Margin Requirements and Equilibrium Asset Prices

Daniele Coen-Pirani*

Graduate School of Industrial Administration, Carnegie Mellon University,
Pittsburgh, PA 15213-3890, USA

Abstract

This paper studies the effect of margin requirements on asset prices and trading volume in a general equilibrium asset pricing model where Epstein-Zin investors differ in their degree of risk aversion. Under the assumptions of unit intertemporal elasticity of substitution and zero net supply of riskless assets, I show analytically that binding margin requirements do not affect stock prices. This result stands in contrast to previous partial equilibrium analysis where fixed margin requirements increase the volatility of stock prices. In this framework, binding margin requirements induce a fall in the riskless rate, increase its volatility, and increase stock trading volume.

Keywords: Margin Requirements, General Equilibrium, Asset Prices, Stock Trading Volume, Volatility

JEL Classification: G11; G12; G18

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1 Introduction

A variety of recent papers has argued that leverage constraints such as margin requirements in the cash or futures market for stock might increase, rather than stabilize, the volatility of stock prices.\(^1\) Chowdry and Nanda (1998) show how fixed margin requirements might induce rational changes in stock prices even when economic fundamentals do not change. Aiyagari and Gertler (1999) and Mendoza and Smith (2000) argue that binding margin requirements might induce stock prices to “overreact” to changes in economic fundamentals.

While these results have been obtained in the context of different models, the economic mechanism behind them is the same. A negative shock to stock prices reduces the value of the collateral pledged by levered investors. A sufficiently large fall in stock prices forces levered investors to liquidate stock in order to meet margin calls. Therefore, the price of stock has to fall even more in order to induce more risk-averse investors to absorb the excess supply of risky assets.

In this paper I argue that this result relies crucially on these models’ assumption that the riskless rate is exogenously fixed.\(^2\) Since the only price that is allowed to adjust when margin requirements are binding is the stock price, these models might fail to properly account for the effects produced by these constraints. The purpose of this paper is not to claim that margin requirements do not play an important role in exacerbating the effect of exogenous shocks on stock prices. It claims, instead, that when the riskless rate is allowed to adjust in response to shocks, “overreaction” of stock prices will occur only in some well defined circumstances, which have not been emphasized by the literature mentioned above: i) a sufficiently large positive net supply of riskless assets; ii) a relatively high intertemporal elasticity of substitution in consumption.

To illustrate these points, I introduce a general equilibrium model where two types of Epstein-Zin investors, heterogeneous in their degree of risk aversion, trade in stock and a riskless asset.

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\(^1\) The empirical evidence does not support the view that in the U.S. official stock margin requirements have significantly contributed to diminish the monthly volatility of stock returns. See Schwert (1989) and Hsieh and Miller (1990). A major limitation of empirical studies is the fact that since 1934 the Federal Reserve Board has modified them only 22 times.

\(^2\) Chowdry and Nanda (1998) assume the existence of a storage technology with given return. Aiyagari and Gertler (1999) impose a market clearing condition for the riskless asset, but, since they assume that the household does not face any costs of adjusting riskless debt in its portfolio, the determination of the riskless rate in their model is the same as in the representative-agent Lucas (1978) model. Mendoza and Smith (2000) assume an exogenous riskless rate because they model a small open economy.
in order to share aggregate dividend risk. In equilibrium, less risk-averse investors buy stock on margin and a margin requirement sets a limit to their use of leverage. Under the assumptions that investors have the same unit elasticity of intertemporal substitution and that the net supply of riskless assets is zero, I show analytically that only the price of the riskless asset adjusts when margin requirements are binding. Under these assumptions the price of stock is unaffected by margin requirements and, in contrast to the partial equilibrium models mentioned above, does not “overreact” nor displays multiple equilibria.

The intuition for this result is as follows: the market-clearing conditions for the stock and the riskless asset jointly imply that changes in the stock price-dividend ratio can only be due to changes in the aggregate propensity to save or to changes in the value of the net supply of riskless assets. Since the latter is, by assumption, equal to zero, margin requirements can affect the price of stock only by inducing a change in the aggregate propensity to save. By restricting leverage, margin requirements set an upper bound to the ratio between the value of the stock an investor buys and the value of his wealth. With homothetic preferences and only capital income, consumption-savings decisions can be separated from portfolio decisions, and are therefore not directly affected by this portfolio constraint. A unit elasticity of substitution implies that an investor’s consumption decision is independent of the risk-corrected mean return on his portfolio. Therefore, when margin requirements bind, the investors’ savings decisions do not change and the price of stock is not affected.

In this model, when constrained levered investors are forced to sell stock, the rate of return on the riskless asset has to fall to induce more risk-averse investors to absorb its excess supply. Consequently, binding margin requirements increase the volatility of the return on the riskless asset as well as stock trading volume. Thus, also in this economy, even if stock prices are unaffected, margin requirements contribute to the instability of asset prices.

I also consider two extensions of the basic model to examine how binding margin requirements affect stock prices when the riskless asset is in positive net supply or the intertemporal elasticity of substitution is different from one.

First, I allow for a positive net supply of riskless assets by introducing government bonds in the model. In this case, aggregate wealth in the economy is equal to the value of stock plus the value of government bonds. With a unit intertemporal elasticity of substitution, the market clearing
condition for consumption implies that aggregate wealth is constant over time. It follows that, when margin requirements bind and the value of government-supplied bonds increases, the price of stock must fall proportionally more than dividends.

Second, I argue that in this general equilibrium model, in the empirically plausible case where the elasticity of intertemporal substitution is less than one, binding margin requirements increase, rather than decrease, the price of stock. When less risk-averse investors become constrained, in fact, the risk-adjusted mean return on their portfolio decreases. If the elasticity of intertemporal substitution is less than one, their propensity to save out of current wealth increases, inducing a higher price of stock. Thus, even when the assumptions of zero net supply of riskless assets and unit intertemporal elasticity are relaxed, the general equilibrium analysis undertaken in this paper suggests a more complex relationship between asset prices and margin requirements than the one implied by a partial equilibrium approach.

The mechanism that induces a fall in stock prices when government bonds are in positive net supply is similar to the one emphasized by Kiyotaki and Moore (1997). They consider a production economy with two types of agents that are heterogeneous in terms of the production technologies they operate. More productive agents borrow from less productive ones to buy the production input, land. The latter plays the double role of input in production and collateral for borrowing. They show how an unanticipated low productivity shock can cause a downward spiral in land prices, exacerbated by the binding collateral constraint. The key to their result is the fact that there is more than one asset in positive net supply in the economy: land and the technology of less productive agents. As explained in more detail in section 5.1, in their model the “overreaction” of land prices to a low productivity shock is the counterpart of the “underreaction” of the value of the technology used by less productive agents to the same shock. Similarly in my model, stock prices fall more than dividends when margin requirements are binding only if the value of government bonds increases.3

Another related paper is Detemple and Murthy (1997) who study the effect of portfolio constraints on the relationship between asset prices and investors’ intertemporal marginal rates of

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3 An important difference between this paper and Kiyotaki and Moore (1997) is that while they focus exclusively on one-time unanticipated shocks, the dividend shocks considered here do not occur with zero probability. Thus, investors are aware of the fact that dividend shocks can occur, and can plan their consumption and portfolio decisions accordingly.
substitution. They consider an economy where logarithmic expected utility agents trade assets because of different beliefs about the aggregate dividend process. The intuition about the relationship between stock prices and margin requirements provided at the beginning of this introduction also applies to their setup where portfolio constraints do not affect equity prices. An important difference between their paper and this one is that they consider a finite-horizon economy. Consequently, they do not address the issue of whether portfolio constraints will ever be binding in a stationary equilibrium.\footnote{This is the case in Aiyagari and Gertler (1999) where margin requirements bind only during the transition to a stationary equilibrium and, over time, investors move towards non-levered positions.} In the model of this paper, instead, the long-run distribution of wealth across agents is non-degenerate and margin requirements are occasionally binding in a stationary equilibrium.

A less directly related literature is the one on asset pricing with incomplete markets (see Heaton and Lucas (1996) and their references). These papers typically analyze general equilibrium economies where financial markets are incomplete and agents face uninsurable labor income risk. This paper differs from the ones in this literature in two important dimensions. First, there is no idiosyncratic risk in the model of this paper, because the literature cited at the beginning has mainly emphasized the role of margin requirements in exacerbating the effect of aggregate risk on asset prices. Second, the literature on incomplete markets and asset prices focuses on a different kind of borrowing constraint than the one considered here. In those models, investors face fixed short-selling constraints on both riskless and risky assets. In particular, the amount that can be shorted does not depend on the endogenous variables of the model, such as the price of risky assets. With a margin requirement, instead, the maximum amount that an investor can short-sell is proportional to his current wealth, and is therefore state-dependent.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 considers an individual investor’s portfolio and savings problems. Section 4 states and proves the main result and discusses the effects of margin requirements on investors’ portfolios, the volatility of riskless returns, and trading volume. Section 5 discusses the robustness of this result when the assumptions of unit intertemporal elasticity and zero net supply of riskless assets are relaxed, and in the presence of multiple stocks. Section 6 concludes. The appendix contains proofs of the propositions and a description of the numerical algorithm used to solve the model.
2 The Model

In this section I introduce a discrete time dynamic general equilibrium model where two types of investors trade in stock and riskless assets in order to share aggregate endowment uncertainty. The two types differ in their degree of risk aversion. A margin requirement limits the amount of leverage an investor can use when purchasing or short selling stock.

Preferences

The economy is populated by two types of investors denoted by \( i = 1, 2 \). The measure of each type is normalized to one. An investor’s preferences are represented by an Epstein-Zin recursive utility function (Epstein and Zin (1989)). In contrast to the standard expected utility preferences, this specification allows for the independent parameterization of attitudes toward risk and intertemporal substitutability. I denote by \( U_{it} \) investor \( i \)'s utility from time \( t \) onward. This is defined recursively as\(^5\)

\[
U_{it} = \left[ (1 - \beta) C_{it}^{\rho} + \beta (E_t [U_{it+1}^{\gamma_i}])^{\frac{\rho}{\gamma_i}} \right]^\frac{1}{\rho}, \text{ if } 0 < \rho < 1 \tag{1}
\]

\[
= \exp \left[ (1 - \beta) \log C_{it} + \frac{\beta}{\gamma_i} \log E_t [U_{it+1}^{\gamma_i}] \right], \text{ if } \rho = 0. \tag{2}
\]

Current utility is obtained by aggregating current consumption, \( C_{it} \), and the certainty equivalent of random future utility, computed using the information available at time \( t \).

Both types of investors are assumed to discount the future at the same rate \( \beta \in (0, 1) \) and to have the same attitude toward intertemporal substitutability. The intertemporal elasticity of substitution between current consumption and the certainty equivalent of future utility is \( (1 - \rho)^{-1} \). The parameter \( \gamma_i < 1 \) measures the degree of risk aversion of an investor of type \( i \), with a higher \( \gamma_i \) denoting lower risk aversion. I make the following assumption regarding investors’ types:

**Assumption 1.** Investors differ only according to their degree of risk aversion. By convention investors of type 1 are characterized by the highest degree of risk aversion, i.e., \( \gamma_1 < \gamma_2 \).

Aggregate Endowment

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\(^5\)The expression for the utility function when \( \rho = 0 \) (unit intertemporal elasticity of substitution) is easily obtained using de L’Hospital’s rule to compute the limit of \( (U_{it}^\rho - 1) / \rho \) for \( \rho \to 0 \). The case \( \gamma_i = 0 \) can be similarly handled.
The aggregate endowment, in each period, consists of a perishable dividend $D_t$ that evolves according to the process

$$D_{t+1} = \lambda_{t+1} D_t. \quad (3)$$

The growth rate $\lambda_{t+1}$ satisfies the following assumption:

**Assumption 2.** The growth rate of the dividend follows an i.i.d. two-state Markov process $\lambda_{t+1} \in \{\lambda_l, \lambda_h\}$, with $\lambda_l < \lambda_h$ and $Pr(\lambda_{t+1} = \lambda_l) = Pr(\lambda_{t+1} = \lambda_h) = 0.5$.

The assumption of statistical independence of dividend growth across periods allows me to highlight the endogenous mechanisms of the model since it implies that all changes in the conditional moments of asset returns are due to the dynamics of the distribution of wealth among investors.

**Markets and Assets**

At each point in time there exists a spot market for the consumption good, which I take to be the numeraire, a market for a riskless asset (“bond”), and a market for a risky asset (“stock”). A share of the stock pays the dividend $D_t$ at time $t$, while the bond trades for one period, with each unit paying one unit of the consumption good. I denote by $B_{it}$ the quantity of bonds held by investor $i$ at the beginning of time $t$ and by $q_t$ their price in terms of the consumption good. For convenience, I also define $b_{it+1}$ to be the ratio $B_{it+1}/D_t$.

I denote by $s_{it}$ the share of the stock held by investor $i$ at the beginning of time $t$ and by $P_t$ the ex-dividend price of one share in terms of the consumption good. I also denote by $p_t = P_t/D_t$ the stock price-dividend ratio. The total number of shares outstanding is normalized to one.\(^6\)

**Government**

The main role of the government in this economy is to supply one-period riskless bonds. Let $B^g_t$ denote the quantity of outstanding government bonds at the beginning of time $t$. Defining $b^g_{t+1}$ to be the ratio $B^g_{t+1}/D_t$, the government budget reads

$$q_t b^g_{t+1} = \frac{b^g_t}{\lambda_t} - \tau_t \quad (4)$$

\(^6\)It is simple to introduce futures contracts in the model, along the lines of Aiyagari and Gertler (1999) and Chowdry and Nanda (1998). In this version of the model, if margin requirements in the futures market are lower than the ones in the cash market, investors would leverage only by trading in futures contracts. It is easy to show that the model with futures contracts is formally analogous to the model of this section.
where $\tau_t \equiv T_t / D_t$ denotes the government’s tax revenue standardized by the aggregate dividend. For convenience, I assume that the government follows the policy of keeping the outside supply of riskless bonds proportional to the aggregate dividend:

$$b^g_t = b^g \quad \text{for } t = 0, 1, 2..., $$

for some $b^g > 0$.\(^7\) Substituting $b^g$ for $b^g_t$ and $b^g_{t+1}$ in (4) it is easy to see that government revenue must adjust in the following way:

$$\tau_t = b^g \left( \frac{1}{\lambda_t} - q_t \right).$$

Finally, I assume that the government raises revenue by means of a proportional tax on dividends, so that $\tau_t$ can be interpreted as the dividend tax rate at time $t$.\(^8\)

**Investors’ Budget Constraints**

Both types of investors maximize their time-zero utility $U_{i0}$ subject to the sequence of budget constraints

$$(1 - \tau_t + p_t) s_{it} + \frac{b_{it}}{\lambda_t} = c_{it} + q_t b_{it+1} + p_t s_{it+1} \quad \text{for } t = 0, 1, 2...$$

(6)

where $c_{it}$ denotes the ratio $C_{it} / D_t$. Without loss of generality, I assume that at time zero $s_{i0} \in [0, 1]$ and $b_{i0} = 0$ for $i = 1, 2$.

**Margin Requirements**

Each investor is subject to the following margin requirement

$$(1 - \tau_t + p_t) s_{it} + \frac{b_{it}}{\lambda_t} - c_{it} \geq \kappa p_t |s_{it+1}| \quad \text{for } t = 0, 1, 2...$$

(7)

for $\kappa \in [0, 1]$. This equation states that an investor must finance a fraction $\kappa$ of his stock purchases (or short sales) out of his own savings. Using equations (6) and (7) it is easy to see that the borrowing limit for an investor that buys or sells stock is proportional to his savings. Formally, for

\(^7\)This assumption guarantees that the supply of government bonds does not become a state variable of the model, which would significantly complicate the analysis.

\(^8\)I ignore taxation of interest income because it can be shown that in this setting it does not affect the price of equity.
$s_{it+1} > 0$, we have

$$-q_t b_{it+1} \leq \left( \frac{1 - \kappa}{\kappa} \right) \left( (1 - \tau_t + p_t) s_{it} + \frac{b_{it}}{\lambda_t} - c_{it} \right),$$

while for $s_{it+1} < 0$ the constraint implies that

$$-p_t s_{it+1} \leq \frac{1}{\kappa} \left( (1 - \tau_t + p_t) s_{it} + \frac{b_{it}}{\lambda_t} - c_{it} \right).$$

The parameter $\kappa$ is exogenously given. I assume, with Aiyagari and Gertler (1999), that it is set by a government agency, like the Federal Reserve in the U.S.\(^9\) Notice that the borrowing constraint (7) differs from the one commonly used in the asset pricing literature with incomplete markets, according to which the amount an investor can borrow is constant over time (see, e.g., Heaton and Lucas (1996)). A margin requirement, instead, implies that this amount depends positively on his savings and therefore on his wealth. As an investor’s wealth changes endogenously over time, so does his borrowing constraint.

**Recursive Equilibrium**

In this economy aggregate demand for assets and the consumption good depends, in general, on the way aggregate wealth is distributed among investors, and so do bonds and stock prices. Let $\Omega_t$ denote aggregate wealth at a given point in time. This can be obtained by adding the investors’ budget constraints (6), imposing the stock and bonds market-clearing conditions, and taking into account the government’s budget constraint (5), to obtain:

$$\Omega_t \equiv (1 - \tau_t + p_t) + \frac{b_{it}^\alpha}{\lambda_t} = 1 + p_t + q_t b_{it}^\alpha.$$

Denote by $\Gamma \in [0, 1]$ the fraction of aggregate wealth held by investors of type 1. Since there are only two types of investors, $\Gamma$ summarizes the distribution of wealth in the economy. I also denote by $\Gamma' = H (\Gamma, \lambda')$ the law of motion for the distribution of wealth. The distribution of wealth tomorrow depends on the distribution of wealth today and next period’s realization of the endowment shock, denoted by $\lambda'$. Let $\omega$ represent individual wealth appropriately standardized by the aggregate dividend $D$.

\(^9\)Some authors, e.g., Schwert (1989), have pointed out that in the U.S. the Federal Reserve might have changed margin requirements in response to changes in stock prices. In this paper, I ignore this channel and concentrate on the effect of exogenous changes in margin requirements on asset prices.
Exploiting the homotheticity of Epstein-Zin preferences, the investors’ problem can then be reformulated in the following recursive form\(^{10}\)

\[
V_i (\omega, \Gamma) = \max_{c,b,s} \left[ (1 - \beta) c^\alpha + \beta \left( E \left[ (\lambda')^{\gamma_i} V_i (\omega', \Gamma') \right] \right)^{\frac{1}{\gamma_i}} \right]^{\rho}
\]

s.t.
\[
\omega \geq c + q (\Gamma) b' + p (\Gamma) s' \\
\omega' = (1 - \tau (\Gamma', \lambda') + p (\Gamma')) s' + \frac{b'}{\lambda'} \\
\omega - c \geq \kappa p (\Gamma) |s'| \\
\Gamma' = H (\Gamma, \lambda'),
\]

where here and in the rest of the paper \(E [.]\) denotes an expectation with respect to the distribution of \(\lambda'\).

I now proceed to define a recursive equilibrium for this economy.

**Definition 1.** A **Recursive Equilibrium** for this economy is represented by

- a law of motion \(H^* (\Gamma, \lambda')\), a tax function \(\tau^* (\Gamma, \lambda)\), pricing functions \(p^* (\Gamma)\) and \(q^* (\Gamma)\);

- the investors’ value functions \(\{V_i^* (\omega, \Gamma)\}_{i=1,2}\);

- the investors’ decision rules \(\{c_i^* (\omega, \Gamma)\}_{i=1,2}, \{b_i^{s'} (\omega, \Gamma)\}_{i=1,2}, \{s_i^{s'} (\omega, \Gamma)\}_{i=1,2}\);

such that:

- \(\{V_i^* (\omega, \Gamma)\}_{i=1,2}, \{c_i^* (\omega, \Gamma)\}_{i=1,2}, \{b_i^{s'} (\omega, \Gamma)\}_{i=1,2}, \{s_i^{s'} (\omega, \Gamma)\}_{i=1,2}\) solve problem (8) for given pricing functions \(q^* (\Gamma), p^* (\Gamma), \tau^* (\Gamma, \lambda)\), and law of motion \(H^* (\Gamma, \lambda')\);

- the markets for the consumption good, the riskless asset and the stock clear:

\[
\begin{align*}
&c_1^* (\Gamma \Omega^* (\Gamma), \Gamma) + c_2^* ((1 - \Gamma) \Omega^* (\Gamma), \Gamma) = 1 \\
b_1^* (\Gamma \Omega^* (\Gamma), \Gamma) + b_2^* ((1 - \Gamma) \Omega^* (\Gamma), \Gamma) = b^g \\
s_1^{s'} (\Gamma \Omega^* (\Gamma), \Gamma) + s_2^{s'} ((1 - \Gamma) \Omega^* (\Gamma), \Gamma) = 1;
\end{align*}
\]

\(^{10}\)In the unit intertemporal elasticity case \(\rho = 0\) equation (8) needs to be appropriately modified along the lines of equation (2).
• the tax rate adjusts to keep the supply of government bonds constant at $b^g$:

$$\tau^* (\Gamma, \lambda) = b^g \left( \frac{1}{\lambda} - q(\Gamma) \right);$$

(10)

• $H^* (\Gamma, \lambda')$ is consistent with equilibrium:

$$H^* (\Gamma, \lambda') = s_i^* (\Omega^* (\Gamma), \Gamma) \left( 1 - \frac{b^g}{\lambda' \Omega^* (H^* (\Gamma, \lambda'))} \right) + \frac{b^* (\Omega^* (\Gamma), \Gamma)}{\lambda' \Omega^* (H^* (\Gamma, \lambda'))}. \quad (11)$$

The following sections are devoted to the characterization of the recursive equilibrium of this economy.

3 The Investors’ Optimization Problem

In this section I consider the investors’ optimization problem (8), for given arbitrary pricing and tax functions, and law of motion of the wealth distribution.

First, by the homotheticity of preferences and the linearity of both the budget constraint and the margin requirement, it follows that the value function for both investors and the decision rules for consumption, bond and stock holdings can be written as the product of individual wealth $\omega$ and a term involving only the distribution of wealth $\Gamma$. For example, the value function and the decision rule for consumption take respectively the form $V_i (\omega, \Gamma) = \omega v_i (\Gamma)$ and $c_i (\omega, \Gamma) = \omega h_i (\Gamma)$, for some positive functions $v_i (.)$ and $h_i (.)$.

Second, the optimization problem in (8) can be recast in a consumption-savings problem and a portfolio decision problem. To do this, denote the fraction of an investor’s savings invested in the stock by $x \equiv p(\Gamma) s'/ (\omega - c)$, and the rate of return on his portfolio by

$$R (x, \Gamma, \lambda') \equiv x R_s (\Gamma, \lambda') + (1 - x) R_b (\Gamma),$$

where the after-tax rates of return on stock and on the riskless asset are, respectively,

$$R_s (\Gamma, \lambda') \equiv \frac{(1 - \tau (\Gamma', \lambda')) D' + P (\Gamma', D')}{P (\Gamma, D)} = \frac{1 - \tau (\Gamma', \lambda') + p (\Gamma')}{p (\Gamma)} \lambda'$$

$$R_b (\Gamma) \equiv \frac{1}{q (\Gamma)}.$$
An investor’s next period wealth can then be re-written as \( \omega' = (\omega - c) R(x, \Gamma, \lambda') / \lambda' \). Using the definition of \( x \), the margin requirement (9) amounts to the following portfolio constraint:

\[
|x| \leq \frac{1}{\kappa}. \tag{12}
\]

Taking into account the linearity of the value function and of the decision rules in \( \omega \), the recursive problem (8) can be reformulated as follows:

\[
v_i(\Gamma) = \max_{h,x} \left[ (1 - \beta) h^\rho + \beta (1 - h)^\rho \left( E \left[ v_i(\Gamma')^{\gamma_i} R(x, \Gamma, \lambda')^{\gamma_i} \right] \right)^{\frac{1}{\gamma_i}} \right]^{\frac{1}{\rho}} \tag{13}
\]

s.t.

\[
|x| \leq \frac{1}{\kappa}
\]

\[
\Gamma' = H(\Gamma, \lambda')
\]

Notice that \( v_i(\Gamma) \) represents the marginal utility of wealth in state \( \Gamma \) for investor \( i \). As it is clear from equation (13), the portfolio-choice problem can be analyzed independently from the consumption-savings decision.
Consumption-Savings Decision

Define the risk-corrected mean return on investor $i$’s portfolio to be

$$\mu_i^*(\Gamma) \equiv \max_x \left( E \left[ v_i \left( H(\Gamma, \lambda') \right)^{\gamma_i} R \left( x, \Gamma, \lambda' \right)^{\gamma_i} \right] \right)^{1 \over \gamma_i} \text{ s.t. } |x| \leq \frac{1}{\kappa}.$$ 

Then, given $\mu_i^*(\Gamma)$, it is easy to see that in state $\Gamma$ an investor of type $i$ will choose to consume a fraction $h_i^*(\Gamma) \in (0, 1)$ of his wealth where

$$h_i^*(\Gamma) = \frac{1}{1 + \mu_i^*(\Gamma)^{1 \over 1-\beta}} \left( \frac{\beta}{1-\beta} \right)^{1 \over 1-\beta}.$$ (14)

An increase in the risk-corrected mean return on the portfolio induces an income and a substitution effect on current consumption. When $\rho > 0$, the substitution effect prevails and $h_i^*(\Gamma)$ decreases when $\mu_i^*(\Gamma)$ increases. The opposite happens if $\rho < 0$. In the case of unit intertemporal elasticity of substitution, $\rho = 0$, these two effects cancel out and $h_i^*(\Gamma) = 1 - \beta$: consumption is “myopic”, in the sense that does not depend on expected returns.

Equation (14) can be substituted back into problem (13) to yield the following simplified recursive equation ($\rho \neq 0$) in the marginal utility of wealth$^{11}$

$$v_i(\Gamma) = \left( (1-\beta)^{1 \over 1-\beta} + \beta^{1 \over 1-\beta} \mu_i^*(\Gamma)^{1 \over 1-\beta} \right)^{1 \over 1-\rho}.$$ (15)

Portfolio Decision

When the margin constraint is not binding, an investor’s optimal portfolio decision is the share $x_i^a(\Gamma)$ that satisfies the following first order condition.$^{12}$

$$E \left[ v_i \left( \Gamma' \right)^{\gamma_i} R \left( x, \Gamma, \lambda' \right)^{\gamma_i-1} \left( R_s(\Gamma, \lambda') - R_b(\Gamma) \right) \right] = 0.$$ (16)

If the margin requirement is binding, instead, the optimal portfolio decision is simply given

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$^{11}$The case $\rho = 0$ can be handled similarly.

$^{12}$An interior solution to this unconstrained optimization problem is obtained if and only if $R_s(\Gamma, \lambda_k) > R_b(\Gamma) > R_s(\Gamma, \lambda_j)$ where $k = h$ and $j = l$ or viceversa. Notice that, since $\gamma_i < 1$, an investor will never choose a portfolio that yields a non-positive return in some state of the world. Thus the possibility of “bankruptcy” is ruled out. It is easy to check that the second order sufficient condition for a maximum is verified.
Proposition 1. Define

\[ y_i(\lambda') = v_i \left( H(\Gamma, \lambda') \right)^{\frac{\gamma}{\gamma - \gamma_i}} \left( |R_s(\Gamma, \lambda') - R_b(\Gamma)| \right)^{\frac{1}{\gamma - \gamma_i}} \]

\[ z(\lambda') = \begin{cases} 
R_b(\Gamma)^{-1} |R_s(\Gamma, \lambda_l) - R_b(\Gamma)| & \text{if } \lambda' = \lambda_l \\
R_b(\Gamma)^{-1} |R_s(\Gamma, \lambda_h) - R_b(\Gamma)| & \text{if } \lambda' = \lambda_h 
\end{cases} \]

Then,

\[ x_i^u(\Gamma) = \frac{E[y_i(\lambda') - y_i(\lambda_l)]}{E[y_i(\lambda') z(\lambda')]} \text{ for } i = 1, 2 \]

and the optimal weight of the stock in investor i’s portfolio is

\[ x_i^*(\Gamma) = \begin{cases} 
\min \left[ x_i^u(\Gamma), \frac{1}{\kappa} \right] & \text{if } x_i^u(\Gamma) \geq 0 \\
\max \left[ x_i^u(\Gamma), -\frac{1}{\kappa} \right] & \text{if } x_i^u(\Gamma) < 0 
\end{cases} \] (17)

Proof. See appendix A. ■

The solution to the investor’s problem is such that, without changes in investment opportunities over time (i.e., constant \( \Gamma \)), margin requirements would be either binding forever or never binding. This feature of the solution is due to the homotheticity of preferences which makes an investor’s optimal portfolio choice independent of his wealth. Thus, in this environment changes in the distribution of wealth among investors are essential to make margin requirements bind only in some states of the world and not in others. I now turn to the general equilibrium effects of these constraints.

4 Margin Requirements and Equilibrium Asset Prices: the Case of Unit Intertemporal Elasticity and Bonds in Zero Net Supply

In this section I describe the effects of margin requirements on equilibrium asset prices when investors have a unit elasticity of intertemporal substitution, \( \rho = 0 \), and the riskless asset is in zero

\footnote{For a partial equilibrium analysis of portfolio decisions with leverage constraints in a continuous time setting, see Grossman and Vila (1992).}
net supply, $b^g = 0$. Under these circumstances binding margin requirements do not affect stock returns, and the asset market adjusts by means of a reduction of the riskless rate.\textsuperscript{14} The intuition for this result is simple. When $\rho = 0$ savings decisions do not depend on expected returns on investment. Margin requirements only constrain an investor’s portfolio decision, and do not directly affect his consumption-savings choice, as equation (12) shows. Thus, when margin requirements are binding the propensity to save out of current wealth of constrained and unconstrained investors alike is just $\beta$. In equilibrium, if $b^g = 0$, aggregate savings must equal the stock price-dividend ratio. The latter is therefore also unaffected by binding margin requirements. The next proposition states and proves this result.

**Proposition 2.** If $\rho = 0$ and $b^g = 0$, the rate of return on stock is independent of the margin requirement $\kappa$ and is given by

$$R_s^* (\lambda^*) = \frac{\lambda^*}{\beta}. \quad (18)$$

**Proof.** Since $\rho = 0$, it is the case that $c_i^* (\omega, \Gamma) = (1 - \beta) \omega$ for $i = 1, 2$. Substitute the equilibrium consumption of the two investors in the market-clearing condition for the consumption good to get

$$(1 - \beta) (\Gamma + (1 - \Gamma)) \Omega^* (\Gamma) = 1. \quad (19)$$

If $b^g = 0$, aggregate wealth is $\Omega^* (\Gamma) \equiv 1 + p^* (\Gamma)$. Replacing this definition into equation (19) yields the equilibrium price-dividend ratio $p^* = \beta (1 - \beta)^{-1}$, and therefore the equilibrium rate of return on stock (18).

Notice that the result of proposition 2 does not rely on the fact that there are only two types of investors in the economy, nor on the assumption that dividend growth is an i.i.d. two-state process. With $N$, instead of two, types of agents and autocorrelated shocks equation (19) would take the form

$$(1 - \beta) \left( \sum_{i=1}^{N} \Gamma_i \right) (1 + p (\Gamma, \lambda)) = 1, \quad (20)$$

where $\Gamma_i$ denotes the share of aggregate wealth held by investors of type $i$, $\Gamma = (\Gamma_1, \Gamma_2, ..., \Gamma_N)$, and $\sum_{i=1}^{N} \Gamma_i = 1$. The result of proposition 2 would then follow.\textsuperscript{15}

\textsuperscript{14}In section 5 I discuss how this result might change when each of these two assumptions is relaxed.

\textsuperscript{15}Notice that equation (20), and therefore proposition 2, holds for an arbitrary number of dividend growth states $\lambda \in (\lambda_1, \lambda_2, ..., \lambda_L)$. 

15
In general, when margin requirements are binding less risk-averse investors are not able to leverage as much as they would like. In equilibrium, asset prices have to change in such a way as to induce more risk-averse investors to hold more stock in their portfolios than in the frictionless economy. In this economy this portfolio reallocation is achieved by means of a lower riskless rate. Under the assumption that \( \gamma_1 = 0 \), it is possible to characterize analytically the investors’ portfolio decisions as well as the equilibrium riskless rate in states of the world where the margin requirement is binding.

**Proposition 3.** If \( \rho = \gamma_1 = 0 \), in states of the world where the margin requirement is binding, i.e. \( x_2^* (\Gamma) \geq 1/\kappa \), the equilibrium riskless rate \( R^*_b (\Gamma) \) is the strictly positive solution to the following quadratic equation in \( R_b (\Gamma) \):

\[
\Gamma x_1^* (\Gamma) + \frac{1 - \Gamma}{\kappa} = 1
\]

where

\[
x_1^* (\Gamma) = \frac{\beta R_b (\Gamma) \left( E \left[ \lambda' \right] - \beta R_b (\Gamma) \right)}{(\beta R_b (\Gamma) - \lambda_l) (\lambda_h - \beta R_b (\Gamma))},
\]

(21)

Moreover,

\[
\frac{\partial R^*_b (\Gamma)}{\partial \kappa} < 0.
\]

**Proof.** See appendix B. ■

Similar results hold for \( \gamma_1 \neq 0 \), as shown in the numerical illustration of figures 1, 2 and 3.¹⁶ The first two figures display the equilibrium portfolio choices \( x_i^* (\Gamma) \) of investors of type 1 and 2 as a function of \( \Gamma \) for different values of the margin parameter \( \kappa \).

¹⁶When I solve the model numerically, I let the length of a period to be roughly a month and I set the parameters as follows: \( \lambda_h = 1.012, \lambda_l = 0.9911, \rho = 0, b^y = 0, \beta = 0.9814, \gamma_1 = -2.0, \gamma_2 = 0.5 \). The aggregate dividend is identified with aggregate consumption and the process for \( \lambda' \) parameterized using Mehra and Prescott (1985)’s values. The values for the degrees of risk aversion and the discount factor are common in the literature. Currently, the official stock margin requirements set by the Federal Reserve Board are 50% (initial margin) and 25% (maintenance margin). See appendix D for a description of the numerical algorithm used to solve the model.
Figure 3 represents the equilibrium riskless rate $R^*_b(\Gamma)$. Notice that

$$R^*_b(0) = \frac{E[(\lambda')^{\gamma_2}]}{\beta E[(\lambda')^{\gamma_2-1}]} > R^*_b(1) = \frac{E[(\lambda')^{\gamma_1}]}{\beta E[(\lambda')^{\gamma_1-1}]}.$$ 

When the less risk-averse type of investor holds all the wealth in the economy, the riskless rate must be higher than at the other extreme because his incentives to purchase the stock on margin are stronger.

Figure 3 approximately here.

When margin requirements bind, the volatility of the riskless rate and stock trading volume increase on impact as constrained investors sell some of the stock in their portfolios to more risk-averse ones inducing an extra drop in the riskless rate with respect to the frictionless economy. Figure 4 plots the ratio between the conditional volatility of the riskless rate in the economy with margin requirements and the conditional volatility of the riskless rate in the frictionless economy. The conditional volatility of the riskless rate in the economy with margin parameter $\kappa$ is defined to be:

$$\sigma^*_{b}(\Gamma) \equiv \left(E \left[\left(R^*_b(\Gamma') - E[R^*_b(\Gamma')]\right)^2\right]\right)^{\frac{1}{2}}.$$

Figures 4 and 5 approximately here.

Figure 5 presents an analogous plot for stock trading volume, defined as:

$$v^*_{\kappa}(\Gamma, \lambda') \equiv |s_2(\Gamma') - s_2(\Gamma)|.$$

These figures also show how for value of $\Gamma$ close enough to 1, margin requirements effectively reduce the volatility of the riskless rate and stock trading volume. By reducing leverage, in fact, margin requirements reduce the extent of wealth redistribution among investors in response to dividend shocks. Wealth redistribution, in turn, drives both changes in riskless returns and stock trading over time. The closer $\Gamma$ is to 1 the more constrained are less risk-averse investors, and the less wealth redistribution takes place in the economy. This effect tends to reduce the volatility of the riskless rate and stock trading volume.

$^{17}$Since $v(\Gamma, \lambda_0)$ is very similar to $v(\Gamma, \lambda_0)$ I only report the former.
The analysis has, thus far, concentrated on the properties of the recursive equilibrium of the economy. In some models, such as Aiyagari and Gertler (1999)’s, the levered investor accumulates over time enough wealth that margin requirements stop binding in a finite number of periods. The model economy of this paper, instead, admits a stationary recursive equilibrium where margin requirements are occasionally binding.\footnote{For more details on the long-run distribution of wealth in economies populated by heterogeneous Epstein-Zin agents, see Coen-Pirani (2001).} Figure 6 shows the stationary distribution of wealth when $\kappa = 0.30$ and the other parameters of the model take the values listed in footnote (16).\footnote{To construct this figure I have generated five million observations for $\Gamma$ using the recursion $\Gamma' = H(\Gamma, \lambda')$, starting from $\Gamma_0 = 0.5$ for ten times. Each time I have changed the seed of the random number generator. For each simulation I have computed an histogram and then averaged histograms across simulations. The figure represents these “averaged” histograms.}

Figure 6 approximately here.

This histogram has some interesting features. First of all, notice that the distribution of wealth is never concentrated in the middle of the range $[0,1]$. In states of the world in which $\Gamma$ is close to either 0 or 1, borrowing by less risk-averse investors is relatively small and the economy will tend to remain there longer. Second, it also shows how, for about 50% of the time, this economy behaves approximately like a representative-agent model populated only by less risk-averse investors. However, and contrary to Aiyagari and Gertler (1999), in the stationary equilibrium margin requirements bind with positive probability. The histogram suggests that, for $\kappa = 0.30$, the economy is in states of the world where the margin requirement binds for about 10% of the time.

5 Extensions

The analysis of the effects of margin requirements has thus far proceeded under the assumptions of zero net supply of riskless assets and unit intertemporal elasticity of substitution. The following two sections discuss the qualitative impact of relaxing each of these two assumptions on the result of proposition 2. Relaxing either one of these two assumptions is enough to yield time-varying expected stock returns. In this case the latter are, in principle, affected by margin requirements.\footnote{I do not perform a quantitative analysis of the effects of margin requirements on asset prices when $b^g \neq 0$ and $\rho \neq 0$. When $b^g$ is reasonably calibrated the effects of margin requirements on stock prices are likely to be small, as}
In section 5.1, I prove that if the net supply of riskless assets is positive, the price of stock falls when margin requirements are binding and the riskless rate decreases. However, the mechanism that causes the price of stock to fall is different from the one stressed by the partial equilibrium literature. In section 5.2, I argue that in the empirically plausible case in which the intertemporal elasticity of substitution is less than unity, binding margin requirements tend to decrease, rather than increase, expected stock returns. In section 5.3, I show how the result of proposition 2 extends naturally to the case of multiple stocks.

Thus, the discussion that follows suggests that in general equilibrium the relationship between margin requirements and stock returns is potentially more complex than what the partial equilibrium analysis suggests.

5.1 Government Bonds in Positive Net Supply

Proposition 2 provides sufficient conditions for margin requirements not to have any impact on stock returns. When investors have a unit elasticity of substitution, the optimal policy is to save a fraction \( \beta \) of their current wealth. Thus, when \( \rho = 0 \), the only way margin requirements can affect the price of stock is through their impact on the composition of aggregate wealth. If the riskless asset is in zero net supply, this cannot happen because aggregate wealth is only a function of the stock price-dividend ratio. In this section I consider the empirically plausible case where the net supply of riskless assets is positive, i.e., \( b^g > 0 \). Under these circumstances, even when \( \rho = 0 \), margin requirements have an impact on the price of stock, as the next proposition shows.

**Proposition 4.** When \( \rho = 0 \) and \( b^g \neq 0 \), in equilibrium, the effect of margin requirements on stock returns is related to their effect on riskless returns by the following equation:

\[
\frac{\partial \log R^*_{s} (\Gamma, \lambda')}{\partial \kappa} = - \left( \frac{(1 - \beta) b^g}{\beta R^*_b (\Gamma) - (1 - \beta) b^g} \right) \frac{\partial \log R^*_b (\Gamma)}{\partial \kappa} \tag{22}
\]

Moreover, if \( b^g > 0 \),

\[
\frac{\partial \log R^*_s (\Gamma, \lambda')}{\partial \kappa} \cdot \frac{\partial \log R^*_b (\Gamma)}{\partial \kappa} < 0.
\]

**Proof.** See appendix C. 

argued in section 5.1. In the case \( \rho \neq 0 \), instead, the model becomes very difficult to solve, because both bond and stock prices have to be found numerically.
The proposition shows that when government bonds are introduced in the model, stock returns become a function of the aggregate state $\Gamma$. The intuition is that in this economy aggregate wealth is not only given by the sum of the dividend and the stock price, but also by the value of the net supply of bonds $b^g$. Since aggregate wealth is a constant equal to $(1 - \beta)^{-1}$, when the value of government debt increases, the price of stock has to adjust in the opposite direction. This implies that, when margin requirements bind and the riskless rate falls to accommodate the necessary portfolio reallocation, the value of government bonds increases, and the stock price has to fall.

Notice that, even if the mechanism described above generates a fall in stock prices when margin requirements bind, it is different from the one emphasized by the partial equilibrium literature. There, when margin requirements bind, the stock price has to fall to clear the stock market because by assumption the riskless rate cannot adjust. In this version of the model with $b^g > 0$, the price of stock falls only when the riskless rate falls, as equation (32) in the proof of proposition 4 shows. In general equilibrium, in fact, the market-clearing condition for the consumption good has to hold, in addition to the stock market-clearing condition. The former imposes a restriction, equation (33) below, on the way risky and riskless returns are related. When margin requirements bind and the riskless rate falls, this restriction dictates that the price of stock has to fall as well.

As mentioned in the introduction, this mechanism is related to the one emphasized by Kiyotaki and Moore (1997) (in the following referred to as KM). To see this it is instructive to compare the market clearing condition for the consumption good in my model (equation 32 in the proof of proposition 4):

$$p^*(\Gamma) = \frac{\beta}{1 - \beta} - \frac{b^g}{R^*_g(\Gamma)},$$

with the corresponding equation in KM (1997, page 146, eq. A4):\footnote{In this comparison I am referring to the version of Kiyotaki and Moore’s model contained in their appendix, where agents have the same discount factor and strictly convex preferences. The original equation A4 in Kiyotaki and Moore’s paper reads

$$(1 - \beta \sigma) (Y_t + q_t K + mV_t) = Y_t,$$

where $Y_t$ is aggregate output, $q_t$ the price of land, $K$ the quantity of land, and $mV_t$ the value of the gatherers’ technology. In the text I have normalized $K$ to 1, rearranged the equation and defined $p_{t}^{km} \equiv q_t/Y_t$ and $v_t^{km} = mV_t/Y_t$. I have also used $\beta$ instead of $\beta \sigma$ because in the economy I consider agents are infinitely lived.}

$$p_{t}^{km} = \frac{\beta}{1 - \beta} - v_t^{km},$$
where $p_t^{km}$ and $v_t^{km}$ in KM denote respectively the price of land and the value of the less productive agents’ technology standardized by aggregate output. In KM, a low productivity shock increases the value of the technology used by more productive agents, relative to aggregate output. Since also in KM aggregate wealth is constant (and equal to $(1 - \beta)^{-1}$), the increase in $v_t^{km}$ must induce a fall in the value of land relative to output, i.e., an “overreaction” of the land price $p_t^{km}$. Notice however, that this “overreaction” is tightly linked to the “underreaction” in the price of the other asset $v_t^{km}$. Similarly, in the version of my model with bonds in positive net supply, the fall in the stock price-dividend ratio is due to the increase in the value of government bonds relative to the aggregate endowment.

It is natural to ask by how much the rate of return on stock is going to increase if, when margin requirements are binding, the riskless rate falls by one percentage point. To obtain an idea of the magnitude of these effects, suppose that the model was parameterized at a monthly frequency, with an annual discount factor $\beta = 0.96$ and an annual riskless rate approximately equal to one percent, i.e., $R^*_b(\Gamma) = 1.01$. The average monthly government debt-consumption ratio for the U.S. in the last 40 years has been around $b^\theta = 7$ (see Aiyagari and Gertler (1991, page 325)). This parameterization implies that the coefficient on the right hand side of equation (22) is

$$\frac{(1 - \beta) b^\theta}{\beta R^*_b(\Gamma) - (1 - \beta) b^\theta} \approx 0.02.$$ 

In this case, if the riskless rate falls by one percentage point when margin requirements are binding, stock returns would increase by only two basis points. While the results of this simple exercise must be interpreted cautiously, they nevertheless suggest that to generate a significant impact of binding margin requirements on stock returns, it might be necessary to postulate a relatively large outstanding supply of riskless assets.

### 5.2 Intertemporal Elasticity of Substitution Less Than One

In this section I briefly discuss the consequences of relaxing the assumption of unit intertemporal elasticity of substitution on the relationship between margin requirements and asset returns. Some authors, e.g., Hall (1988), have in fact pointed out that the intertemporal elasticity of substitution in consumption is likely to be closer to zero ($\rho \to -\infty$) than to unity ($\rho = 0$).

To better understand the effect of margin requirements on stock returns when $\rho < 0$, it is
instructive to consider a representative-agent version of the model \((\Gamma = 0 \text{ or } \Gamma = 1)\). In this circumstance, the sign of the parameter \(\rho\) determines the relationship between the risk-adjusted expected rate of return on the investor’s portfolio and equilibrium stock returns. It is easy to show, in fact, that if the investor with risk aversion \(\gamma_i\) holds all the wealth in the economy and \(b^g = 0\) stock returns are given by

\[ R_s^\ast (\lambda') = \frac{\lambda'}{\beta (E [(\lambda')^{\gamma_i}] - \mu_i)}. \]

If \(\rho < 0\), as the risk-corrected mean yield on the investor’s portfolio, \((E [(\lambda')^{\gamma_i}])^{\frac{1}{\gamma_i}}\), decreases, the investor would like to decrease consumption and increase savings because the income effect prevails. In a representative-agent economy, in equilibrium, the only way to save is to buy stock. Since the supply of stock is fixed, an increase in the price of stock, i.e., a decrease in its rate of return, is needed to clear the stock market.

This intuition can help understand what would happen in the economy with two types of investors when the less risk-averse type becomes constrained by the margin requirement. When less risk-averse investors become constrained the risk-corrected mean return on their portfolio, \(\mu_2^\ast (\Gamma)\), decreases. By equation (14), when \(\rho < 0\), their propensity to consume out of current wealth, \(h_2^\ast (\Gamma)\), goes down. These investors react to binding margin requirements by saving more. In this way they partially compensate the effect of margin calls on their stock holdings.\(^{23}\) The additional demand for stock that originates from the decrease in \(h_2^\ast (\Gamma)\) tends to increase the price-dividend ratio and therefore decrease expected stock returns. To summarize, when \(\rho < 0\), binding margin requirements tend to decrease expected stock returns, rather than increase them through a drop in the price of stock as suggested by the partial equilibrium literature.

\[ \text{5.3 Multiple Stocks} \]

In this section I discuss how the result of proposition 2 extends to the case where there are many stocks rather than only one, as assumed so far.\(^ {24}\) Suppose there are \(m = 1, 2, \ldots, M\) stocks in the economy, with the total number of shares of each stock normalized to 1. Let \(P_t^m\) and \(D_t^m\)


\(^{23}\)Notice that the margin requirement constrains only the investor’s portfolio choice and not his total stock holdings.

\(^{24}\)For simplicity, I abstract from government bonds in this section.
respectively denote the ex-dividend price and the dividend paid by stock $m$ at time $t$. I assume that the latter can be specified as:

$$D^m_t = \eta^m_t D_t,$$

where $\eta^m_t$ is a finite state Markov process and $D_t$ is simply the aggregate dividend, i.e.,

$$\sum_{m=1}^{M} \eta^m_t = 1. \tag{23}$$

The evolution of the aggregate dividend $D_t$ follows the stochastic process in equation (3). Standardizing all growing variables by $D_t$ one obtains the following generalized version of the margin requirement (7):

$$\sum_{m=1}^{M} \left( p^m_t + \eta^m_t \right) s^m_{it} + \frac{b_{it}}{\lambda_t} - c_{it} \geq \sum_{m=1}^{M} \kappa^m p^m_t \left| s^m_{it+1} \right|, \tag{24}$$

where $p^m_t = P^m_t / D_t$, $s^m_{it+1}$ are the shares of stock $m$ purchased by investor $i$, and $\kappa_m \in [0, 1]$ denotes the fraction of the value of stock $m$ that the investor must finance using his own savings.

As in the model of section 2, the investors’ problem can then be separated into a consumption-savings decision and a portfolio decision, where the latter now also involves choosing a portfolio of stocks. In the case of unit intertemporal substitution $\rho = 0$, the investors’ propensity to consume out of current wealth is again $1 - \beta$. Since aggregate wealth at time $t$ is given by $\sum_{m=1}^{M} \left( p^m_t + \eta^m_t \right)$, the market clearing condition for the consumption good implies that in equilibrium:

$$\sum_{m=1}^{M} p^m_t = \frac{\beta}{1 - \beta}. \tag{25}$$

This result intuitively generalizes the one obtained in section 4. Equation (25) states that in equilibrium the sum of stock prices in this economy is a constant proportion of the aggregate dividend. Let $R^m_{t,t+1}$ denote the rate of return on stock $m$ between $t$ and $t + 1$:

$$R^m_{t,t+1} = \frac{p^m_{t+1} + \eta^m_{t+1}}{p^m_t} \lambda_{t+1}.$$

Then, equations (23) and (25) imply that the rate of return on the the value-weighted stock market
portfolio in this economy is:

\[
\sum_{m=1}^{M} \frac{p_t^m}{\sum_{m=1}^{M} p_t^m} R_t^m = \frac{\lambda_{t+1}}{\beta}.
\]

Thus, in the case of multiple stocks the result of proposition 2 applies to the rate of return on the value-weighted market portfolio. Specifically, while binding margin requirements might affect the rate of return on individual stocks, the rate of return on the market portfolio is independent of these constraints.

6 Concluding Comments

In this paper, I have argued that the claim, advanced in several theoretical papers, that fixed margin requirements contribute to increase the volatility of stock prices, might be very sensitive to these papers’ assumption that the riskless rate is exogenously given. I have shown, in the context of a simple general equilibrium model with heterogeneous investors, how binding margin requirements might not have any impact on stock returns. In this case the economy adjusts by a decrease in the riskless rate, which therefore becomes more volatile with respect to the frictionless economy.

The results of this paper point to two natural extensions of the analysis. First, the numerical results of section 4 suggest that when margin requirements are binding the volatility of the riskless rate displays a significant increase with respect to the frictionless model. It would be interesting to calibrate the model of this paper in order to fully assess its quantitative implications for the relationship between margin requirements and asset prices. Second, this paper also suggests that margin requirements might have a larger impact on the volatility of the prices of riskless rather than risky assets. However, most applied studies relate margin requirements to measures of stock return volatility. It would be interesting to study empirically the behavior of bond prices at times when stock prices exhibit large drops and margin requirements are more likely to bind. I leave both these explorations to future research.
A Proof of Proposition 1

The starting point of the proof is equation (16). Solving this equation for \( x \) yields the unconstrained optimum \( x_i^u (\Gamma) \). To solve for \( x_i^u (\Gamma) \), notice that equation (16) can be rewritten as:

\[
0.5 v_i (H(\Gamma,\lambda_h))^\gamma_i [R_b (\Gamma) + x (R_s (\Gamma,\lambda_h) - R_b (\Gamma))]^{\gamma_i - 1} (R_s (\Gamma,\lambda_h) - R_b (\Gamma)) = 0.5 v_i (H(\Gamma,\lambda_i))^\gamma_i [R_b (\Gamma) + x (R_s (\Gamma,\lambda_i) - R_b (\Gamma))]^{\gamma_i - 1} (R_b (\Gamma) - R_s (\Gamma,\lambda_i)).
\]

Notice that since I am focusing on an interior solution I assume without loss of generality that

\[
R_s (\Gamma,\lambda_h) > R_b (\Gamma) > R_s (\Gamma,\lambda_i).
\]

Rearrange equation (26) to get:

\[
\frac{R_b (\Gamma) + x (R_s (\Gamma,\lambda_h) - R_b (\Gamma))}{R_b (\Gamma) + x (R_s (\Gamma,\lambda_i) - R_b (\Gamma))} = \left( \frac{v_i (H(\Gamma,\lambda_h))^\gamma_i (R_s (\Gamma,\lambda_h) - R_b (\Gamma))}{v_i (H(\Gamma,\lambda_i))^\gamma_i (R_b (\Gamma) - R_s (\Gamma,\lambda_i))} \right)^\frac{1}{\gamma_i - 1}.
\]

Defining \( \phi_i (\Gamma) \) as

\[
\phi_i (\Gamma) \equiv \left( \frac{v_i (H(\Gamma,\lambda_h))^\gamma_i (R_s (\Gamma,\lambda_h) - R_b (\Gamma))}{v_i (H(\Gamma,\lambda_i))^\gamma_i (R_b (\Gamma) - R_s (\Gamma,\lambda_i))} \right)^\frac{1}{\gamma_i - 1},
\]

equation (27) can be rewritten as

\[
\frac{R_b (\Gamma) + x (R_s (\Gamma,\lambda_h) - R_b (\Gamma))}{R_b (\Gamma) + x (R_s (\Gamma,\lambda_i) - R_b (\Gamma))} = \phi_i (\Gamma).
\]

In turn, this equation can be solved for \( x \), yielding

\[
x = \frac{R_b (\Gamma) (\phi_i (\Gamma) - 1)}{R_s (\Gamma,\lambda_h) - R_b (\Gamma) + \phi_i (\Gamma) (R_b (\Gamma) - R_s (\Gamma,\lambda_i))}.
\]

Substituting back \( \phi_i (\Gamma) \) from (28) and simplifying gives the expression for \( x_i^u (\Gamma) \) in proposition 1:

\[
x_i^u (\Gamma) = \frac{0.5 y_i (\lambda_h) - 0.5 y_i (\lambda_i)}{0.5 y_i (\lambda_h) z (\lambda_h) + 0.5 y_i (\lambda_i) z (\lambda_i)},
\]

where \( y_i (\lambda) \) and \( z (\lambda) \) are defined in the text of the proposition.
The rest of the proof consists of taking into account the constraints on \( x_i^* (\Gamma) \) represented by the margin requirement. That is, whenever \( x_i^u (\Gamma) > 1/\kappa \), I pick \( 1/\kappa \) to be the solution of the constrained optimization problem, and similarly whenever \( x_i^u (\Gamma) < -1/\kappa \), I take \(-1/\kappa \) to be the solution. Hence equation (17) in the text of the paper.

**B Proof of Proposition 3**

If \( \rho = 0 \), by proposition 2, \( E \left[ R^*_b (\lambda') \right] = E \left[ \lambda' \right] / \beta \). Moreover, if \( \gamma_1 = 0 \), proposition 1 implies that \( x_1^* (\Gamma) \) is given by (21). Market clearing in the stock market requires that

\[
\Gamma x_1^* (\Gamma) + (1 - \Gamma) x_2^* (\Gamma) = 1. \tag{29}
\]

When margin requirements are binding \( x_2^* (\Gamma) = \kappa^{-1} \). Substituting \( x_1^* (\Gamma) \) and \( x_2^* (\Gamma) \) into equation (29) yields a quadratic equation in the unknown \( R^*_b (\Gamma) \). This quadratic equation has two solutions, a strictly positive one and a strictly negative one. The strictly positive solution is \( R^*_b (\Gamma) \). To prove that the equilibrium riskless rate is decreasing in \( \kappa \), it is enough to apply the implicit function theorem to equation (29). This yields

\[
\frac{\partial R^*_b (\Gamma)}{\partial \kappa} = \frac{1}{\kappa^2} \left( 1 - \frac{\Gamma}{\Gamma} \right) \left( \frac{\partial x_1^* (\Gamma)}{\partial R_b (\Gamma)} \right)^{-1}. \tag{30}
\]

It is easy to show, from equation (21), that

\[
\frac{\partial x_1^* (\Gamma)}{\partial R_b (\Gamma)} < 0,
\]

which completes the proof.

**C Proof of Proposition 4**

As in proposition 2, since \( \rho = 0 \), it is the case that \( c_i^* (\omega, \Gamma) = (1 - \beta) \omega \) for \( i = 1, 2 \). Substitute the equilibrium consumption of the two investors in the market-clearing condition for the consumption good to get

\[
(1 - \beta) \left( \Gamma + (1 - \Gamma) \right) \Omega^* (\Gamma) = 1. \tag{31}
\]
By definition, \( \Omega^* (\Gamma) = (1 + p^* (\Gamma) + b\theta q^* (\Gamma)) \), with \( b\theta \neq 0 \). Replacing \( \Omega^* (\Gamma) \) into (31) and rearranging this equation yields

\[
p^* (\Gamma) = \frac{\beta}{1 - \beta} - \frac{b\theta}{R^*_b (\Gamma)}.
\]

(32)

To obtain the rate of return on stock, substitute the tax rate (10), \( p^* (\Gamma) \) and \( p^* (\Gamma') \) from (32) in the definition of \( R^*_s (\Gamma, \lambda') \) to obtain

\[
R^*_s (\Gamma, \lambda') = \frac{1 - \tau (\Gamma', \lambda') + p^* (\Gamma')}{p^* (\Gamma)} \lambda' = \frac{R^*_b (\Gamma) \left( \lambda' - b\theta (1 - \beta) \right)}{\beta R^*_b (\Gamma) - b\theta (1 - \beta)}.
\]

(33)

To prove the first claim, take logs of both sides of (33) and differentiate the resulting expression:

\[
\frac{\partial \log R^*_s (\Gamma, \lambda')}{\partial \log R^*_b (\Gamma)} = \frac{-(1 - \beta) b\theta}{\beta R^*_b (\Gamma) - (1 - \beta) b\theta}.
\]

(34)

Last, replace (34) in the following equation:

\[
\frac{\partial \log R^*_s (\Gamma, \lambda')}{\partial \kappa} = \frac{\partial \log R^*_s (\Gamma, \lambda')}{\partial \log R^*_b (\Gamma)} \frac{\partial \log R^*_b (\Gamma)}{\partial \kappa}.
\]

The denominator of equation (34) is always positive, because \( p^* (\Gamma) > 0 \) implies

\[
R^*_b (\Gamma) > \frac{b\theta (1 - \beta)}{\beta}.
\]

Therefore, as long as \( b\theta > 0 \), the numerator of (34) is negative and the derivative of \( \log R^*_s (\Gamma, \lambda') \) with respect to \( \log R^*_b (\Gamma) \) is negative. This proves the second claim.

D Numerical Algorithm

In this appendix I describe the numerical algorithm I have used to solve the model and generate Figures 1-6. Notice that in all these figures \( \rho = 0 \) and \( b\theta = 0 \), so the algorithm does not need to find the equilibrium stock price-dividend ratio \( p (\Gamma) \), which is always equal to \( \beta / (1 - \beta) \), as implied by proposition 2. The algorithm features the following three main steps.

Step 1: Solving the Investors’ Problem

Solving the agents’ problem involves solving the recursive equation (13) in order to obtain the
function \( v_i(\cdot), i = 1, 2 \), for given initial guesses for the functions \( q(\cdot) \) and \( H(\cdot, \cdot) \). Equation (13) can be solved by replacing the optimal decision rules (14) and (17) into the Bellman equation (13) and by iterating on \( v_i(\cdot) \) until convergence. To iterate on (13), I discretize the state space (the unit interval \([0, 1]\)) for \( \Gamma \) and construct a grid \( \{\Gamma_k\}_{k=1}^K \), using \( K = 200 \) evenly spaced grid points, with \( \Gamma_1 = 0 \) and \( \Gamma_{200} = 1 \). Since in general \( \Gamma' = H(\Gamma_k, \lambda') \) does not lie on the grid, I approximate \( v_i(\Gamma') \) using cubic splines which produce a twice continuously differentiable interpolated function (see Press et al. (1996), chapter 3). Convergence of the value function \( v_i(\cdot) \) is reached when the maximum (across grid points) percentage difference between the value function at two successive iterations is smaller than \( 10^{-7} \).

**Step 2: Finding the Market-Clearing Bond Pricing Function**

The second step of the algorithm is to use the bond demand functions \( b'_i(\Gamma_k(1 + p(\Gamma_k)), \Gamma_k) \) produced by Step 1 to update the pricing function \( q(\cdot) \) at the grid points \( \{\Gamma_k\}_{k=1}^K \). Since there are \( K \) grid points this amounts to solving a system of \( K \) equations, corresponding to the market clearing conditions for the riskless asset at each \( \Gamma_k \) on the grid:

\[
\sum_{i=1}^{2} b'_i(\Gamma_k(1 + p(\Gamma_k)), \Gamma_k) = 0, \text{ for } k = 1, 2, \ldots, K.
\]

The \( K \) unknowns in this system of equations are the bond prices \( \{q(\Gamma_k)\}_{k=1}^K \).\textsuperscript{25} To solve this problem I use Broyden’s algorithm which operates in the following way (for a more detailed description, see Press et al. (1996), chapter 9). First, it numerically approximates the Jacobian matrix associated with the non-linear system. It then uses this approximate Jacobian to find an updated vector of prices by implementing the Newton step, which guarantees quadratic convergence if the initial guess is close to the solution. If the Newton step is not “successful”, the algorithm tries a smaller step by backtracking along the Newton dimension. When an acceptable step is determined, prices are updated and the algorithm can proceed in the way described above, once an updated Jacobian has been obtained. Since the numerical computation of the Jacobian can be costly (and in this model it is), the Jacobian at the new prices is iteratively approximated using Broyden’s formula. The non-linear solver stops when the maximum excess demand for bonds divided by the total asset

\textsuperscript{25}Notice that every time the algorithm tries a different price vector, it has to go back to Step 1 and recompute the agents’ decision rules in order to evaluate the excess demands at those prices.
demand (in absolute value) is smaller than $10^{-4}$.

**Step 3: Updating the Law of Motion for the Wealth Distribution**

The third step consists of updating the law of motion for the distribution of wealth $H(\Gamma, \lambda')$ at the grid points $\{\Gamma_k\}_{k=1}^K$. That is, given the law of motion $H_j^j (\Gamma_k, \lambda')$ found at the $j$-th iteration on Step 3, use equation (11) to find $H_{j+1}^j (\Gamma_k, \lambda')$:

$$H_{j+1}^j (\Gamma_k, \lambda') = s_1' (\Gamma_k (1 + p (\Gamma_k)), \Gamma_k) + \frac{b_1' (\Gamma_k (1 + p (\Gamma_k)), \Gamma_k)}{(1 + p (H_j^j (\Gamma_k, \lambda')))} \lambda',$$

where, as noted above, since $\rho = 0$ and $b^g = 0$:

$$p (\Gamma_k) = p (H_j^j (\Gamma_k, \lambda')) = \frac{\beta}{1 - \beta}.$$

After $H_{j+1}^j (\Gamma_k, \lambda')$ has been computed, the algorithm goes back to Step 1 with a new law of motion for the wealth distribution. The algorithm stops when the convergence criteria for Steps 1 and 2 have been satisfied and the maximum distance (across grid-points) between two successive laws of motion for the distribution of wealth is smaller than $10^{-5}$. 
References


Legend for Figure 1:
Margin requirements and the portfolio choice of type 1 investors

Legend for Figure 2:
Margin requirements and the portfolio choice of type 2 investors

Legend for Figure 3:
Margin requirements and asset returns

Legend for Figure 4:
Margin requirements and the volatility of the riskless rate

Legend for Figure 5:
Margin requirements and stock trading volume

Legend for Figure 6:
The stationary distribution of wealth when $\kappa = 0.30$
Figure 1 - Margin requirements and the portfolio choice of type 1 investors
Figure 2 - Margin requirements and the portfolio choice of type 2 investors
Figure 4 - Margin requirements and the volatility of the riskless rate
Figure 5 - Margin requirements and stock trading volume
Figure 6 - The stationary distribution of wealth when $\kappa=0.30$