PRIME NUMBERS AND THE RIEMANN HYPOTHESIS

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This minicourse has two main goals. The first is to carefully define the Riemann zeta function and explain how it is connected with the prime numbers. The second is to elucidate the Riemann Hypothesis, a famous conjecture in number theory, through its implications for the distribution of the prime numbers.

1. The Riemann Zeta Function

Let $\mathbb{C}$ denote the complex numbers. They form a two dimensional real vector space spanned by 1 and i where i is a fixed square root of -1, that is,

$$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}.$$

Definition 1. The *Riemann zeta function* is the function of a complex variable

$$\zeta(s) := \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} n^{-s}, \quad s \in \mathbb{C}.$$

It is conventional to write $s = \sigma + it$ where $s = \sigma + it$ and $\sigma, t \in \mathbb{R}$

While this definition may seem straightforward, perhaps you've not thought of what complex powers are before. We are probably comfortable with integral powers of a complex number – we do this in PROMYS at least in the case of $\mathbb{Z}[i] \subset \mathbb{C}$ – but it is by no means obvious what a complex power of a real number should be.

While more could certainly said, we'll appeal to standard rules of exponentials and the Euler identity

$$e^{i\theta} = \cos\theta + i \sin\theta, \qquad \text{for } \theta \in \mathbb{R}.$$

Then for $n \in \mathbb{R}$ we have that

$$n^{-s} = n^{-\sigma - it} = n^{-\sigma} n^{-it} = n^{-\sigma} e^{-it \log n} = n^{-\sigma}\big(\cos(-t \log n) + i \sin(-t \log n)\big).$$

We note that by standard trigonometric identities, the quantity in the rightmost parentheses has absolute value 1. Thus the mantra is the the real part of the exponent controls the magnitude of an exponential, while the imaginary part of the exponent controls the angle of the ray drawn from the origin to the result. This is like the "polar form" of complex numbers, i.e. an element of $\mathbb{C}$ may be written as

$$re^{i\theta}, \text{ where } r \in \mathbb{R}_{\geq 0}, \theta \in [0, 2\pi).$$

This way of writing a complex number is unique unless $r = 0$.

Now that we understand what each term in the Riemann zeta function's definition means, the next thing that we must ask is whether or for which $s \in \mathbb{C}$ the sum defining $\zeta(s)$ converges. A useful definition for us is the following

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Definition 2. Let $(a_i) \subset \mathbb{C}$ be a sequence of complex numbers. We say that the sum

$$\sum_{i=1}^{\infty} a_i$$

converges absolutely or *is absolutely convergent* provided that the sum

$$\sum_{i=1}^{\infty} |a_i|$$

converges in $\mathbb{R}$, where $|\cdot|$ denotes the complex absolute value.

Example 3. The sequence $(a_i)_{i\geq 1}$ defined by $a_i = (-1)^{i+1}/i$ converges in $\mathbb{C}$ but is not absolutely convergent.

Absolute convergence is useful because of the following

Proposition 4. *If a sum is absolutely convergent, then it is convergent.*

Proof. Exercise. □

Applying the idea of absolute convergence and the fact from calculus class that for $\sigma \in \mathbb{R}$ the sum $\sum_{n\geq 1} n^{-\sigma}$ converges for $\sigma > 1$, we deduce that $\zeta(s)$ converges for $\sigma > 1$.

We might ask if $\zeta(s)$ converges when $\sigma = 1$, or perhaps for some $\sigma < 1$. However, for $s = 1$, we find that the zeta function gives the harmonic series, which does not converge. Therefore the statement that $\zeta(s)$ converges for $\sigma > 1$ is best possible.

2. The Riemann Zeta Function and the Primes

Why would one think that the primes had anything to do with this function? Well, Euler was fond of writing such equalities as

$$\begin{aligned}
&\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots \\
&= \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right)\left(1 + \frac{1}{5} + \cdots\right)\cdots
\end{aligned}$$

as a proof of the infinitude of the primes. We can quickly make this more rigorous. As a consequence of unique prime factorization, we can write the zeta function (in its region of absolute convergence) as

$$\begin{aligned}
\zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \cdots\right)\left(1 + \frac{1}{5^s} + \cdots\right)\cdots \\
&= \prod_p \sum_{k=0}^{\infty} p^{-ks} \\
&= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},
\end{aligned}$$

where the last equality applies the geometric series formula $\sum_{i=0}^{\infty} r^i = (1-r)^{-1}$ for $|r| < 1$ and $\prod_p$ signifies a product over the primes in $\mathbb{N}$. As the equality on the second line follows from unique prime factorization, we can say that the equation

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

is an "analytic statement of unique prime factorization." It is known as the *Euler product*. This gives us a first example of a connection between the zeta function and the primes.

Now lets take the limit of $\zeta(s)$ as $s \to 1^+$, that is, as s approaches 1 from the right along the real line. Clearly this limit does not converge, as we know that the harmonic series diverges. Yet the limit of each factor in the Euler product is the real number

$$\left(1 - \frac{1}{p}\right)^{-1} = \frac{p}{p-1}.$$

Since the product of these real numbers over the primes diverges, we may then conclude the following

Theorem 5. *There are infinitely many primes.*

Thus using calculus concepts and the unique prime factorization theorem, the zeta function has yielded another proof of the most ancient theorem about the primes. This is another sign that the zeta function and the primes are connected.

The Euler product may also be applied to solve this problem.

Exercise 6. Call a positive integer m *nth power free* provided that for all $\ell \in \mathbb{Z}^+$,

$$\ell^n \mid m \Longrightarrow \ell = 1.$$

Now for $x \in \mathbb{R}^+$ and integers $n \geq 2$ define $f_n(x)$ as

$$f_n(x) = \#\{m \in \mathbb{Z}^+ : m \leq x \text{ and } m \text{ is } n\text{th power free}\}.$$

Prove that for integers $n \geq 2$,

$$\lim_{x \to \infty} \frac{f_n(x)}{x} = \zeta(n)^{-1}.$$

Ok, what else can we find out about the primes by applying calculus concepts to the Euler product? Take the (modified) Taylor series for the log function at 1, that is, the series

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots, \qquad \text{for } x \in \mathbb{C}, |x| < 1.$$

Applying this to the Euler product for the zeta function we find that

$$\begin{aligned}
\log \zeta(s) = \log \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} &= -\sum_p \log(1 - p^{-s}) \\
&= \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \\
&= \sum_p \frac{1}{p^s} + \left(\sum_p \sum_{m=2} \frac{1}{mp^{ms}}\right).
\end{aligned}$$

Now try you hand at bounding sums in the following

Exercise 7. The sum on in the parentheses in the final equality converges for $s \geq 1$.

This has a very pleasant consequence.

Corollary 8. *The sum of the reciprocals of the primes diverges.*

Proof. Consider the equation for $\log \zeta(s)$ above as $s \to 1^+$. We know that $\log \zeta(s) \to +\infty$ because $\lim_{s\to 1^+} \zeta(s) = +\infty$. However, the quantity on the right hand side in parentheses remains bounded as $s \to 1^+$. Therefore the sum $\sum_p p^{-s}$ becomes arbitrarily large as $s \to 1^+$, allowing us to conclude that the sum

$$\sum_p \frac{1}{p}$$

does not converge. $\square$

Using more subtle properties of the zeta function involving techniques like those that will be "black boxed" later in this talk, one can prove that

$$\sum_{p\leq x} \frac{1}{p} \sim \log \log x.$$

3. The Properties of the Zeta Function

In the previous talk this evening, Josh discussed on the Chebyshev bounds on the prime counting function

$$\pi(x) := \sum_{p\leq x} 1$$

where p varies over prime numbers no more than x, and the related function

$$\psi(x) := \sum_{p^n \leq x} \log p$$

where p ranges over primes and n ranges over positive integers such that $p^n \leq x$. We wil rewrite the second function as

$$\psi(x) = \sum_{n\leq x} \Lambda(n),$$

where Λ is the von Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some } m \\ 0 & \text{otherwise.} \end{cases}$$

Be sure to observe that the two formulations of $\psi(x)$ are equivalent.

Mathematicians in the nineteenth century worked toward the prime number theorem, which is a statement about the asymptotic behavior of $\pi(x)$. We often use the $\sim$ symbol to describe asymptotics. It has the following meaning: let $f(x)$ and $g(x)$ be two complex functions of a real variable x. We say that

$$f(x) \sim g(x)$$

provided that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1.$$

With this notation in place we can state the prime number theorem.

Theorem 9 (Prime Number Theorem)**.** *Let $\pi(x)$ be the prime counting function defined above. Then*

$$\pi(x) \sim \frac{x}{\log x}.$$

Other ways of stating the prime number theorem are that the probability of a randomly chosen positive integer no more than x being prime approaches $1/\log x$, or that the probability of a randomly chosen positive integer near x being prime is $1/\log x$. Note that these probabilities are not trivially the same - these quantities are similar because of the log function's differential properties.

The prime number theorem was proven in 1896 by Hadamard and Vallée Poussin independently. Each completed the proof by constraining the complex numbers s such that $\zeta(s) = 0$. The next main ideas develop several main ideas having to do with these zeros, leaving complex analysis as a "black box."

The zeta function is often called the "Riemann zeta function" because Riemann instigated serious study of it. He proved the following items, in bold.

• **$\zeta(s)$ has an "analytic continuation" to the rest of $\mathbb{C}$, which is complex differentiable except for the "simple pole" at $s = 1$.**

An *analytic continuation* is an important concept in complex analysis, but one has already encountered this function in calculus class. Consider the function

$$1 + x + x^2 + x^3 + \cdots ,$$

which converges only in the disk $\{z \in \mathbb{C} : |z| < 1\}$ and on some points on the border of the disc. This is the Taylor expansion at 0 for the function

$$\frac{1}{1-x},$$

which we know perfectly well can be defined for x outside the disk, even though its series representation cannot. Analytic continuation is much like this: it extends a function that is a priori defined on a constrained domain to a larger domain of definition in a sensible way. However, our intuition from real functions will fail us here, because there are many real differentiable functions like

$$f(x) := \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

which are once or twice differentiable, but not smooth (i.e. infinitely differentiable). However, complex differentiable functions have the amazing property that they are infinitely differentiable whenever they are once differentiable. Any differentiable (thus infinitely differentiable) complex function is termed "holomorphic" or "analytic." This special property of complex differentiable functions is critical for showing that functions such as $\zeta(s)$ have unique analytic continuations. The continuation of $\zeta(s)$ is differentiable at every $s \in \mathbb{C}$ except $s = 1$; here it looks locally like $\frac{1}{s-1}$, a situation which is called "having a simple pole" at $s = 1$.

We won't talk about what the analytic continuation of the zeta function looks like on much of $\mathbb{C}$, because of the next item that Riemann proved limits zeros rather seriously.

• **The zeta function obeys a functional equation**, namely

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s),$$

where

$$\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx \text{ for } \sigma > 0.$$

and the cos function is defined on $\mathbb{C}$ by

$$\cos(s) = \frac{e^{is}+e^{-is}}{2}.$$

Note that the restriction on the convergence of the Γ-function does not stop us from relating any two values of the zeta function (however, the Γ-function also admits an analytic continuation!).

This functional equation relates the zeta functions value at a complex number s to its value at the point given by reflecting s across the point $1/2 \in \mathbb{C}$. Thus as we know the behavior of the zeta function for $\sigma > 1$ relatively well, we also know its behavior for $\sigma < 0$. We will demonstrate this by characterizing all of the zeros of $\zeta(s)$ just as Riemann did.

Consider the functional equation and let s vary over $A = \{s \in \mathbb{C} : \Re(s) > 1\}$. Let us find those values of s where the right side of the functional equation has a zero. First of all, $\zeta(s) \neq 0$ for all $s \in A$. This is the case because the logarithm of $\zeta(s)$ converges for all $s \in A$, which, since $x \to 0$ in $\mathbb{C}$ implies that the real part of $\log x$ approaches $-\infty$, means that $\zeta(s)$ cannot be zero. All of the remaining factors are never zero for any s in $\mathbb{C}$. Thus the right side of the functional equation is zero only when $\cos(\pi s/2)$ is zero, that is, when s is an odd integer at least 3. Therefore we have that $\zeta(1-s)$ is zero for all odd integers that are at least three, i.e.

$$\zeta(s) = 0 \text{ and } \Re(s) < 0 \Longrightarrow s = -2, -4, -6, -8, \dots.$$

These zeros at the negative even integers are called the *trivial zeros* of $\zeta(s)$.

Now we have limited the unknown zeros of $\zeta(s)$ to the region $0 \leq \Re(s) \leq 1$. This region is called the *critical strip*. We will see the arithmetic significance of this shortly, but having gone this far in characterizing the zeros, it is time to state the Riemann Hypothesis.

Conjecture 10 (Riemann Hypothesis)**.** *The non-trivial zeros of $\zeta(s)$ have real part one-half, i.e.*

$$\zeta(s) = 0 \textit{ and } 0 \leq \Re(s) \leq 1 \Longrightarrow \Re(s) = \frac{1}{2}.$$

Let's make a few observations about the zeros of $\zeta(s)$ in the critical strip. Note that via our computations of complex exponentials at the outset, the complex conjugate $\overline{n^{-s}}$of n^{-s} is equal to $n^{-\overline{s}}$. Therefore $\overline{\zeta(s)} = \zeta(\overline{s})$ for all $s \in \mathbb{C}$. Therefore if $s = \sigma + it$ is a zero of zeta in the critical strip, its complex conjugate $s = \sigma - it$ is a zero as well. Recall also that zeros in the critical strip will reflect over the point 1/2. Therefore if the Riemann hypothesis is false and $\rho = \beta + i\gamma$ is a zero in the critical strip but off the *critical line* $\Re(s) = 1/2$, there are zeros to the right of the critical line, two of them being in the list of distinct zeros

$$\rho = \beta + i\gamma; \quad \overline{\rho} = \beta - i\gamma; \quad 1-\rho = (1-\beta) - i\gamma; \quad \overline{1-\rho} = (1-\beta) + i\gamma.$$

Many many zeros of the Riemann zeta function have been calculated, and all of them are on the critical line. We should also remark that all zeros of the Riemann zeta function are simple zeros (i.e. they look like the zero of $f(x) = x$ at zero, not the the zero of $f(x) = x^n$ where $n > 1$), but this is not worth getting into.

But now, what can we do to connect the primes to the zeros of the zeta function? The key is complex analysis.

4. The Zeros of the Zeta Function and Prime Distribution

Complex analytic tools are good at finding poles - after all, poles are the interruptions in the strong condition of complex differentiability. Since we have been claiming that the zeros of the zeta function are significant for prime distribution, it makes sense that we will further examine the log of the zeta function. The log has poles where zeta has zeros or poles. More precisely, we will examine its logarithmic derivative, i.e. the derivative of its log, which also will have poles where zeta has poles or zeros. We calculate

$$\begin{aligned}\frac{d}{ds}(\log \zeta(s)) &= -\frac{d}{ds}\sum_p \log(1-p^{-s}) \\ &= \sum_p (1-p^{-s})^{-1}p^{-s}\log p \\ &= \sum_p \sum_{m\geq 1} \log p p^{-ms}.\end{aligned}$$

Recalling the definition of the von Mangoldt function $\Lambda(n)$ from the beginning of section 3, we find that the last equality can be immediately translated into

$$\frac{d}{ds}\log \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{-s}}.$$

Now comes in the critically important tool from complex analysis, which we must leave as a black box. In English, the idea is this: given a reasonably well behaved (e.g. not exponentially growing) function $f : \mathbb{N} \to \mathbb{C}$, the value of

$$F(x) = \sum_{n\leq x} f(n)$$

is intimately tied to the poles of

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

via *Perron's formula.*

In our case, we want to discuss $\psi(x) = \sum_{n\leq x}\Lambda(n)$. Recall that the prime number theorem states that $\psi(x) \sim x$.

In fact, Perron's formula will even interpolate the points of discontinuity of $\psi(x)$, that is we will get a formula for

$$\psi_0(x) := \begin{cases} \psi(x) & \text{if } \psi \text{ is continuous at } x \\ \frac{\psi(x^+)+\psi(x^-)}{2} & \text{if } \psi \text{ is not continuous at } x.\end{cases}$$

Perron's formula states that

$$\psi_0(x) = x - \sum_{\substack{\zeta(\rho)=0 \\ 0\le\Re(\rho)\le 1}} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1-x^{-2}),$$

that is, as we add non-trivial zeros ρ to the sum on the right hand side, the right hand side will coverge pointwise to $\psi_0(x)$ for every $x \in \mathbb{R}_{\ge 0}$. This equality is known as the *explicit formula* for ψ.

We remark that the first term of the right side (namely x) comes from the pole of $\zeta(s)$ at 1, the second term comes from the non-trivial zeros, the third term comes from the properties of $\zeta(s)$ at 0, and the final term comes from the trivial zeros. Notice that each exponent of x is the coordinates of a pole – this is part of Perron's formula.

Finally we've come up with an expression for $\psi(x)$ depending on the zeros of x. The prime number theorem states that $\psi(x) \sim x$, i.e. that the first term of the right side above dominates all of the others as $x \to \infty$. Let's check this, term by term.

The rightmost term, $\frac{1}{2}\log(1-x^{-2})$, will shrink to 0 as x gets large, so we need not worry about it. The next term is a constant, and so does not matter when x gets large either. This leaves the non-trivial zeros to contend with the x term on the left. We should remark that it is not obvious that the sum over the non-trivial zeros converges, but estimates for zero density in the critical strip first done by Riemann are sufficient to show that the sum converges.

Recall from the outset that when we take a real number x to the complex power ρ, the real part of ρ controls the size of x^ρ and its imaginary part controls the angle of the ray from 0 to x^ρ. In fact, because zeros of the zeta function come in conjugate pairs, we may pair up the zeros in the sum over non-trivial zeros; one pair $\rho = \beta + i\gamma$ and $\overline{\rho}$ where $\rho = |\rho|\, e^{i\theta_\rho}$ gives us

$$\begin{aligned}\frac{x^\rho}{\rho} + \frac{x^{\overline{\rho}}}{\overline{\rho}} &= \frac{x^\beta(e^{i\gamma\log x}\overline{\rho} + e^{-i\gamma\log x}\rho)}{\beta^2+\gamma^2} \\ &= \frac{x^\beta}{|\rho|}(e^{i(\gamma\log x - \theta_\rho)} + e^{-i(\gamma\log x-\theta_\rho)}) \\ &= 2\frac{x^\beta}{|\rho|}\cos(\gamma\log x - \theta_\rho).\end{aligned}$$

As θ_ρ is very close to $\pm\pi/2$, this result looks a lot like a constant times $x^\beta\sin(\gamma\log x)$, and this is a fine function to graph or look at in order to get an idea of what each conjugate pair of zeros contributes to the $\psi(x)$ function. You may find a picture of what such a function looks like at a website to be discussed below. Note also that the fact that imaginary parts cancel when we sum over conjugate pairs makes a lot of sense, since $\psi(x)$ is a real function.

With this in place, we can say what Hadamard and Vallée Poussin proved about the zeros of the zeta function in order to have the prime number theorem as a corollary: *they proved that there are no zeros of the zeta function on the line* $\Re(s) = 1$. Let's think about why this is sufficient.

Let's say that $\rho = \beta + i\gamma$ is a zero in the critical strip that is not on the critical line, that is, a counterexample to the Riemann hypothesis. Then there exists a zero that is to the right of the critical line, i.e. with $\beta > 1/2$, by the discussion of the symmetry of zeros above.

Without loss of generality, say that ρ is this zero. Then ρ and its conjugate $\overline{\rho}$ generate a term of the sum which reaches a magnitude as high as x^β times a constant. If somehow the four zeros generated by ρ and the zero-symmetries were the only counterexamples to the Riemann hypothesis, then there would be crazy variations in prime density: if x were such that $\cos(\gamma \log x - \theta_\rho)$ is near 1, then by examining the explicit formula for $\psi(x)$ we see that there would be fewer primes near to this x. This is the case because the wave

$$2\frac{x^\beta}{|\rho|}\cos(\gamma \log x - \theta_\rho)$$

generated by ρ and its conjugate will be much bigger than the waves generated by any of the other non-trivial zeros. Likewise, when x is such that $\cos(\gamma \log x - \theta_\rho)$ is close to -1, there will be many more primes near x. We can see how the truth of the Riemann hypothesis makes x as good of an estimate for $\psi(x)$ as it can be.

When we consider the prime number theorem, we realize that our aberrant zero ρ will not create trouble as long as $\beta < 1$; for we calculate

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = \lim_{x\to\infty} \frac{x - 2x^\beta \cos(\gamma \log x - \theta_\rho)/\left|\rho\right|}{x} = 1,$$

provided that $\beta < 1$. If there were to be a zero ρ with real part equal to 1, we can see via the explicit formula that the prime number theorem would not be true.

I want to conclude by giving you the idea that the Riemann hypothesis does not just make $\psi(x)$ as close to x as possible, but also is very beautiful. If it is true, it means that primes are generated by a bunch of different frequencies of waves that grow at the same rate. If some waves grow faster than others, we get havoc.

To observe this all in action, check out the following website:

http://www.secamlocal.ex.ac.uk/people/staff/mrwatkin/zeta/encoding2.htm

This website contains graphs of the components of the explicit formula coming from each zero, so that you can see what a function like $2\frac{x^\beta}{|\rho|}\cos(\gamma \log x - \theta_\rho)$ looks like. But most importantly, it has an applet at the bottom that shows the ψ-function in yellow and shows the right side of the explicit formula as additional non-trivial zeros are added to the sum. Check it out!