HIGHER YONEDA PRODUCT STRUCTURES AND IWASAWA ALGEBRAS MODULO p

CARL WANG-ERICKSON

ABSTRACT. We give answers to three questions posed by Sorensen [Sor20]. These concern the relationship between a modulo p Iwasawa algebra of a torsionfree pro-p group and A_∞-algebra structures on its Yoneda algebra.

CONTENTS

1. INTRODUCTION

Let p be a prime number. The subject of this paper is the modulo p representation theory of p-adic Lie groups G. This is of natural interest relative to the proposed p-adic local Langlands correspondence. For an introduction to this connection, see [Har16].

In light of the fact that p divides the pro-order of any positive-dimensional p-adic Lie group, the usual functor $\mathcal{F}$ between modulo p smooth representations of G and modules for the Hecke algebra is not exact and loses information. This situation is similar to the failure of Maschke's theorem for representations of a finite group over a field whose characteristic is not relatively prime to the order of the group. The presence of non-semi-simple objects obstructs $\mathcal{F}$ from being an equivalence. Schneider has proposed a *derived Hecke algebra* whose appropriate module categories should not lose information, and has completely described the Hecke algebra in the case $G = \mathbb{Z}_p$ [Sch15].

Recently, Sorensen [Sor20] produced a generalization of Schneider's description of the derived Hecke algebra to G that are merely required to be pro-p, hence compact, and torsionfree. This crucially relies on the notion of an A_∞-algebra structure on the derived Hecke algebra, along with a category of A_∞-modules. He also asked some questions regarding whether his results could be made more precise, proving stronger characterizations of G and its category of smooth modulo p representations in terms of the derived Hecke algebra. We remark that Sorensen's description is completely explicit only in the case that G is abelian (cf. his remark in [Sor20, §1, pg. 155]), and that this limitation is part of what what his questions address.

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The point of this paper is to record positive answers to these questions. A secondary goal is to illustrate that the author's recent work [WE18, Part 2] furnishes an efficient framework to apply the the role of A_∞-algebras in non-commutative deformation theory in order to answer the questions. We recall results from [WE18] in §2, followed by the application toward positive answers in §3. In preparation to state these answers, the rest of this introduction is occupied with introducing the results and questions of Sorensen [Sor20].

As Sorensen mentions [Sor20, §12], these positive answers had been confirmed to him by experts. Work of Positselski, which provides another approach to positive answers, are discussed at the end of this introduction (§1.5).

1.1. **The setting of** [Sor20]**.** We recall the setting of Sorensen's paper [Sor20]. We will use some common notions about p-adic Lie groups without giving definitions here, referring the reader to [Sor20], where they are clearly explained. Schneider's book [Sch11] is a thorough exposition of this background material.

Let G be a p-adic Lie group that is torsionfree and pro-p. Let k be a finite field of characteristic p. Let $\Omega = k[\![G]\!]$ be the completed group algebra, the *Iwasawa algebra* of G, which is a local associative k-algebra equipped with its standard profinite topology. Let $D(\Omega)$ denote the derived category of the category $\mathrm{Mod}(\Omega)$ of pseudocompact left Ω-modules, which, as Sorensen explains [Sor20, §3], is anti-equivalent to the category of smooth k-linear representations of G. Let $\Omega^!$ denote the opposite algebra of the Yoneda algebra of Ω, recalling that the Yoneda algebra in the category $\mathrm{Mod}(\Omega)$,

$$\mathrm{Ext}^\bullet_\Omega(k,k) := \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \mathrm{Ext}^i_\Omega(k,k),$$

is a canonical $\mathbb{Z}$-graded k-algebra under the cup product. We call $\Omega^!$ the *Koszul dual* k-algebra of Ω; it plays the role of the derived Hecke algebra, for reasons explained in [Sor20, §1].

For concreteness, and in order to recall Sorensen's results, we set up a narrower class of groups G where the structure of $\Omega^!$ is well-understood (see especially [Sor20, §§7-8] and [Sch11] for reference). When G is equipped with a *valuation*, there arises a graded *k-Lie algebra of G* that we denote by $\mathfrak{g}$ (see e.g. [Sch11, §§23-25]). When there is a *basis* for G whose elements have the same valuation $t \in \mathbb{R}_{>1/(p-1)}$, G is called *equi-p-valued* and $\mathfrak{g}$ is concentrated in degree t; in particular, $\mathfrak{g}$ is abelian. Sorensen especially focuses on the case where G is a *uniform* pro-p group, which implies that G is equi-p-valued and that the valuation can be chosen so that $\mathfrak{g}$ is concentrated in degree 1.

Combining a theorem of Lazard [Laz65] in the equi-p-valued case with consequences of the straightforward nature of $\mathfrak{g}$ in the uniform case, one has a canonical $\mathbb{Z}$-graded k-algebra isomorphism of [Sor20, Cor. 8.3],

$$\Omega^! \xrightarrow{\sim} \bigwedge_k \mathfrak{g}^* := \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \wedge^i_k(\mathfrak{g}^*), \tag{1.1.1}$$

where $(-)^*$ denotes k-linear duality. A particular consequence of (1.1.1) is that $\dim_k \mathrm{Ext}^i_\Omega(k,k) = \binom{\dim_k \mathfrak{g}}{i}$ for integers $i, 0 \leq i \leq \dim_k \mathfrak{g}$.

This discussion makes it clear that the passage $\Omega \mapsto \Omega^!$ loses information: there are equal-dimensional uniform pro-p groups that are not isomorphic, and thus their

Iwasawa algebras are not isomorphic. Yet the k-Lie algebras of uniform pro-p groups are determined up to isomorphism by their dimension alone [Sor20, §7.1].

1.2. **The results of** [Sor20]**.** Sorensen proves that there exists an A_∞*-algebra* structure enriching the graded algebra $\Omega^!$ that recovers the lost information, in the following sense. We denote such a structure by m, and write $(\Omega^!, m)$ for the resulting A_∞-algebra. For an introduction to A_∞-algebras, see the references given in §1.6, where the notion of "enrichment" is also discussed.

We emphasize that m is unique up to non-unique isomorphism, which is typical for A_∞-algebra structures. That is, whilst m is not canonical, the isomorphism class of $(\Omega^!, m)$ is canonical. In particular, m is called *trivial* when it carries no more information than $\Omega^!$; triviality of $(\Omega^!, m)$ is well-defined up to isomorphism.

There is a derived category of strictly unital left A_∞-modules of $(\Omega^!, m)$, denoted $D_\infty(\Omega^!, m)$. The main result of [Sor20] is that there is an equivalence of triangulated categories

$$D(\Omega) \xrightarrow{\sim} D_\infty(\Omega^!, m). \qquad \text{([Sor20, Thm. 1.1])}$$

And when G is a uniform pro-p group, this can be rephrased as

$$D(\Omega) \xrightarrow{\sim} D_\infty(\textstyle\bigwedge \mathfrak{g}^*, m). \qquad \text{([Sor20, Thm. 1.2])}$$

1.3. **The questions.** Sorensen asks whether and how the relationship between Ω and the isomorphism class of $(\Omega^!, m)$ can be made more precise [Sor20, §12]. We quote his questions verbatim; the only changes arise from

- writing $G' \subset G$ as a subgroup instead of $H \subset G$, and from
- following this paper's convention of writing $(\Omega^!, m)$ for an A_∞-algebra structure on $\Omega^!$ that extends its inherent graded algebra structure, leaving $\Omega^!$ to denote the underlying dg-algebra (with trivial differential).

Here are the questions.

(a) *By* [Sor20, Thm. 1.1]*, one can recover $\Omega = k[\![G]\!]$ up to derived equivalence from the A_∞-algebra $(\Omega^! = \mathrm{Ext}^\bullet_\Omega(k,k)^{\mathrm{op}}, m)$. Does $(\Omega^!, m)$ determine Ω up to isomorphism?*

(b) *Is there a converse to* [Sor20, Thm. 1.2] *in the sense that G must be abelian if the A_∞-structure on $\bigwedge \mathfrak{g}^*$ is trivial?*

(c) *Suppose $G' \subset G$ is an open subgroup. Then $\Omega(G)$ is finite free over the subalgebra $\Omega(G')$ and we have the restriction map* $\mathrm{Mod}(\Omega(G)) \to \mathrm{Mod}(\Omega(G'))$ *which induces a map $D(\Omega(G)) \to D(\Omega(G'))$. Is there a morphism of A_∞-algebras $(\bigwedge \mathfrak{g}^*, m) \to (\bigwedge \mathfrak{g}'^*, m')$ inducing the corresponding map on D_∞ via "extension of scalars" along this map?*

1.4. **The answers.** We answer these questions affirmatively in §3. We show that question (a) has an affirmative answer in a particularly strong way – there exists a presentation for Ω in terms of A_∞-structures related to $(\Omega^!, m)$ – which is then used to answer (b) and (c). We review background from [WE18] in §2, which culminates in a presentation of Ω in terms of an A_∞-algebra structure m' enriching the graded algebra structure on $\Omega^!$ (Theorem 2.5.1). In order to apply this presentation, the key technical requirement, satisfied in Corollary 3.1.5, is an explicit isomorphism of A_∞-algebras between $(\Omega^!, m)$ and $(\Omega^!, m')$. Both m and m' arise quite naturally from a single set of choices, but they are quite different and are reconciled using a result of Segal [Seg08] recorded as Proposition 3.1.4.

1.5. **Related works.** As discussed in [WE18, §4.5], the fact that a choice of A_∞-algebra structure m on $\Omega^!$ determines a presentation for Ω was proved by Segal [Seg08, Thm. 2.14] in an analogous situation when k has characteristic zero. (This was also proved in the case of a graded algebra in place of Ω in [LPWZ09].) The extension to general characteristic is given in [WE18, Part 2]. In addition, the amplification of [Seg08] given in [WE18, Cor. 6.2.6] especially clarifies the given answer to question (c).

L. Positselski has previously answered these questions positively, in the sense that positive answers follow from the isomorphism (1.1.1) and rather immediate consequences of his work. Namely, Positselski has studied equivalences of module categories that accompany bar-cobar equivalences of categories of dg-algebras and A_∞-algebras, from which positive answers can be derived.

- A positive answer to question (a) follows from [Pos11, §6.10, part (b) of Theorem, pg. 76]. It is also recorded as [Pos17, Thm. 3.3].
- A positive answer to question (b) may be found in [Pos17, end of Ex. 6.3, pg. 225].
- A positive answer to question (c) follows from [Pos11, §6.9, part (a) of Proposition, pg. 74].

1.6. **Conventions and definitions.** We work with complexes, graded algebras, dg-algebras, and A_∞-algebras over a finite field k of positive characteristic p. All gradings in the remainder of this paper are indexed by $\mathbb{Z}$, and the differentials have graded degree $+1$.

Remark 1.6.1. The assumption that p is odd is used in [Sor20, §2] in order to relate the Yoneda algebra $\Omega^!$ to the k-Lie algebra $\mathfrak{g}$ via the isomorphism (1.1.1). We will not require this since we will simply work directly with the Yoneda algebra $\Omega^!$. This approach comes along with the minor caveat that when $p = 2$, we are actually answering versions of questions (b) and (c) with $\bigwedge \mathfrak{g}^*$ replaced by $\Omega^!$. The main theorems of this paper, in §3, are phrased accordingly.

We let $\hat{T}_k V$ denote the free completed tensor algebra on a graded k-vector space V. We let V^* denote the graded degree-wise k-linear dual of V, that is, $(V^*)^n = (V^{-n})^*$. This dual operation extends to complexes.

We use Σ to denote suspension of a (differential) graded k-vector space. This is mainly used to move elements of graded vector spaces from degree 1 to degree 0, so that we can consider algebras involving them as (classical) k-algebras (as opposed to graded k-algebras). The symbol ΣV^* should be read as $(\Sigma V)^*$, first suspending and then applying the graded dual.

An A_∞-algebra over k is an algebra in graded k-vector spaces over the A_∞-operad. In this article, we call these "A_∞-algebras," not mentioning k. See the article of Keller [Kel01] for the full definition of the category of A_∞-algebras, matching the convention we use here. Here, we give summary definitions. In particular, when B, B' are graded k-vector spaces, we use (B, m), where $m = (m_n)_{n\geq 1}$, to denote an A_∞-algebra structure on B, i.e.

$$m_n : B^{\otimes n} \longrightarrow B, \text{ for } n \geq 1, \text{ of graded degree } 2 - n$$

satisfying certain compatibility conditions. Likewise, $f = (f_n)_{n\geq 1} : (B, m) \to (B', m')$ denotes a morphism of A_∞-algebras, where

$$f_n : B^{\otimes n} \longrightarrow B', \text{ for } n \geq 1, \text{ of graded degree } 1 - n.$$

We also refer to terms describing A_∞-algebras (minimal, formal) and A_∞-morphisms (quasi-isomorphism, etc.) that can be found in [Kel01]. We emphasize that an A_∞-algebra (B, m) is called *minimal* when $m_1 = 0$.

We will treat dg-algebras $(C, d_C, m_{2,C})$, where d_C is the differential and $m_{2,C}$ is the multiplication, as A_∞-algebras. The A_∞-structure is $m = (m_n)_{n\geq 1}$ where $m_1 = d_C$, $m_2 = m_{2,C}$, and $m_n = 0$ for $n \geq 3$. In contrast, we say that an A_∞-algebra structure m *enriches* a dg-algebra $(C, d_C, m_{2,C})$ when $m_1 = d_C$ and $m_2 = m_{2,C}$; for enrichments m, there is no restriction on m_n for $n \geq 3$. We remark that enrichments of graded algebras, considered to be a dg-algebra with a trivial differential, are minimal by definition. For example, the A_∞-algebra enrichments of the Yoneda algebra $\Omega^!$ discussed earlier in this introduction are minimal.

The works of Keller [Kel01, Kel02, Kel06] are useful introductions to A_∞-algebras in relation to representations of algebras, with respect to the perspective and conventions of this paper.

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2. Recalling a result from [WE18]

In this section, our goal is to state an application of [WE18, Cor. 6.2.6] to the Iwasawa algebra Ω, which is recorded here as Theorem 2.5.1. We first recall background that is presented at greater length in [WE18, §5].

2.1. **Hochschild cohomology.** Firstly we recall a continuous version of the standard *Hochschild cochain complex*

$$C^\bullet(\Omega, k) := \bigoplus_{i\in\mathbb{Z}_{\geq 0}} C^i(\Omega, k) := \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \mathrm{Hom}_k(\Omega^{\otimes i}, k)$$

of Ω, where the (Ω, Ω)-bimodule structure on k is trivial and where $\mathrm{Hom}_k(\Omega^{\otimes i}, k)$ consists of those k-linear maps that are continuous under the topology induced by the profinite topology carried by Ω. This is naturally a dg-algebra, where the multiplication comes from the multiplication operation on k and the standard cup product of Hochschild cochains. In what follows, we presume continuity of all Hochschild cochains and pass over topological conditions in silence.

Likewise, denote the cohomology of the Hochschild cochain complex, which we will call *Hochschild cohomology*, by

$$H^\bullet(\Omega, k).$$

This is a graded k-algebra.

It is standard that $H^\bullet(\Omega, k)$ is canonically isomorphic, as a graded k-algebra, to the Yoneda algebra $\mathrm{Ext}^\bullet_\Omega(k, k)$. We apply this isomorphism without further comment in the sequel.

2.2. **Minimal models for dg-algebras.** Let $(C, d_C, m_{2,C})$ be a dg-k-algebra with graded cohomology algebra $H = (H^\bullet(C), 0, m_2)$. It is a result of Kadeishvili [Kad82], which may also be found recorded [WE18, §2.2], that there exists an A_∞-algebra structure $m = (m_n)_{n\geq 1}$ on the cohomology of a dg-algebra such that

- it enriches the graded algebra structure on H, in the sense that
$$m_1 = 0 \quad \text{and} \quad m_2 \equiv m_{2,C} \pmod{B^\bullet(C)}$$
where $B^\bullet(C)$ represents the graded vector space of coboundaries in C.
- there exists a quasi-isomorphism of A_∞-algebras
$$f = (f_n)_{n\geq 1} : H \to C$$
where f_1 sends each cohomology class to a choice of representative cocycle. In order to interpret this map in the A_∞ category, recall that dg-algebras may be taken to be A_∞-algebras with trivial higher multiplications, as discussed in §1.6.

This data (m, f) is unique up to non-unique isomorphism.

Because an A_∞-algebra (A, m) is called minimal when $m_1 = 0$, we call a (H, m) produced by Kadeishvili a *minimal model* of $(C, d_C, m_{2,C})$, as $f : (H, m) \to (C, d_C, m_{2,C})$ is a quasi-isomorphism. We call such (m, f) a *minimal model structure* of H relative to $(C, d_C, m_{2,C})$.

Subsequent work of Kontsevich–Soibelman [KS00] established the existence of minimal models for A_∞-algebras and clarified that a homotopy retract structure on (C, d_C) relative to $(H, 0)$ gives rise to a choice of (m, f) producing the minimal model.

Definition 2.2.1. Let (A, d_A), (C, d_C) be complexes. We call (A, d_A) a *homotopy retract* of (C, d_C) when they are equipped with maps

$$\begin{tikzcd} C \arrow[loop left, "h"] \arrow[r, shift left, "p"] & A \arrow[l, shift left, "i"] \end{tikzcd}$$

such that p and i are morphisms of complexes, $h : C \to C[1]$ is a morphism of graded vector spaces, $\mathrm{id}_C - ip = d_C h + h d_C$, and i is a quasi-isomorphism.

Proposition 2.2.2 (Kontsevich–Soibelman [KS00])**.** *Let (C, m') be an A_∞-algebra. A homotopy retract (i, p, h) between $(H^\bullet(C), 0)$ and (C, m'_1) induces, via explicit formulas, a minimal model structure (f, m). That is, there are formulas in (i, p, h) and m' that produce the minimal A_∞-algebra structure m on $H^\bullet(C)$ and the quasi-isomorphism $f : (H^\bullet(C), m) \to (C, m')$.*

Proof. See [LV12, Thm. 9.4.14] or [WE18, Thm. 5.2.5]; both of these references record the formulas. □

Applying this to the case where (C, m') is a dg-algebra (i.e. $m'_n = 0$ for $n \geq 3$) implies Kadeishvili's result on A_∞-algebra minimal models for dg-algebras.

Remark 2.2.3. Merkulov set up the same formulas in a more concrete way [Mer99], which the author learned from work of Lu–Palmieri–Wu–Zhang [LPWZ09]. These formulas may be found in [WE18, Ex. 5.2.8], and we give some information here for the reader's convenience.

Following Merkulov, we note that a homotopy retract between the cohomology $(H,0)$ and the complex it arose from, (C,d_C), amounts to a direct sum decomposition

$$C^n = B^n \oplus \tilde{H}^n \oplus L^n \text{ for all } n \geq 0, \tag{2.2.4}$$

where B^n denotes the subspace of C^n consisting of n-coboundaries, $\tilde{H}^n$ is a complement to B^n in the subspace Z^n of C^n consisting of n-cocycles, and L^n is a complement to Z^n in C^n. Then f_1 in degree n is a map $H^n \to C^n$ lifting each cohomology class to a choice of representing cocycle. This is specified by the decomposition above as follows: f_1 is the inverse of the natural isomorphism $\tilde{H}^n \xrightarrow{\sim} H^n$. Similarly, $f = (f_n)_{n\geq 1}$ and $m = (m_n)_{n\geq 1}$ are given inductively by formulas in C^n using the decomposition above and the isomorphism $f_1 : H^n \xrightarrow{\sim} \tilde{H}^n$.

It will also be useful to have an inverse quasi-isomorphism to the f of the minimal model structure.

Proposition 2.2.5. *Let (C,m') and (i,p,h) as in Proposition 2.2.2, so that we have the minimal model structure (f,m) described there. Then p extends to a quasi-isomorphism of A_∞-algebras, in the following sense: there exists a quasi-isomorphism $g = (g_n)_{n\geq 1} : (C,m') \to (H^\bullet(C),m)$ such that $g_1 = p$. Moreover, g is a left inverse to f, in that $g\circ f : (H^\bullet(C),m) \to (H^\bullet(C),m)$ is the identity map. That is, $g\circ f$ is an A_∞-isomorphism, where $(g\circ f)_1$ is the identity map $\mathrm{id}_{H^\bullet(C)}$ and $(g\circ f)_n = 0$ for $n \geq 2$.*

Proof. This follows from [CL19, Thm. 3.9(2)]. □

2.3. **The bar equivalence.** We recall a dualized version of the bar equivalence, which is described at more length in [WE18, §2].

Let (A,m) be an A_∞-algebra. Taking the suspension of the graded dual of $m_n : A^{\otimes n} \to A$ as described in §1.6, we get

$$m_n^* : \Sigma A^* \longrightarrow (\Sigma A^*)^{\otimes n}, \text{ of graded degree 1.}$$

Taking the product over the codomain, we produce

$$m^* = \prod_{n\geq 1} m_n^* : \Sigma A^* \longrightarrow \hat{T}_k \Sigma A^*.$$

By applying the Leibniz rule, we uniquely extend this map to a derivation

$$m^* : \hat{T}_k\Sigma A^* \longrightarrow \hat{T}_k\Sigma A^*.$$

Note that nothing in the construction of m^* depends on m satisfying the compatibility conditions demanded of an A_∞-algebra structure on A. In fact, m gives an A_∞-algebra structure if and only if the derivation m^* is a differential, i.e. $(m^*)^2 = 0$. This is a consequence of the bar equivalence, which is an isomorphism of categories between A_∞-algebras and co-free co-complete co-dg-algebras. The above "dualed" version of the bar equivalence restricts to an equivalence on those A_∞-algebras A such that A^n is finite-dimensional for all $n \in \mathbb{Z}$.

Thus, when (A,m) is an A_∞-algebra, we write

$$\mathrm{Bar}^*(A,m) := (\hat{T}_k\Sigma A^*, m^*, s)$$

for the complete dg-algebra given by the differential m^* and the standard multiplication s of $\hat{T}\Sigma A^*$. In words, we call this the *dual bar construction of* (A,m).

2.4. **The classical hull.** There is a natural inclusion functor from k-algebras to dg-k-algebras, sending a k-algebra D to a dg-k-algebra $D[0]$ concentrated in degree zero and with a trivial differential. This functor has a left adjoint on dg-k-algebras. This functor sends a dg-k-algebra $(B, d_B, m_{2,B})$ to its quotient $\mathcal{A}(B) = \mathcal{A}(B, d_B, m_{2,B})$ by the ideal generated by

$$\bigoplus_{n \in \mathbb{Z} \smallsetminus \{0\}} B^n \text{ and } d_B(B^{-1}),$$

which we call the *classical hull* of B.

We are especially interested in the case of the dg-algebra $B = \mathrm{Bar}^*(A, m)$. Because its underlying complete graded algebra is freely generated by ΣA^*, one may readily compute that the classical hull is presented as

$$\mathcal{A}(B) = \frac{\hat{T}_k(\Sigma(A^1))^*}{(m^*(((\Sigma(A^2))^*))}.$$

2.5. **A result from** [WE18]**.** Recall from the introduction that Ω is the Iwasawa algebra of G over k and $\Omega^!$ is the opposite algebra of the Yoneda algebra $\mathrm{Ext}^\bullet_\Omega(k, k)$. The main result that we wish to recall from [WE18, §6] gives a presentation of Ω in terms of a choice of decomposition of $C^\bullet(\Omega, k)$ as in (2.2.4). We state it in terms of its application to Ω.

Theorem 2.5.1. *Choose a homotopy retract structure on $(H^\bullet(\Omega, k), 0)$ relative to $(C^\bullet(\Omega, k), d_C)$, or, equivalently, a decomposition of $C^\bullet(\Omega, k)$ as in (2.2.4). This determines the additional data (f, m) as explained in §2.2. These data determine an isomorphism*

$$\rho^u : \Omega \xrightarrow{\sim} \mathcal{A}(\mathrm{Bar}^*(H^\bullet(\Omega, k))) \cong \frac{\hat{T}_k \Sigma H^1(\Omega, k)^*}{(m^*(\Sigma H^2(\Omega, k)^*)}$$

given by, for $x \in \Omega$,

$$\rho^u : x \mapsto \bar{x} + \sum_{i=1}^{\infty} (\underline{e} \mapsto (f_i(\underline{e}))(x)),$$

where $\underline{e}$ is a generic element of $(\Sigma H^1(\Omega, k))^{\otimes i}$ and $x \mapsto \bar{x}$ denotes reduction modulo the unique maximal ideal of Ω.

We explain how it is that $\underline{e} \mapsto (f_i(\underline{e}))(x)$ denotes an element of $(\Sigma H^1(\Omega, k)^*)^{\otimes i}$, where $i \geq 1$. Notice first that the fixed f_i, having graded degree $1 - i$, maps $(H^1(\Omega, k))^{\otimes i}$ to $C^1(\Omega, k)$. As $C^1(\Omega, k)$ consists of functions from Ω to k, evaluating $f_i(\underline{e})$ at a fixed choice of $x \in \Omega$ results in the desired map $H^1(\Omega, k)^*)^{\otimes i} \to k$.

Proof of Theorem 2.5.1. The statement of Theorem 2.5.1 is an application of [WE18, Cor. 6.2.6(1)], where

- Ω replaces $k[G]^\wedge_{\ker \rho}$,
- an assumption that $H^n(\Omega, k)$ is finite-dimensional for all n is dropped, since this follows from G being finite-dimensional as a p-adic Lie group, and
- there are some other simplifications because $\rho : k[G] \to k$ is the trivial representation in the present case.

Indeed, because G is pro-p, the completed group algebra Ω of G over k is canonically isomorphic to the completion of $k[G]$ at the kernel of the trivial representation $\rho : k[G] \to k$. □

Remark 2.5.2. We see in the presentation of Theorem 2.5.1 what data in the A_∞-algebra $(H^\bullet(\Omega, k), m)$ does not obviously influence the presentation of Ω. Namely, we see that the groups $H^1(\Omega, k)$ and $H^2(\Omega, k)$ along with the A_∞-products $m_n : H^1(\Omega, k)^{\otimes n} \to H^2(\Omega, k)$ determine the isomorphism class of the k-algebra Ω. For example, the groups $H^i(\Omega, k)$ for $i \geq 3$ are not obviously involved. Since, conversely, the isomorphism class of $(H^\bullet(\Omega, k), m)$ is determined by Ω, it would be interesting to determine whether and how the entire A_∞-algebra structure m is determined by $m_n : H^1(\Omega, k)^{\otimes n} \to H^2(\Omega, k)$ in the case of a uniform pro-p group G.

3. Answers and proofs

In this section, we answer the questions of §1.3. These answers are applications of Theorem 2.5.1, which gives a presentation of Ω in terms of a choice of a homotopy retract between the Hochschild cochain complex and the Yoneda algebra (which is Hochschild cohomology of the trivial Ω-bimodule k).

The main obstacle in the way of directly addressing the questions is that an alternative dg-algebra to the Hochschild cochain complex is used in [Sor20] to induce the same Yoneda algebra and an A_∞-algebra structure on its opposite. Namely, the endomorphism dg-algebra of the (projective) bar resolution of k is used. While these algebras are, of course, quasi-isomorphic (as complexes) with naturally compatible induced graded-homogeneous multiplication operations on their cohomology – that is, the Yoneda algebra – we must account for their distinctiveness because the construction of A_∞-algebra structures on their cohomology depends on the dg-algebra structure. We accomplish this using an explicit quasi-isomorphism between these two dg-algebras, due to Segal [Seg08] (see Proposition 3.1.4).

3.1. **Compatibility of the two endomorphism dg-algebras.**

We begin with a definition of compatibility of a homotopy retract. For the definition of homotopy retracts, see e.g. [WE18, Defn. 5.2.1].

Definition 3.1.1. Choose a homotopy retract (i, p, h) (resp. (i', p', h')) between a cochain complex C (resp. C') and its cohomology $H = H^\bullet(C)$ (resp. $H' = H^\bullet(C')$). Let $\Psi : C \to C'$ be a quasi-isomorphism, which therefore induces an isomorphism $H^\bullet(\Psi) : H \xrightarrow{\sim} H'$. We say that Ψ is *compatible* with these two homotopy retracts when we have commutative squares

$$\begin{tikzcd} H \arrow[r, "i"] \arrow[d, "H^\bullet(\Psi)"] & C \arrow[d, "\Psi"] \\ H' \arrow[r, "i'"] & C' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "p"] \arrow[d, "\Psi"] & H \arrow[d, "H^\bullet(\Psi)"] \\ C' \arrow[r, "p'"] & H' \end{tikzcd} \qquad \begin{tikzcd} \Sigma C \arrow[r, "h"] \arrow[d, "\Sigma\Psi"] & C \arrow[d, "\Psi"] \\ \Sigma C \arrow[r, "h'"] & C' \end{tikzcd}$$

Remark 3.1.2. This may be too specific of a definition of a compatible homotopy retract for a general theory, but it suffices for the situation at hand.

We will use the following pair of compatible homotopy retracts, given a quasi-isomorphism and extra maps.

Lemma 3.1.3. *Let Ψ be a quasi-isomorphism of complexes $\Psi : C \to C'$ that admits a left inverse quasi-isomorphism $\Phi : C' \to C$, i.e. $\Phi \circ \Psi = \mathrm{id}_C$. Let (i', p', h') be a homotopy retract between C' and H'. Then the following natural formulas produce a compatible homotopy retract (i, p, h) between C and H.*

$$i = \Phi \circ i' \circ H^\bullet(\Psi), \qquad p = H^\bullet(\Phi) \circ p' \circ \Psi, \qquad h = \Phi \circ h' \circ \Sigma\Psi.$$

Proof. The homotopy retract property of (i, p, h) follows by direct computation. The compatibility follows directly from the formulas. $\square$

Let $E^\bullet(\Omega, k)$ denote the endomorphism dg-algebra of the bar resolution of k. As usual, $C^\bullet(\Omega, k)$ denotes the dg-algebra of Hochschild cochains of the Ω-bimodule k.

Proposition 3.1.4. *There exists an explicit quasi-isomorphism of dg-algebras $\Psi : C^\bullet(\Omega, k) \to E^\bullet(\Omega, k)$ that admits a (non-multiplicative) left inverse of cochain complexes $\Phi : E^\bullet(\Omega, k) \to C^\bullet(\Omega, k)$. Under the identification of graded vector spaces $H^\bullet(C^\bullet(\Omega, k)) \cong H^\bullet(E^\bullet(\Omega, k))$ (using the fact that they both compute the same* Ext*-functors), $H^\bullet(\Psi)$ is the identity map.*

Proof. This is the content of [Seg08, Lem. 2.6]. $\square$

The following corollary sums up the relationship between the two A_∞-algebras we have seen, and which we now recall. We use the notation $H^\bullet(\Omega, k)$ for Hochschild cohomology and $(\Omega^!)^{\mathrm{op}} \cong H^\bullet(E^\bullet(\Omega, k))$ for the Yoneda Ext-algebra, which are canonically identified via $H^\bullet(\Psi)$ according to Proposition 3.1.4.

Corollary 3.1.5. *Let Ψ, Φ, $C^\bullet(\Omega, k)$, $E^\bullet(\Omega, k)$ be as in Proposition 3.1.4. Choose a homotopy retract (i', p', h') between $E^\bullet(\Omega, k)$ and $(\Omega^!)^{\mathrm{op}}$. These choices produce, via explicit formulas,*

- *a compatible homotopy retract (i, p, h) of $C^\bullet(\Omega, k)$ by Lemma 3.1.3.*
- *(f_H, m_H) (resp. $(f^{\mathrm{op}}, m^{\mathrm{op}})$) be the minimal model structure induced by (i, p, h) (resp. (i', p', h')) according to the formulas of Proposition 2.2.2.*
- *$g^{\mathrm{op}} : E^\bullet(\Omega, k) \to ((\Omega^!)^{\mathrm{op}}, m^{\mathrm{op}})$ be the left inverse to f^{op} given by Proposition 2.2.5.*

In addition, the isomorphism of graded algebras $H^*(\Psi) : H^\bullet(\Omega, k) \xrightarrow{\sim} (\Omega^!)^{\mathrm{op}}$ extends to an isomorphism of A_∞-algebras determined by

$$\Upsilon : (H^\bullet(\Omega, k), m_H) \xrightarrow{\sim} ((\Omega^!)^{\mathrm{op}}, m^{\mathrm{op}}),$$

given by

$$\Upsilon = g^{\mathrm{op}} \circ \Psi \circ f_H$$

Proof. In view of the formulas, we may let $\iota = (\iota_n)_{n \geq 1} : (H^\bullet(\Omega, k), m_H) \to ((\Omega^!)^{\mathrm{op}}, m^{\mathrm{op}})$ be determined by $\iota_1 = H^*(\Psi)$ and $\iota_n = 0$ for $n \geq 2$. $\square$

Now we combine the foregoing corollary with the presentation of Ω in terms of cohomological data given in Theorem 2.5.1.

Corollary 3.1.6. *Choose a homotopy retract (i', p', h') between $E^\bullet(\Omega, k)$ and $(\Omega^!)^{\mathrm{op}}$. This choice induces a presentation of Ω in terms of $(\Omega^!, m)$ and other data induced by (i', p', h') enumerated in Corollary 3.1.5. The presentation is given by*

$$\Omega \xrightarrow{\sim} \mathcal{A}(\mathrm{Bar}^*((\Omega^!)^{\mathrm{op}}, m^{\mathrm{op}})) = \frac{\hat{T}_k \Sigma((\Omega^!)^1)^*}{(m^{\mathrm{op}*}(\Sigma((\Omega^!)^2)^*)}$$

$$\rho^u : x \mapsto \bar{x} + \sum_{i=1}^{\infty} (\underline{e} \mapsto ((f \circ \Upsilon^{-1})_i(\underline{e}))(x)),$$

where $\underline{e}$ is a generic element of $(\Sigma(\Omega^!)^1)^{\otimes i}$ and $x \mapsto \bar{x}$ denotes reduction modulo the unique maximal ideal of Ω.

The meaning of $(\underline{e} \mapsto ((f \circ \Upsilon^{-1})_i(\underline{e}))(x))$ is just as explained after Theorem 2.5.1, keeping in mind that Υ^{-1} is an A_∞-isomorphism $((\Omega^!)^{\mathrm{op}}, m^{\mathrm{op}}) \xrightarrow{\sim} (H^\bullet(\Omega, k), m_h)$. Note also that $(\Omega^!)^i = \mathrm{Ext}^i_\Omega(k,k)$.

Proof. This is a combination of Corollary 3.1.5 and Theorem 2.5.1. □

3.2. **Question (a): Characterizing the Iwasawa algebra with A_∞-products.** We prove that the A_∞-enrichment of the Yoneda algebra of Ω characterizes Ω up to isomorphism.

Theorem 3.2.1. *The isomorphism class of the A_∞-algebra $(\Omega^!, m)$ determines Ω up to isomorphism.*

Proof. We see in Corollary 3.1.6 that when m is determined by a homotopy retract between $(\Omega^!, 0)$ and $E^\bullet(\Omega, k)$, then the classical hull of its dual bar construction admits an isomorphism from Ω. Because the isomorphism class of the k-algebra $\mathcal{A}(\mathrm{Bar}^*(A, m))$ does not depend on the choice of a A_∞-algebra (A, m) within its minimal isomorphism class, we have the theorem. □

3.3. **Question (b): Formal A_∞-algebra model and abelianness.** A minimal A_∞-structure m is called *formal* when $m_n = 0$ for $n \geq 3$.

Theorem 3.3.1. *If the A_∞-algebra structure m enriching $\Omega^! \cong \bigwedge \mathfrak{g}^*$ has trivial higher multiplications $m_n, n \geq 3$, then G is abelian. In particular, $\Omega \simeq k[\![x_1, \ldots, x_n]\!]$, where n is the k-dimension of $\mathrm{Ext}^1_\Omega(k,k)$.*

The converse to this theorem was proven in [Sor20, Thm. 1.2].

Proof. Because G injects into the units of the completed group algebra Ω, it suffices to prove that Ω is commutative.

When $(\Omega^!, m)$ is trivial, then, by [Sor20, Thm. 1.2], it is A_∞-isomorphic to the Iwasawa algebra $\Omega(\mathbb{Z}_p^d)$ of $\mathbb{Z}_p^d$, where $d = \dim G$. Then, by Theorem 3.2.1, we know that $\Omega \simeq \Omega(\mathbb{Z}_p^d)$, so Ω is commutative. □

Remark 3.3.2. The author thanks Claus Sorensen for suggesting the efficient proof above upon seeing an earlier version of this paper. For the purpose of illustrating what calculations lie below the result, the following more explicit argument still may be instructive as to the role of the A_∞-structures.

Alternate proof of Theorem 3.3.1. By Corollary 3.1.6, we have a presentation for Ω in terms of $\mathcal{A}(\mathrm{Bar}^*((\Omega^!)^{\mathrm{op}}, m^{\mathrm{op}}))$, where $m_n^{\mathrm{op}} = 0$ for $n = 1$ or $n \geq 3$, and m_2^{op} is given by the isomorphism $\Omega^! \cong \bigwedge \mathfrak{g}^*$ of (1.1.1). Recall from §2.3 that the expression $m^{\mathrm{op}*}$ determining $\mathcal{A}(\mathrm{Bar}^*((\Omega^!)^{\mathrm{op}}, m^{\mathrm{op}}))$ in Corollary 3.1.6 is the product over n of the suspended linear duals $m_n^{\mathrm{op}*} : \Sigma\mathrm{Ext}^2_\Omega(k,k)^* \to (\Sigma\mathrm{Ext}^1_\Omega(k,k)^*)^{\otimes n}$ of the A_∞-structure maps $m_n^{\mathrm{op}} : \mathrm{Ext}^1_\Omega(k,k)^{\otimes n} \to \mathrm{Ext}^2_\Omega(k,k)$. Thus we are only concerned with the degree 2 contribution

$$m_2^{\mathrm{op}*} : \Sigma\mathrm{Ext}^2_\Omega(k,k)^* \to (\Sigma\mathrm{Ext}^1_\Omega(k,k)^*)^{\otimes 2}.$$

To calculate $m_2^{\mathrm{op}*}$, we note that the isomorphism $\Omega^! \cong \bigwedge \mathfrak{g}^*$ supplies a canonical isomorphism

$$\wedge^2 \mathrm{Ext}^1_\Omega(k,k) \xrightarrow{\sim} \mathrm{Ext}^2_\Omega(k,k)$$

such that the multiplication m_2^{op} in the graded algebra $(\Omega^!)^{\mathrm{op}}$, restricted to $\mathrm{Ext}^1_\Omega(k,k)$, is the composition of this map with the standard projection

$$\mathrm{Ext}^1_\Omega(k,k)^{\otimes 2} \twoheadrightarrow \wedge^2 \mathrm{Ext}^1_\Omega(k,k).$$

Therefore, the image of $m_2^{\mathrm{op}*}$ in $(\Sigma\mathrm{Ext}^1_\Omega(k,k)^*)^{\otimes 2}$ is precisely the alternating tensor subspace. This completes this alternate proof of Theorem 3.3.1. $\square$

3.4. **Question (c): Change of group.** In this section, we work with an open subgroup $G' \subset G$. Correspondingly, we write $\Omega(G), \Omega(G')$ for their Iwasawa algebras. And for all objects discussed in previous sections with respect to G, we use their "primed version" with respect to G', e.g. f' instead of f.

Theorem 3.4.1. *Let $G' \subset G$ be an open subgroup. Then there is a morphism of A_∞-algebras $(\Omega(G)^!, m) \to (\Omega(G')^!, m')$ compatible with the restriction map $\mathrm{Mod}(\Omega(G)) \to \mathrm{Mod}(\Omega(G'))$, where "compatible" means that we have associated this map of A_∞-algebras to the right-hand vertical arrow in this diagram of presentation maps*

$$\begin{tikzcd} \Omega(G') \arrow[r, "\sim"] \arrow[d] & \frac{\hat{T}_k \Sigma\mathrm{Ext}^1_{\Omega(G')}(k,k)^*}{(m^*((\Sigma\mathrm{Ext}^2_{\Omega(G')}(k,k)^*))} \arrow[d] \\ \Omega(G) \arrow[r, "\sim"] & \frac{\hat{T}_k \Sigma\mathrm{Ext}^1_{\Omega(G)}(k,k)^*}{(m^*((\Sigma\mathrm{Ext}^2_{\Omega(G)}(k,k)^*))} \end{tikzcd} \tag{3.4.2}$$

and that the diagram commutes ***up to inner automorphism*** *in $\Omega(G')$.*

Because the diagram commutes up to inner automorphism, it induces the map of module categories required by question (c).

The proof relies upon using Hochschild cohomology to produce the diagram above, and then applying the isomorphism Υ of Corollary 3.1.5 at the end.

Proof. Choose two (independent) homotopy retracts as in Corollary 3.1.5, one for objects associated to G, and one for objects associated to G'. This results in the objects enumerated there, which we will now use. In addition, we require a left inverse g'_H to f'_H, as in Proposition 2.2.5.

We link the objects associated to G to those associated to G' by via the natural map of Hochschild cochains $C^\bullet(G,k) \to C^\bullet(G',k)$ induced by restricting functions of $G^{\times i}$ to its subgroup $G'^{\times i}$. We have a morphism of A_∞-algebras $H_G \to H_H$ resulting from the composite

$$\eta_H : H^\bullet(G,k) \xrightarrow{f_H} C^\bullet(G,k) \xrightarrow{\text{restr.}} C^\bullet(G',k) \xrightarrow{g'_H} H^\bullet(G',k). \tag{3.4.3}$$

Subsequently, we produce $\eta : ((\Omega(G)^!)^{\mathrm{op}}, m^{\mathrm{op}}) \to ((\Omega(G')^!)^{\mathrm{op}}, m'^{\mathrm{op}})$ by $\eta := \Upsilon \circ \eta_H \circ \Upsilon^{-1}$. This is the opposite A_∞-morphism to the desired morphism in the statement of the theorem.

This morphism η is compatible with the natural restriction map of quasi-compact module categories $\mathrm{Mod}(\Omega(G)) \to \mathrm{Mod}(\Omega(G'))$ because – we claim – (3.4.2) commutes up to inner automorphism by the domain $\Omega(G')$. This claim of commutativity follows from the following facts:

- the right-hand downward map in (3.4.2) is induced by η, simply by applying the (functorial) dual-bar construction and the classical hull construction (described in §§2.3-2.4) to η
- the A_∞-quasi-isomorphisms

$$f_H : H^\bullet(G,k) \xrightarrow{\sim} C^\bullet(G,k), \quad f'_H : H^\bullet(G',k) \xrightarrow{\sim} C^\bullet(G',k)$$

  are used to produce the presentations appearing as the horizontal pair of arrows in the theorem statement, via the formula of Theorem 2.5.1.

Thus the clockwise map in (3.4.2) corresponds to $f'_H \circ \eta_H : H^\bullet(G,k) \to C^\bullet(G',k)$, while the counter-clockwise map in (3.4.2) corresponds to the composition of the leftmost two maps of (3.4.3), which we now denote by $t_H := (\text{restr.}) \circ f_H$. Expressed in terms of t_H, the two presentation maps correspond to $f'_H \circ g'_H \circ t_H$ and t_H, respectively. From [WE18, Thm. 6.2.3], we know that f'_H and $f'_H \circ g'_H \circ f'_H$ result in a pair of isomorphisms

$$\Omega(G') \xrightarrow{\quad\quad\sim\quad\quad} \frac{\hat{T}_k \Sigma \mathrm{Ext}^1_{\Omega(G')}(k,k)^*}{(m^*((\Sigma \mathrm{Ext}^2_{\Omega(G')}(k,k)^*))},$$

as in the top horizontal arrow of (3.4.3), that differ by an inner automorphism of $\Omega(G')$. $\square$

References

[CL19] J. Chuang and A. Lazarev. On the perturbation algebra. *J. Algebra*, 519:130–148, 2019.

[Har16] Michael Harris. Speculations on the mod p representation theory of p-adic groups. *Ann. Fac. Sci. Toulouse Math. (6)*, 25(2-3):403–418, 2016.

[Kad82] T. V. Kadeishvili. The algebraic structure in the homology of an $A(\infty)$-algebra. *Soobshch. Akad. Nauk Gruzin. SSR*, 108(2):249–252 (1983), 1982.

[Kel01] Bernhard Keller. Introduction to A-infinity algebras and modules. *Homology Homotopy Appl.*, 3(1):1–35, 2001.

[Kel02] Bernhard Keller. A-infinity algebras in representation theory. In *Representations of algebra. Vol. I, II*, pages 74–86. Beijing Norm. Univ. Press, Beijing, 2002.

[Kel06] Bernhard Keller. A-infinity algebras, modules and functor categories. In *Trends in representation theory of algebras and related topics*, volume 406 of *Contemp. Math.*, pages 67–93. Amer. Math. Soc., Providence, RI, 2006.

[KS00] Maxim Kontsevich and Yan Soibelman. Deformations of algebras over operads and the Deligne conjecture. In *Conférence Moshé Flato 1999, Vol. I (Dijon)*, volume 21 of *Math. Phys. Stud.*, pages 255–307. Kluwer Acad. Publ., Dordrecht, 2000.

[Laz65] Michel Lazard. Groupes analytiques p-adiques. *Inst. Hautes Études Sci. Publ. Math.*, (26):389–603, 1965.

[LPWZ09] D.-M. Lu, J. H. Palmieri, Q.-S. Wu, and J. J. Zhang. A-infinity structure on Ext-algebras. *J. Pure Appl. Algebra*, 213(11):2017–2037, 2009.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.

[Mer99] S. A. Merkulov. Strong homotopy algebras of a Kähler manifold. *Internat. Math. Res. Notices*, (3):153–164, 1999.

[Pos11] Leonid Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Mem. Amer. Math. Soc.*, 212(996):vi+133, 2011.

[Pos17] Leonid Positselski. Koszulity of cohomology = $K(\pi,1)$-ness + quasi-formality. *J. Algebra*, 483:188–229, 2017.

[Sch11] Peter Schneider. *p-adic Lie groups*, volume 344 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[Sch15] Peter Schneider. Smooth representations and Hecke modules in characteristic p. *Pacific J. Math.*, 279(1-2):447–464, 2015.

[Seg08] Ed Segal. The A_∞ deformation theory of a point and the derived categories of local Calabi-Yaus. *J. Algebra*, 320(8):3232–3268, 2008.

[Sor20] Claus Sorensen. Koszul duality for Iwasawa algebras modulo p. *Represent. Theory*, 24:151–177, 2020.

[WE18] Carl Wang-Erickson. Deformations of residually reducible Galois representations via A_∞-algebra structure on Galois cohomology. arXiv:1809:02484v2 [math.NT], 2018.

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA
E-mail address: `carl.wang-erickson@pitt.edu`