In *Working Notes of the AAAI 1998 Spring Symposium on Prospects for a Commonsense Theory of Causation*, pages 24-30, Stanford University, Stanford, California, March 23-25, 1998

Reversible Causal Mechanisms in Bayesian Networks

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Abstract

Causal manipulation theorems proposed by Spirtes *et al.* (1993) and Pearl (1995) in the context of directed probabilistic graphs, such as Bayesian networks, do not model so called reversible causal mechanisms, i.e., mechanisms that are capable of working in several directions, depending on which of their variables are manipulated exogenously. An example involving reversible causal mechanisms is the power train of a car: normally the engine moves the transmission which, in turn, moves the wheels; when the car goes down the hill, however, the driver may want to use the power train to slow down the car, i.e., let the wheels move the transmission, which then moves the engine.

Reversible causal mechanisms are modeled quite naturally in the context of equilibrium structural equation models. In this paper, we investigate whether Bayesian networks are capable of representing reversible causal mechanisms. Building on the result of Druzdzel and Simon (1993), which shows that conditional probability tables in Bayesian networks can be viewed as descriptions of causal mechanisms, we study the conditions under which a conditional probability table can represent a reversible causal mechanism.

Introduction

For over a decade, artificial intelligence researchers have tried to incorporate a principled account of causality into existing formalisms for uncertain reasoning, for example in the area of planning (Davidson & Fehling 1994), inferring causal structure from observations (Pearl & Verma 1991; Spirtes, Glymour, & Scheines 1993), explaining causal assumptions in decision analytic models (Heckerman & Shachter 1995), experimental design (Pearl 1995), and in modeling counterfactual reasoning (Balke & Pearl 1995).

An explicit representation of causality is important in artificial intelligence because of several reasons.[1] The foremost of these is that causal models allow intelligent agents to predict the effects of their actions. A causal model frees the agent from the need to store a combinatorially large set of pairs *action* $\Rightarrow$ *effect*. Result of external manipulation on the model variables can be predicted directly from the model. In the context of directed graphical probabilistic models, such as Bayesian networks, two foremost formalizations of manipulation are due to Spirtes *et al.* and Pearl. Spirtes *et al.* (1993) proposed a theorem, known as *causal manipulation theorem*, specifying the effect of imposing externally a value on any node in a graphical model. Pearl (1995), who built his recent work on structural equation models, has an equivalent formalization of manipulation. While these formalizations are very useful and bring much clarity into statistics by making a clear distinction between observation and manipulation, they stop short of modeling reversible causal mechanisms. Both approaches imply that if the effect of an external manipulation of a node n imposes a deterministic value on it (for the sake of simplicity, we will consider only such interventions in this paper), the effect of this manipulation on other nodes in the network can be predicted by removing the arcs from the parents of n to n. This operation, popularly known as *arc cutting semantics*, is based on the premise that all mechanisms in the model represent an asymmetric relationship among their variables. This poses a limitation on modeling reversible causal mechanisms, i.e., mechanisms that are capable of working in several directions, depending on which of their variables are manipulated exogenously. An example of a reversible causal mechanism is the power train of a typical car: when a car goes up the hill, the engine moves the transmission which, in turn, moves the wheels; when the car goes down the hill, the driver may want to use the power train to slow down the car, i.e., let the wheels move the transmission which then moves the engine. The interactions between the engine and the transmission and between the transmission and the wheels are independent of how the power train is actually used. Knowledge of these interactions can be reused whenever these mechanisms are parts of a larger system.

Reversible mechanisms are modeled quite naturally in the context of equilibrium structural equation models, as nothing in a structural equation implies asymmetry among its variables. It is the context of a model that introduces asymmetry among variables. Druzdzel and Simon (1993) related the notion of causality in directed probabilistic graphs to structural equations and structural equation models in econometrics. They showed

[1] (Druzdzel & Simon 1993) list and discuss several important reasons for an explicit representation of causality in intelligent systems.

that it is possible to reproduce the joint probability distribution modeled by a Bayesian network (Pearl 1988) by a system of simultaneous equations in such a way that its independence graph coincides with the causal ordering of the system. They made it quite clear that conditional probability tables in Bayesian networks can be viewed as descriptions of causal mechanisms and, thereby, be equivalent to structural equations. It is natural to ask the question whether they can, similarly to structural equation models, represent reversible mechanisms.

In this paper, we build on the result of Druzdzel and Simon (1993), studying the algebraic conditions under which a conditional probability table can represent a reversible causal mechanism. We demonstrate that representing reversible causal mechanisms in Bayesian networks is possible, although reversibility implies strong constraints on the conditional probability tables.

We will use lower case letters, such as v, to denote variables, and capital letters, such as V, to denote a sets of variables. Furthermore, n_v will denote the number of values of a variable v or, more in general, the number of elements in a finite set. $\pi(v)$ will denote the set of parents of v in the independence graph of a Bayesian network $\mathcal{B}$. The values of a variable will be called *configurations*, the set of all configurations of a variable v denoted by $C(v)$, while an element of this set will be denoted by c_v. A configuration of a set of variables is an assignment of a value to each of them; the set of all configurations of a set of variables V will be written as $C(V)$, one particular configuration is c_V. The *restriction* $c_{|V}$ of a configuration c to a set of variables V is the part of c that assigns values to variables in V.

Structural Equation Models

Systems of simultaneous structural equations, known as *structural equation models* (SEMs) are used throughout science to model so-called *equilibrium systems*, reasonably isolated parts of the real world that are known or assumed to reach a stable state if outside influences on the system remain invariant. Such outside influences are called *exogenous*. Their values are determined outside the system either because we are not willing or are not able to account for their behavior. The values of dependent variables, called also *endogenous* variables, are determined inside the model.

A system of simultaneous equations can be easily transformed into another system that has the same set of solutions, for example by combining two or more equations together. In general, there are infinitely many systems that are extensionally equivalent (that is, have the same set of solutions). However, a system of equations that is a model of a system in the real world usually has one form that has more intuitive appeal because it portrays the causal structure of the real system. In such a model, each equation represents a conceptually distinct causal mechanism (Simon 1953). The concept of a *structural equation* is a semantic one; it is in general impossible to decide syntactically whether an equation is structural or not, though we can obtain strong clues from experimentation and, to a lesser extent, from observation (Wold 1954).

A generic form of an equation that we will use throughout this paper is

$$f(x_1, x_2, \ldots, x_n, \mathcal{E}) = 0\ , \tag{1}$$

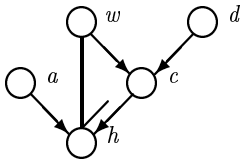
where f is some algebraic function, its arguments x_1, x_2, ..., x_n are various system variables, and $\mathcal{E}$ is an error variable. This form is usually called an *implicit function*. In order to obtain a variable x_i $(1 \leq x_i \leq n)$ as a function of the remaining variables, we must solve the equation (1) for x_i. We say that the function

$$x_i = g(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \mathcal{E}) \tag{2}$$

found in this way is defined *implicitly* by (1) and that the solution of this equation gives us the function explicitly. Often, the solution can be stated explicitly in terms of elementary functions. In other cases, the solution can be obtained in terms of an infinite series or other limiting process; that is, one can approximate (2) as closely as desired.

Example: The following system of equations is our perception of the relationship between heart disease (h), blood cholesterol level (c), wine consumption (w), average age (a), and amount of fat in the diet (d) of a population.

$$\begin{cases} w = c_0 \\ d = c_1 \\ a = c_2 \\ c = \alpha d + \beta w + c_3 \\ h = \gamma w + \delta c + \varepsilon a + c_4 \end{cases}$$



□

The relations among variables in a self-contained model of structural equations are asymmetric. For example, if we increase w, h will be affected; but modifying the last equation will leave w untouched. This is because *heart disease* (h) depends (indirectly) on *wine consumption* (w) but not vice versa. Examining the dependencies captured by equations, and asymmetries that they imply for the model variables, leads to an ordering of the variables that is represented by the graph on the right hand side. Simon (1953) developed an algorithm for explicating this ordering, and argued that, if all equations in the model are structural and all exogenous variables in the model are truly exogenous in the system, the resulting graph can be interpreted causally.

The advantage of a structural form is that it supports prediction of the effects of changes to the system. Such changes are modeled by replacing or modifying those mechanisms that are affected by the intervention. For example, a government campaign to increase wine consumption would be modeled by increasing c_0 while invention of a new medicine to increase resistance to heart disease would be modeled by modifying the last equation. The ability of SEMs to predict the effect of manipulation, or, as it is often called, *change in structure*, has been known in econometrics since the inception of the concept of structural equation models. The seminal work of Simon explicated this property and gave it

a causal interpretation. We will refer to this view of causality as *mechanism based*.

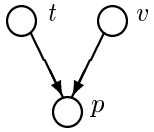
Causal Reversibility

The formalizations proposed by Spirtes *et al.* (1993) and Pearl (1995) assume that imposing a value on a variable renders that variable independent of its direct causes. This means that taking action on a variable can never be relevant to that variable's causes. This may seem evident in situations where there is a strong asymmetric relationship between the variable and its causes; for example, wearing a raincoat protect from getting wet but it does not make the rain go away. But in many areas where a cause and effect variable are more on a par with each other, one cannot be sure the cause variable will not be affected by tampering with the (former) effect variable. For example, acting on the wheels of a car does not normally break their link to transmission; a modern electric train will use its engine as a generator when braking, thereby feeding electricity back to the power lines.

Structural equation models have no difficulty with modeling this type of dependencies. Replacement of one equation by another can in general reverse the direction of causal ordering between variables. To model a decision, or an action, we add an equation that describes the intervention. When the intervention is so strong that it sets the manipulated variable to a specific value, we add an exogenous equation, i.e., an equation that sets a variable to a constant value. After adding this equation, the system is overconstrained and needs to be relaxed by removing another equation, in general not the one that determined the value of the manipulated variable in the original causal graph. Depending on the choice of this relaxing equation, the causal ordering of the modified system can be quite different; in particular, the direction of causation between any pair of variables can be reversed.

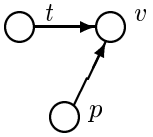
Example: Consider a cylinder filled with ideal gas. The condition of the gas can be described by its temperature (t), pressure (p), and volume (v), and the relationship between these three is given by the perfect gas law $pv/t = C$, where C is a constant. Suppose we set the volume and the temperature externally. This situation is captured by the following system of equations and its corresponding causal graph:

$$\begin{cases} t = c_0 \\ v = c_1 \\ pv/t = c_3 \end{cases}$$



Causal ordering will reveal that both temperature and volume affect pressure. Changing temperature, for example, will affect pressure but not volume. However, if we change the context in which the mechanism that ties temperature, volume, and pressure is embedded, for example by allowing the cylinder to expand so that the pressure will be equal to the outside pressure, the resulting system will be:

$$\begin{cases} t = c_0 \\ p = c_2 \\ pv/t = c_3 \end{cases}$$



In this system, we have replaced the equation that fixed the volume with an equation that fixes the pressure, leaving the mechanism that ties them intact. Now the new volume can be computed by substituting the first two equations in the third and solving for v. The resulting new causal ordering will show that volume has become endogenous, being affected by temperature and pressure. In effect, manipulation has led to reversal of the direction of causation. □

Algebraically necessary condition for causal reversibility is that the resulting model is unambiguous with respect to its solutions. This comes down to the existence of explicit functions for the causal ordering that results from manipulation. The following definitions, and especially the definition of injectivity, formalize the algebraic conditions that are necessary for a structural equation to model a reversible causal mechanism.

Definition 1 (equation system) *An* equation system *is a set $\mathcal{S} = \{e_1, \ldots, e_n\}$ where each e_1 is a relation on a set of discrete finite variables $V(e_i) = \{v_1, \ldots, v_m\}$, $e_i \subseteq C(v_1) \times \cdots \times C(v_m)$. The union of all variables appearing in $\mathcal{S}$ will be denoted by $V(\mathcal{S}) = \bigcup_{e_i \in \mathcal{S}} V(e_i)$.*

Note that an equation can be interpreted simply as the set of configurations that are its solutions. In the sequel, we will take that point of view.

Definition 2 (solution) *A* solution *of an equation system $\mathcal{S}$ is a configuration $c \in C(V(\mathcal{S}))$ such that for each equation $e \in \mathcal{S}$ we have that $c_{|V(e)} \in e$.*

Definition 3 (compatibility) *Configurations c_1 and c_2 are* compatible *if they assign the same values to the variables in their intersection, that is, if for each $v \in V(c_1) \cap V(c_2)$, we have $c_{1,|\{v\}} = c_{2,|\{v\}}$.*

Definition 4 (substitution) *If e is an equation,* substituting *a configuration c into e means removing from e configurations not compatible with c and subsequently removing the variables appearing in c from each configuration in e.*

Substitution is useful for obtaining solutions of a system by recursively substituting configurations of variables that precede an equation in causal ordering. Note that nothing will happen if c and e do not have a variable in common.

In the sequel, please note that the order in which variables are specified is not significant and is exploited for notational convenience only.

Definition 5 (injectivity) [2] *Let e be an equation on $V(e) = \{v_1, \ldots, v_n\}$. e is said to be* injective *with respect to v_1 if for each configuration $c \in C(v_2, \ldots, v_n)$,*

[2] This definition differs from the traditional notion of in-

there is exactly one configuration on $V(e)$ compatible with c.

Injectivity with respect to a variable x means that given an equation and values for all variables but x, we can find a unique value for x to satisfy the equation.

Finally two last elements useful in formalizing the algebraic conditions for causal reversibility.

Definition 6 (partial solution set) *The* partial solution set *of a complete subset C of a self-contained system of equations $\mathcal{S}$ is its solution set if it is a root in causal ordering; otherwise, it is the set of configurations of $C(V(C))$ that simultaneously satisfy the equations in C and the partial solution sets of C's parents in causal ordering over $\mathcal{S}$.*

The following definition formalizes the idea that in order for a system to admit a causal interpretation, it must have a unique solution for each choice of its exogenous variables.

Definition 7 (causal unambiguity) *A system is* causally unambiguous *if the partial solution sets of its complete subsets are singleton for each configuration of its exogenous variables.*

When reordering a manipulated system of equations, in general mechanisms will be reversed. This means the configuration of exogenous and endogenous variables with respect to an equation will change. The following theorem states the implications that Definition 7 has on reversibility; that is, what property equations have to satisfy in order for reordering to render a system causally unambiguous.

Theorem 1 (reversibility) *Let $\mathcal{S}$ be an unambiguous system of equations and $\mathcal{S}'$ be a system of equations that is the result of some intervention on $\mathcal{S}$. $\mathcal{S}'$ is causally unambiguous if each equation $e \in \mathcal{S}'$ is injective with respect to its endogenous variable.*

Proof: For root nodes, the theorem follows trivially since injectivity of an exogenous equation means it has a unique solution so it has a singleton partial solution.

Assume an equation e on $V(e) = v_1, \ldots, v_n$ is injective with respect to its endogenous variable v_1, and assume its parent partial solution sets are singleton. It follows that there is only one configuration $c_{2,\ldots,n} \in C(v_2, \ldots, v_n)$ compatible with the parent partial solutions.

According to Definition 5, there is exactly one configuration $c_{1,\ldots,n}$ compatible with $c_{2,\ldots,n}$ and thus the partial solution is unique. The theorem now follows by induction. □

Theorem 1 is merely a reformulation of the constraint put on reordering in the previous section. It says that, in order for reordering after an intervention to yield an unambiguous system, equations need to be expressed as an explicit function from the exogenous variables to the endogenous variable.

jectivity (e.g., (Rudeanu 1974)) in that it implies that e represents a function that is complete on its domain.

Bayesian Networks

Informally, the graph of a Bayesian network (Pearl 1988) encodes qualitative knowledge about relevance relationships in a domain. Its nodes represent discrete, finite stochastic variables and contain numerical information about the probabilities of their values conditional on the values of their parents in the graph. By exploiting the graph as a means of specifying the conditional independencies that must hold between sets of nodes, a Bayesian network can be used to economically represent a joint probability distribution on the nodes of the graph. Efficient algorithms exist for absorbing evidence about the values of nodes into the network and updating marginal probabilities accordingly. Bayesian networks can thus be used for probabilistic inference.

In what follows, we will consider Bayesian networks with discrete variables that can take on any finite number of values.

Definition 8 (Bayesian network) *A Bayesian network $\mathcal{B}$ is a tuple (G, F) where*

1. *$G = (V(G), A(G))$ is an acyclic directed graph with vertices $V(G) = \{v_1, \ldots, v_n\}, n \geq 1$, and arcs $A(G)$, and*
2. *$F = \{f_v | v \in V(G)\}$ is a set of real-valued nonnegative functions $f_v : C(v) \times C(\pi(v)) \rightarrow [0, 1]$, called* conditional probability assessment functions*, such that for each configuration $c_{\pi(v)}$ of $\pi(v)$, we have*

$$\sum_{c_v \in C(v)} f_v(c_v, c_{\pi(v)}) = 1 \tag{3}$$

A joint probability distribution can also be represented by a system of simultaneous equations. Druzdzel and Simon (1993) were able to reproduce the joint distribution over a Bayesian network by representing each conditional probability table by an equation with error terms, and have subsequently proven that causal ordering of this system yields a graph that is isomorphic to the independence graph of the Bayesian network. Interpreting the independence graph as a causal model then amounts to assuming that the corresponding equations are structural.

The representation of probability tables as equations used in (Druzdzel & Simon 1993) was for the sake of simplicity of exposition limited to Boolean variables. To allow for a convenient and compact representation of conditional probabilities of multiple-valued variables, we introduce the concept of a probability matrix.

Definition 9 (probability matrix) *Let $\mathcal{B}$ be a Bayesian network (G, F) on variables $V = v_1, \ldots, v_n$, and let $v \in V$. The* probability matrix *M_v of v is a two-dimensional matrix, where columns are indexed by the configurations $C(\pi(v))$ of parents of v while the rows are indexed by $C(V)$, the values of v.*

Each cell of M_v is assigned an interval of $[0, 1]$ in such a way that

1. *all intervals in a column of M_v are mutually exclusive,*

	v_1	**true**		**false**	
	v_2	**true**	**false**	**true**	**false**
v_3	**true**	0.9	0.8	0.7	0.1
	false	0.1	0.2	0.3	0.9

	v_1	**true**		**false**	
	v_2	**true**	**false**	**true**	**false**
v_3	**true**	$[0, 0.9]$	$[0, 0.8]$	$[0, 0.7]$	$[0, 0.1]$
	false	$(0.9, 1]$	$(0.8, 1]$	$(0.7, 1]$	$(0.1, 1]$

Figure 1: An example of a probability table $\Pr(v_3|v_1v_2)$ (upper table) and its corresponding probability matrix (lower table).

2. *the union of all intervals in a column is equal to* $[0, 1]$, *and*
3. *the length of each interval is equal to the corresponding conditional probability.*

In other words, the intervals in a column of the matrix partition the interval [0..1] in such a way that we have, for each configuration $c_v \in C(v)$ and $c_{\pi(v)} \in C(\pi(v))$

$$\Pr(\epsilon \in M_v(c_v, c_{\pi(v)})) = f_v(c_v, c_{\pi(v)}) \; . \tag{4}$$

Example: The only difference between a probability table and a probability matrix is the fact that in the latter, probabilities have been replaced by subsets of the intervals $[0, 1]$. Figure 1 shows an example of a probability table and its corresponding probability matrix. The cells of the latter are assigned intervals in one possible way to satisfy the criteria of Definition 9. Note that an interval in a column need not be contiguous and that neither the upper cell need to include 0 nor the lowest cell need to include 1. □

The following theorem, a sightly different framing of a theorem proposed by Druzdzel and Simon (1993), describes how the joint probability distribution defined by a Bayesian network can be represented by a system of simultaneous equations with latent variables.

Theorem 2 (representability) *Let $\mathcal{B} = (G, F)$ be a Bayesian network on nodes $V = v_1, \ldots, v_n$ as in Definition 8, that defines a probability distribution $\Pr_{\mathcal{B}}$. There exists a system of simultaneous equations $\mathcal{S}$ on V that defines a distribution $\Pr_{\mathcal{S}}$ such that $\Pr_{\mathcal{B}}(c) = \Pr_{\mathcal{S}}(c)$ for all $c \in C(V)$.*

Proof: (Sketch) The proof is by demonstrating a procedure for constructing $\mathcal{S}$. In a Bayesian network, the probability of a configuration $c \in C(V)$ is simply the product of the probabilities for each node v of its configuration c_v conditional on the configuration of its parents, $c_{\pi(v)}$. Therefore, it suffices to replicate the conditional probabilities for each node given its parents.

We will construct one equation for each of the variables. Each equation will include an independent, continuous latent variable ϵ_v, uniformly distributed over the interval $[0, 1]$. Note that $\forall x (0 < x \leq 1) \Pr(\epsilon_v \leq x) = x$.

We first consider a node v without predecessors. Its probability table contains prior probabilities over its configurations $C(v)$. The following deterministic function reproduces these priors:

$$f_v(\epsilon_v) = c \quad \text{if} \quad \epsilon_v \in M_v(c) \quad \text{for all} \quad c \in C(v) \tag{5}$$

A node v that does have predecessors is treated analogously, except that the function that determines the value of v takes as arguments not just the latent variable but v's parents $\pi(v)$ as well:

$$f_v(c_{\pi(v)}, \epsilon_v) = c_v \quad \text{if} \quad \epsilon_v \in M_v(c_v, c_{\pi(v)}) \tag{6}$$
$$\text{for all} \quad c_{\pi(v)} \in C(\pi(v)) \quad \text{and} \quad c_v \in C(v)$$

□

From the way the system of simultaneous equations corresponding to a Bayesian network is constructed, it follows immediately that the causal ordering of the system is equivalent to the independence graph of the Bayesian network. This allows us to use the causality framework developed for structural equation models in interpreting the structure of Bayesian networks.

Restructuring Bayesian Networks

Mechanism-based approach in the context of structural equation models supports true structural changes that can impact the system in such a way that the causal graph changes completely. Knowledge encoded in terms of conditional probability distributions in a directed probabilistic graph, such as a Bayesian network, on the other hand, seems to be valid only in the particular context, in one mode of operation (e.g., going up the hill or going down the hill in the car power train example) in which the system is studied.

A necessary condition for supporting true structural changes in the framework of directed probabilistic graphs, such as Bayesian networks, is their ability to represent reversible causal mechanisms. In order to be able to represent reversible mechanisms, the conditional probability tables have to fulfill algebraic conditions analogue to injectivity in equation-based systems.

We will illustrate this point and the possibility of restructuring a probabilistic graph with an example. Consider the following system of equations:

$$\begin{cases} x = x_0 \\ f(x, y, \mathcal{E}) = 0 \end{cases} \tag{7}$$

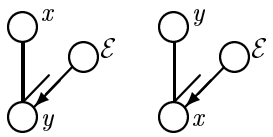
From the system (7), we can easily derive the conditional probability $\Pr(y|x)$. To accomplish this, we need to replace x in f by x_0 and derive the explicit function for y in terms of $\mathcal{E}$ (note that $\mathcal{E}$, a random variable, is expressed in terms of a probability distribution).

Let us now imagine that we want to model the effect of acting on y rather than on x without disturbing the mechanism that ties x and y. The new system of structural equations will take the following form:

$$\begin{cases} y = y_0 \\ f(x, y, \mathcal{E}) = 0 \end{cases} \tag{8}$$

Please, note that only the first equation in (7) had to be replaced, as the second equation describes a mechanism that is unaffected by our intervention. If the function f is injective with respect to x, we can derive the conditional probability $\Pr(x|y)$ from it. To accomplish this, we need to replace y in f by y_0 and derive the explicit function for x in terms of $\mathcal{E}$. We assume that the intervention has left the probability distribution of $\mathcal{E}$ unaffected and, therefore, known as a part of knowledge about the mechanism. This assumption is implied by the assumption that f described a mechanism and $\mathcal{E}$ was exogenous and independent of any other exogenous variable. Note that all that was needed to derive the conditional probability distributions $\Pr(y|x)$ or $\Pr(x|y)$ was the functional form f of the equation binding x, y and $\mathcal{E}$ and the probability distribution of the error variable $\mathcal{E}$.

The causal ordering applied to (7) and (8) yields the graphs of Figure 2 (a) and (b) respectively. Imagine a



(a) (b)

Figure 2: Causal ordering for the system (7) (a) and the system (8) (b).

transition from the network (a) to the network (b) in Figure 2 that is similar to the transition between the systems (7) and (8). As proposed in (Druzdzel & Simon 1993), a node in a causal Bayesian network and its direct predecessors can be seen as a mechanism and the conditional probability distribution of this node on its direct predecessors can be seen as a description of the mechanism when the functional form is unknown. From this point of view, construction of the network (b) given that we have a fully quantified network (a) and the fact that the mechanism binding x, y, and $\mathcal{E}$ is reversible, resembles a change in structure in a structural equation model. Under what algebraic constraints on the conditional probability table $\Pr(y|x)$ would we be able to use $\Pr(y|x)$ to derive the conditional probability distribution $\Pr(x|y)$ of the new graph?

Reversible Mechanisms in Bayesian Networks

We will consider a conditional probability table with associated equation and investigate what it means to choose a different endogenous variable. The lower table in Figure 1 gives an example in the form of a probability matrix as defined in Definition 9.

Recall from the proof of Theorem 2 that this table can be translated into the following equation

$$f_{v_3}(c_{\{v_1,v_2\}}, \epsilon_{v_3}) = c_{v_3}$$
$$\text{if} \quad \epsilon_{v_3} \in M_{v_3}(c_{v_3}, c_{\{v_1,v_2\}})$$

	v_2	**true**	**false**	**maybe**
v_1	**true**	$[0, 0.3]$	$(0.4, 0.6]$	$(0.3, 0.4] \cup (0.6, 1]$
	false	$(0.3, 0.5]$	$(0.6, 1]$	$[0, 0.3] \cup (0.5, 0.6]$
	maybe	$(0.5, 1]$	$[0, 0.4]$	$(0.4, 0.5]$

Table 1: A probability matrix that is sound with respect to both variables v_1 and v_2 appearing in it.

	v_2	**true**	**false**	**maybe**
v_1	**true**	0.3	0.2	0.5
	false	0.2	0.4	0.4
	maybe	0.5	0.4	0.1

Table 2: A "doubly normalized" probability table associated with Table 1.

where ϵ_{v_3} is the error term associated with v_3. Now if, by some intervention, v_1 becomes the endogenous variable and v_3 exogenous, then we will have to compute a new probability table in which v_1 is the dependent variable. This corresponds to calculating the explicit form f'_{v_3} of f_{v_3} for v_1. Unfortunately, this explicit form does not exist because f_{v_3} is not injective with respect to v_1 which is shown for example by existence of the tuples $f_{v_3}(v_1 = \textbf{true}, v_2 = \textbf{true}, \epsilon_{v_3} = 0.5) =< v_3 = \textbf{true} >$ and $f_{v_3}(v_1 = \textbf{false}, v_2 = \textbf{true}, \epsilon_{v_3} = 0.5) =< v_3 = \textbf{true} >$.

The following definition provides us with a concept that will express the sufficient and necessary condition for reversibility of a causal mechanism captured by a Bayesian network.

Definition 10 (soundness) *Let e be an equation on $v_1, \ldots, v_n$ in an equation system $\mathcal{S}$ associated with a Bayesian network $\mathcal{B}$ where v_1 is the endogenous variable. The probability matrix M_{v_1} is* sound *with respect to v_2 if after transposing the table with respect to v_2 it is still a probability matrix.*

Note that a probability matrix corresponding to a conditional probability table $\Pr(x|\pi(x))$ is by definition sound with respect to x. Table 1 shows an example of a probability matrix that is sound with respect to both variables appearing in it. Consistency of a probability matrix with respect to more than one variable seems like a stern condition because intervals cannot overlap with intervals in the same column as well as in the same row; however, a probability table in which the rows of an exogenous variable v_i add up to 1 can always be translated into a matrix consistent with respect to v_i. Table 2 shows such a "doubly normalized" probability table that can be translated into the probability matrix shown in Table 1.

The following theorem proves that soundness is a necessary and sufficient condition for reversibility.

Theorem 3 (reversibility in Bayesian networks) *Let $\mathcal{B}$ be a Bayesian network and $\mathcal{S}$ its corresponding equation system. An equation $e \in \mathcal{S}$ on variables*

$V(e) = \{v_1, \ldots, v_n\}$ *is injective with respect to a variable* v_1 *if and only if* e*'s probability matrix* M_e *is sound with respect to* v_1.

Proof: $\Rightarrow$ Assume e is injective with respect to v_1. Now suppose M_e is not sound with respect to v_1. This means that if we transpose M_e with respect to v_1, there must be an $\epsilon \in [0..1]$ such that either (1) a column exists in which ϵ is an element of two cells, or (2) a column exists in which ϵ is not an element of any cell.

In case of (1), we can conclude from the way equations are created in Theorem 2 that there must be a configuration $c \in C(\{v_2, \ldots, v_n\})$ and configurations $c_{v_1,1}$ and $c_{v_1,2}$ in $C(\{v_1\})$ such that both $< c_{v_1,1}, c, \epsilon > \in e$ and $< c_{v_1,2}, c, \epsilon > \in e$ which makes e non-injective and thus we have a contradiction.

In case of (2), there must be a configuration $c \in C(\{v_1, \ldots, v_n\})$ for which there is no tuple $< c, \epsilon > \in e$ again contradicting injectivity.

$\Leftarrow$ Assume M_e is sound with respect to v_1. Now if we transpose M_e with respect to v_1, again from the way equations are constructed, there must be exactly one tuple for each combination of a configuration for the remaining variables and an $\epsilon \in [0..1]$. □

Soundness is a condition equivalent to injectivity. When a conditional probability table is "doubly normalized," it can be restructured. Suppose $\mathcal{B}$ is a Bayesian network with node v on which we wish to impose an intervention. We can do this by translating $\mathcal{B}$ to a system $\mathcal{S}$ and adding an equation fixing v to one of its values, subsequently relaxing $\mathcal{S}$ by choosing a suitable equation on a path from v to one of its roots in causal ordering over $\mathcal{S}$, to yield $\mathcal{S}'$. Suitability of this equation depends on semantic reversibility of those equations that are reversed when restructuring $\mathcal{S}'$. In order to be able to translate $\mathcal{S}'$ back to a manipulated network $\mathcal{B}'$, the probability tables of $\mathcal{B}$ should be sound with respect to their endogenous variables in $\mathcal{B}'$. The probability tables of $\mathcal{B}'$ can then be calculated by transposing those of $\mathcal{B}$.

Conclusion

Causal manipulation theorems proposed by Spirtes *et al.* (1993) and Pearl (1995) in the context of directed probabilistic graphs, such as Bayesian networks, do not model so called reversible causal mechanisms, i.e., mechanisms that are capable of working in several directions, depending on which of their variables are manipulated exogenously. Reversible mechanisms are modeled quite naturally in the context of equilibrium structural equation models. In this paper, we addressed the question whether Bayesian networks are capable of representing reversible causal mechanisms. Building on the result of Druzdzel and Simon (1993), which shows that conditional probability tables in Bayesian networks can be viewed as descriptions of causal mechanisms, we demonstrated that representing reversible causal mechanisms in Bayesian networks is possible, although this implies strong constraints on the conditional probability tables.

Acknowledgments This research was supported by the Air Force Office of Scientific Research under grant number F49620–97–1–0225 to University of Pittsburgh, and by the National Science Foundation under Faculty Early Career Development (CAREER) Program, grant IRI–9624629, to Dr. Druzdzel. While we are taking full responsibility for any errors in this paper, we would like to thank Herb Simon for enlightening discussions of the topic of causality.

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