducing the two representations above as a sub (resp. quotient), i.e. of the form

$$\begin{pmatrix} \rho_1 & e_{12} & f_{13} & ? \\ & \rho_2 & e_{23} & f_{24} \\ & & \rho_3 & e_{34} \\ & & & \rho_4 \end{pmatrix},$$

if and only if a "higher cup product" $m_3(e_{12}, e_{23}, e_{34}) \in \mathrm{Ext}^2_G(\rho_4, \rho_1)$ vanishes. Notice that the existence of some way to fill in "?" does not depend on the choice of f_{13} or f_{24}.

Going on as above, one will find the following result, which we state as follows.

Proposition 3.0.1. *A set of non-zero extensions $e_{i,i+1} \in \mathrm{Ext}^1_G(\rho_{i+1}, \rho_i)$, $1 \leq i < d$, arises as subquotients of a length d representation with unique Jordan-Hölder filtration with ordered graded pieces $\rho_1, \rho_2, \dots, \rho_d$, i.e. there exists a representation of the form*

$$\begin{pmatrix} \rho_1 & e_{12} & \cdots & f_{1d} \\ & \rho_2 & \ddots & f_{2d} \\ & & \ddots & \vdots \\ & & & \rho_d \end{pmatrix}, \tag{3.0.2}$$

if and only if, for every $\ell, 1 \leq \ell \leq d$, and every sequence $e_i, e_{i+1}, \dots, e_{i+\ell}$ of consecutive extensions, $m_\ell(e_i, e_{i+1}, \dots, e_{i+\ell}) \in \mathrm{Ext}^2_G(\rho_{i+\ell}, \rho_i)$ vanishes.

Thus we see that the behavior of m_n for any n has a representation-theoretic consequence. Compare [Kel01, Example 7.8].

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