Chapter 1. Equations of a line:

(a) **Standard Form:** $\boxed{A\,y + B\,x = C}$.

(b) **Point-slope Form:** $\boxed{y - y_0 = m\,(x - x_0)}$, where m is the slope and (x_0, y_0) is a point on the line.

(c) **Slope-intercept:** $\boxed{y = m\,x + b}$, where m is the slope and b is the y-intercept.

The equation of a **circle** centered at (h, k) and radius r is given by,

$$\boxed{(x - h)^2 + (y - k)^2 = r^2} \; .$$

The **distance** between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by,

$$\boxed{d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \; .$$

The **midpoint** between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by,

$$\boxed{midpt(A, B) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)} \; .$$

Defining the Domain and Range:

Domain: The Domain of a function is the set of (well-defined) x-values, or inputs.

Range: The Range (or Image) of a function is the set of y-values, or outputs, for which there is at least one input x that maps to y.

Parallel and **Perpendicular** lines:

(a) Parallel lines have the *same* slopes, i.e. $\boxed{m_1 = m_2}$.

(b) Perpendicular lines have *negative reciprocal* slopes, i.e.

$$\boxed{m_1\, m_2 = -1 \iff m_1 = -\frac{1}{m_2} \iff m_2 = -\frac{1}{m_1}} \; .$$

Principles for solving inequalities:

Addition Principle:
$$a < b \implies a + c < b + c$$

Multiplication Principles:

$$a < b \text{ and } c > 0 \implies ac < bc$$

$$a < b \text{ and } c < 0 \implies ac > bc \quad \text{(inequality changes direction)}$$

Chapter 2.

The difference quotient or average rate of change:

$$\boxed{\dfrac{f(x+h) - f(x)}{h}}.$$

The composition of two functions f and g is defined as

$$\boxed{(f \circ g)(x) = f(g(x))}.$$

Remark: You should simplify compositions before defining their domain.

Even and **Odd** functions:

(a) **Even** functions are symmetric with respect to the **y-axis** and satisfy

$$\boxed{f(-x) = f(x)}.$$

(b) **Odd** functions are symmetric with respect to the **origin** and satisfy

$$\boxed{f(-x) = -f(x)}.$$

Horizontal (i), (ii) and Vertical (iii), (iv) Translations:

For $\boxed{a > 0}$, we have (i) the graph of $y = f(x + a)$ is a shift of the graph of $f(x)$ a-units to the **left** , and (ii) the graph of $y = f(x - a)$ is a shift of the graph of $f(x)$ a-units to the **right** .

For $\boxed{b > 0}$, we have (iii) the graph of $y = f(x) + b$ is a shift of the graph of $f(x)$ b-units **up** , and (iv) the graph of $y = f(x) - b$ is a shift of the graph of $f(x)$ b-units **down**.

The transformation $f(-x)$ is a **reflection** of the graph of f across the y-axis, and $-f(x)$ is a reflection across the x-axis.

The transformation $f(a\,x)$, with $a > 0$

  (i) makes the graph of f 'stretch' (wrt the x-direction) when $0 < a < 1$.

  (ii) makes the graph of f 'shrink' (wrt the x-direction) when $a > 1$.

The transformation $a\,f(x)$, with $a > 0$

  (i) makes the graph of f 'shrink' (wrt the y-direction) when $0 < a < 1$.

  (ii) makes the graph of f 'stretch' (wrt the y-direction) when $a > 1$.

Direct and Inverse Variation:

(a) **Direct variation:** $f(x) = k\,x$ or $y = k\,x$ where $k > 0$, $x > 0$.

(b) **Inverse variation:** $f(x) = \frac{k}{x}$ or $y = \frac{k}{x}$ where $k > 0$, $x > 0$.

Chapter 3.

Complex Numbers:

The number i is defined as

$$i = \sqrt{-1} \implies i^2 = -1 \ .$$

Any complex number z can be written in the form

$$a + b\,i$$

where a , $b \in \mathbb{R}$. We say the real part of z is a and the imaginary part of z is b , and we write

$$Re\,(z) = a \quad Im\,(z) = b \ , \ \text{respectively.}$$

The **Complex Conjugate** of the complex number $z = a + b\,i$ is

$$\bar{z} = a - b\,i \ .$$

Remark: If $z = a + b\,i$ is a complex number, then the multiplication $z \cdot \bar{z} = \bar{z} \cdot z = a^2 + b^2$, which is a real number! This property of complex numbers is helpful to use when simplifying a given number (e.g. the ratio of two complex numbers) to the form $a + b\,i$.

Quadratic Equations:

Any **quadratic equation** with real coefficients can be written in the form

$$a\,x^2 + b\,x + c = 0 \ . \tag{1}$$

where a , b , c are real numbers.

Any **quadratic function** can be written in the form

$$f(x) = a\,x^2 + b\,x + c \ , \tag{2}$$

Note that when finding the zeros of a quadratic *function*, we obtain a quadratic *equation*. We solve quadratic equations either by factoring, completing the square, or using the quadratic formula, all in the name of finding the roots. If the quadratic equation is written with coefficients a, b, c in the form of Eq. (2), the quadratic formula is

$$\boxed{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}} \ .$$

where a , b , c are real numbers.

Note that a quadratic is defined by two roots (or zeros), say x_1, x_2, and a point on the graph, say $f(x_0) = y_0 \neq 0$, where $x_0 \neq x_1, x_2$. The polynomial can then be constructed as follows: Let c be an unknown, nonzero constant, and write

$$f(x) = c\,(x - x_1)\,(x - x_2) \ .$$

Then we can evaluate the function at the point x_0 to solve for the unknown constant c. We have $y_0 = f(x_0) = c\,(x_0 - x_1)\,(x_0 - x_2)$, and since $x_0 \neq x_1, x_2$, then we can divide by the quantity and obtain,

$$f(x) = \frac{(x - x_1)\,(x - x_2)}{(x_0 - x_1)\,(x_0 - x_2)}\,y_0 \ .$$

Remark: In general, for a polynomial of degree n, we would need n zeroes, say $x_1\,x_2 \ \ldots \ , x_n$ and another point on the graph such that $y_0 = f(x_0) \neq 0$, where $x_0 \neq x_1\,x_2$ (see Chapter 4).

Graphing a quadratic function:

Recall, any quadratic written in the form of Eq. (2), can be written

$$f(x) = a\,(x - h)^2 + k \ ,$$

where $\boxed{h = -\dfrac{b}{2\,a}}$ and $\boxed{k = c - \dfrac{b^2}{4\,a}}$, by completing the square of the *function*.
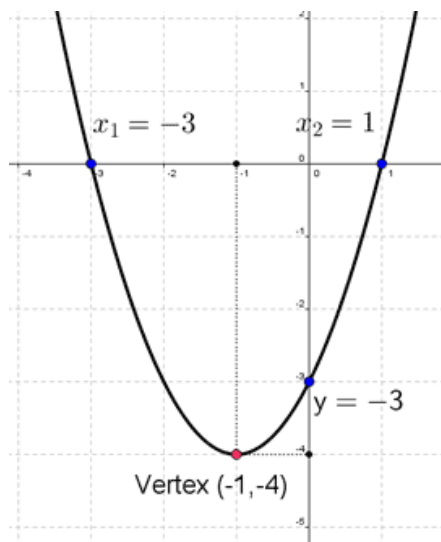
Figure 1: The x-coordinate of the vertex, $x = -1$, gives the axis of symmetry. It is the midpoint or average of the two zeros $x_1 = 1$ and $x_2 = -3$.

The vertex of a parabola written in the form of Eq. (2) is

$$(h, k) = \left(-\frac{b}{2\,a}, f\left(-\frac{b}{2\,a} \right) \right) = \left(-\frac{b}{2\,a}, \; c - \frac{b^2}{4\,a} \right)$$

Solving inequalities with absolute value:

Let $a > 0$. Then

$$\boxed{|x| < a \implies -a < x < a}$$

and

$$\boxed{|x| > a \implies x < -a \text{ or } x > a}$$

Similar statements hold for '$\leq$' and '$\geq$' .

Chapter 4.

Polynomial Functions:

A **polynomial function** $p(x)$ is given by

$$p(x) = a_n\, x^n + a_{n-1}\, x^{n-1} + a_{n-2}\, x^{n-2} + \cdots + a_1\, x + a_0$$

where the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are real numbers and the exponents are whole numbers.

The **Degree** of the polynomial $p(x)$ is the highest power of x, in this case,

$$\mathbf{Deg}(p) = n\,.$$

$\rightarrow$ The degree of a polynomial tells you the end behaviors.

The **Leading Coefficient** of the polynomial $p(x)$ is the coefficient for the highest power term,
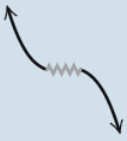
$$\mathbf{LC}(p) = a_n\,.$$

$\rightarrow$ The sign of this term determines the orientation. The Leading coefficient is not to be confused with the **Leading Term** of the polynomial $p(x)$, which is

$$\mathbf{LT}(p) = a_n\, x^n\,.$$

Figure 2:

Using the **Intermediate Value Theorem (IVT)** to show whether a polynomial has a real root:

Let $p(x)$ be a polynomial as in Fig. 3. Suppose that for $a \neq b$, $p(a)$ and $p(b)$ have opposite signs. Then, since $p(x)$ is a polynomial, and polynomials are continuous, there is a real root between a and b, say $c \in (a, b)$. We say that c is a root (or zero) of the function $p(x)$, since we have shown that $p(c) = 0$ by the IVT.
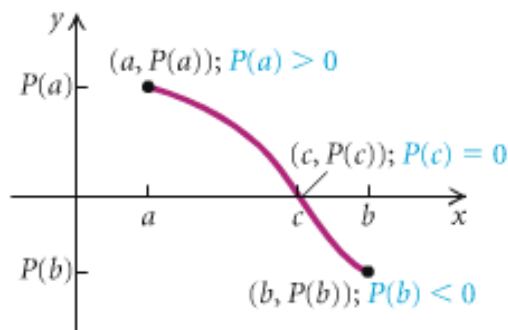

Figure 3:

The Remainder Theorem Let $f(x)$ be a polynomial. Then by dividing $f(x)$ by $x - c$ (using either synthetic or long division), we can write

$$f(x) = (x - c)\, q(x) + R,$$

where $q(x)$ is the <u>quotient</u> (of degree one less than $f(x)$) and R is the <u>remainder</u>. The Remainder Theorem states that

$$f(c) = (c - c)\, q(c) + R = 0\, q(c) + R = 0 + R = R.$$

The Factor Theorem gives a relationship between roots and linear factors. If the polynomial $f(x)$ has a root (or zero) at $x = c$, then

$$\boxed{f(c) = 0 \iff f(x) = (x - c)\, q(x)},$$

i.e. $x - c$ is a factor of $f(x)$ and the remainder is $R = 0$. This is an immediate consequence of the Remainder Theorem.

A **rational function** is a function f that is a quotient of two polynomials. That is,

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and where $q(x)$ is not the zero polynomial. The domain of f consists of all inputs x for which $q(x) \neq 0$.

Remark. So finding the zeroes of the denominator, i.e. $q(x) = 0$, gives you the 'problem childs'. These are the things you have to leave out (hopefully, not how you would actually parent problem children) of the domain of $f(x)$,

$$D_f = \{x \in \mathbb{R} \,|\, q(x) \neq 0\} = \mathbb{R}\} = \mathbb{R} \setminus \{x \,|\, q(x) = 0\}$$

Vertical Asymptotes. If $p(x)$ and $q(x)$ have no common factors, then the Vertical Asymptotes (VA) correspond to the x-values for which $q(x) = 0$. If there are any common factors, these correspond to a hole in the graph, since you still have to leave these points out of the domain.

Horizontal Asymptotes. Consider the rational function $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials with no common factors. There are 3 cases to consider for horizontal asymptotes:

(1) Suppose $\mathbf{deg}(p(x)) < \mathbf{deg}(q(x))$, then the horizontal asymptote occurs at $\boxed{y = 0}$.

(2) Suppose $\mathbf{deg}(p(x)) = \mathbf{deg}(q(x))$ and let the $\mathbf{LC}(p) = a$ and the $\mathbf{LC}(q) = b$, then the horizontal asymptotes occur at the $\boxed{y = \dfrac{a}{b}}$.

(3) Suppose the $\mathbf{deg}(p(x)) = 1 + \mathbf{deg}(q(x))$, then by the remainder theorem, rewrite $p(x) = m(x)\, q(x) + R$, where $m(x)$ is a linear term (i.e. $\mathbf{deg}(m(x)) = 1$) and R is the remainder. Note that $\mathbf{deg}(m(x)) + \mathbf{deg}(q(x)) = 1 + \mathbf{deg}(q(x)) = \mathbf{deg}(p(x))$. Then we can write

$$\boxed{\frac{p(x)}{q(x)} = \frac{m(x)\, q(x) + R}{q(x)} = m(x) + \frac{R}{q(x)}}$$

As the absolute value of x gets large, i.e. $x \to \pm\infty$, $\frac{R}{q(x)} \to 0$.

Then the **Oblique Asymptote** is $\boxed{m(x)}$, since

$$\boxed{\frac{p(x)}{q(x)} \sim m(x)} \text{ as } |x| \to \infty .$$

Chapter 5. Obtaining a Formula for the Inverse of a function. Let $f(x)$ be a one-to-one function so that the inverse not only *exists*, but also is a *function*. Then to find the inverse function, denoted $f^{-1}(x)$...

Step 1. Let $y = f(x)$.

Step 2. Replace $x \longleftrightarrow y$ so that you have $x = f(y)$.

Step 3. Solve for y in terms of x and let this y be $y = f^{-1}(x)$.

Congrats, you have found the inverse! To check your algebra, you can show that that

$$\left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x$$

Remark. Here $\left(f \circ f^{-1}\right)(x) = f(f^{-1}(x))$ means the composition of the function with it's inverse, NOT MULTIPLICATION !

Geometrically, inverse functions correspond to a reflection of $f(x)$ across the line $y = x$, so for $f^{-1}(x)$ to exist, $f(x)$ must pass the horizontal line test (i.e. one-to-one).
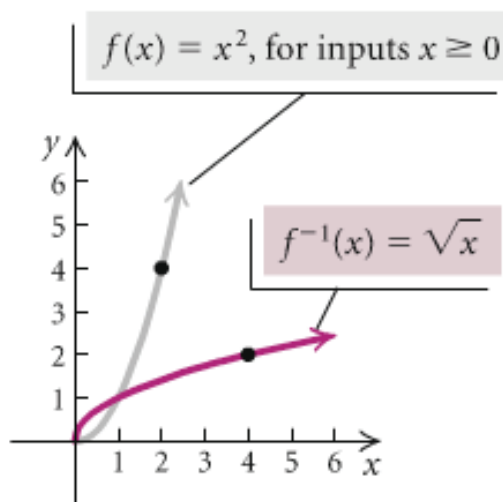

Figure 4:

An **exponential function** has the form $f(x) = a^x$, where x is a real number, $a > 0$ and $a \neq 1$. The function $f(x)$ is called the exponential function with base 'a'. Note that $f(x) > 0$ for all x.
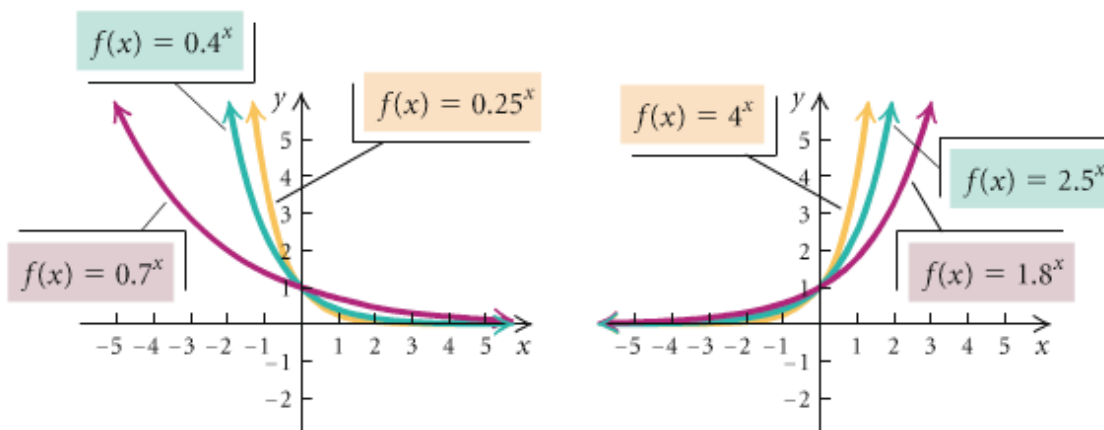

Figure 5:

(**Left**) The graphs of exponential functions with base $0 < a < 1$, $f(x) \to 0$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.

(**Right**) The graphs of exponential functions with base $a > 1$, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to 0$ as $x \to -\infty$.

The logarithmic function

We define $y = \log_a(x)$ as that number y such that $x = a^y$, where $x > 0$, $a > 0$ and $a \neq 1$. For instance, we read '$y = \log_2(x)$' as 'y equals the log base 2 of x' and we should ask ourselves 2 to what power gives me x?
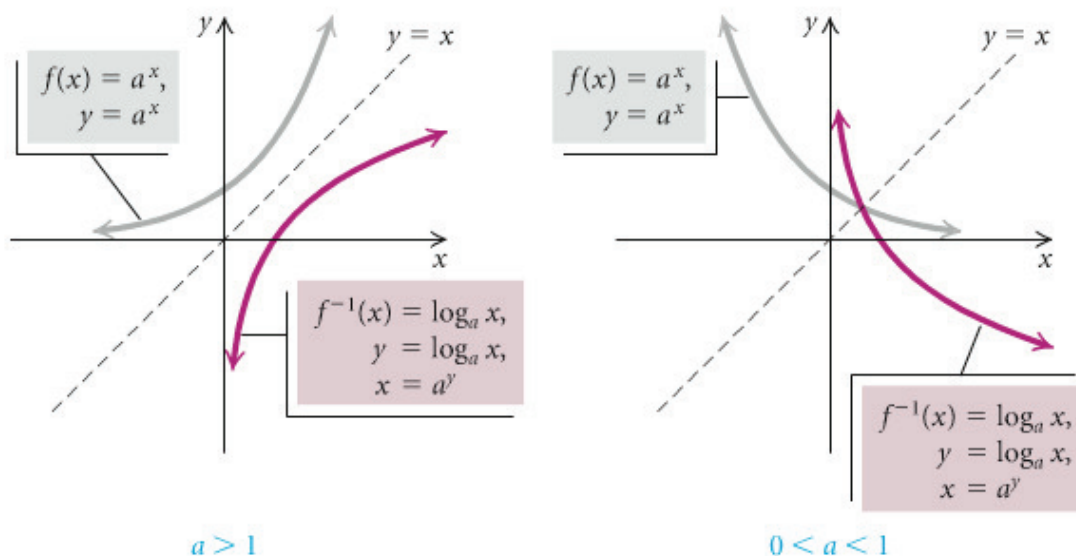
In general,

$$y = \log_a(x) \iff x = a^y$$

So the functions $f(x) = a^x$ and $f^{-1}(x) = \log_a(x)$ are inverses of one another. That is,

$$\left(f \circ f^{-1}\right)(x) = f(\log_a(x)) = \boxed{a^{(\log_a(x))} = x = \log_a(a^x)} = f^{-1}(a^x) = \left(f^{-1} \circ f\right)(x) .$$

Let's look at the graphs of $f(x) = a^x$ and $f^{-1}(x) = \log_a x$ for $a > 1$ and $0 < a < 1$.

Note that the graphs of $f(x)$ and $f^{-1}(x)$ are reflections of each other across the line $y = x$.

Figure 6:

Rules of Logarithms.

Let $a, b > 0$ and $m, n \in \mathbb{Z}$. Then

1. $\log_a(1) = 0$ and $\log_a(a) = 1$ (Since $a^0 = 1$)

2. $\log_a(M \cdot N) = \log_a(M) + \log_a(N)$ (Product Rule)

3. $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$ (Quotient Rule)

4. $\log_a(M^x) = x \log_a(M)$ (Power Rule)

5. $\log_a(a^x) = x$ and $a^{(\log_a(x))}$ (Inverses)

Exponential Growth is a model for, amongst other things, population growth and (continuously) compounded interest. The function model is

$$P(t) = P_0\, e^{k\,t}\ ,\quad k > 0\ .$$

- P_0 - the initial population (or amount of money) invested.
- k - the exponential *growth rate*
- t - the unit of time
- $P(t)$ - the population (or amount of money) at time t

The doubling time. We want to find the amount of time it takes for the population (or money) to double. The *doubling time* t_d satisfies

$$2\,P_0 = P(t_d) = P_0\, e^{k\,t_d}$$

$$\Longleftrightarrow\ 2 = e^{(k\,t_d)}$$

$$\Longleftrightarrow\ \ln(2) = \ln(e^{(k\,t_d)}) = k\,t_d$$

$$\Longleftrightarrow\ \boxed{t_d = \frac{1}{k}\ln(2)}$$

where $\ln(x) = \log_e(x)$ is the natural log or log base $e \approx 2.714\dots$.

Exponential decay is a model for, amongst other things, radioactive decay. The function model is

$$P(t) = P_0\, e^{-k\,t}\ ,\quad k > 0\ .$$

- P_0 - the initial amount of radioactive material (e.g. Bismuth has a half-life of 5 days)
- k - the exponential *decay rate*
- t - the unit of time
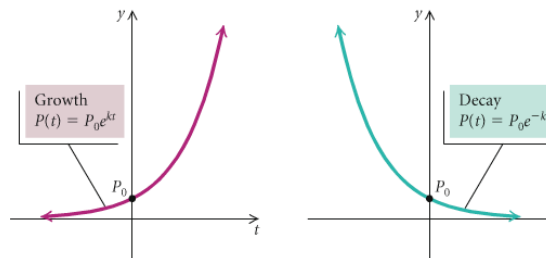- $P(t)$ - the amount of radioactive material at time t

Figure 7:

The Half-Life. We want to find the amount of time it takes for the material to decay to half it's initial amount P_0. The *half-life* time t_h satisfies

$$\frac{P_0}{2} = P(t_d) = P_0 \, e^{-k \, t_h}$$

$$\Longleftrightarrow \quad \frac{1}{2} = e^{-k \, t_h}$$

$$\Longleftrightarrow \quad \ln\left(\frac{1}{2}\right) = \ln(e^{-k \, t_h}) = -k \, t_h$$

$$\Longleftrightarrow \quad \ln\left(\frac{1}{2}\right) = \ln(2^{-1}) = (-1)\ln(2) = -k \, t_h$$

$$\Longleftrightarrow \quad \boxed{t_h = \frac{1}{k}\ln(2)}$$

The half-life for an exponential decay model with decay rate k satisfies the same equation as the doubling time for an exponential model with growth rate k !

Chapter 6. Some methods for solving systems of linear equations:

Substitution - For the 2 by 2 system, pick an equation, solve for one variable in terms of the other. *Substitute* this into the other equation and solve. Then *Back Substitute* in to either of the equations to obtain the other solution variable. Substitution is still do-able in a 3 by 3 system, but we typically use some sort of elimination technique.

Elimination - In this method, *Eliminate* one of the variables by adding two equations. For a 3 by 3 system, if you eliminate twice (using all 3 equations) and in the right way, you can obtain a 2 by 2 system and begin substitution. Then back substitute in the 2 by 2 system and further back substitute for the third variable.

Gaussian elimination. Keep track of system coefficients (with he augmented matrix). The goal is to obtain **Row-echelon form**, which satisfies

1. If a row does not consist entirely of 0's, then the first nonzero entry in the row is a 1.
2. For any two successive nonzero rows, the leading 1 in the lower row should be farther to the right than the leading 1 in the higher row.
3. All the rows consisting entirely of 0's are at the bottom of the matrix.

   If an additional property is satisfied, the matrix is said to be in **Reduced Row-echelon form**:

4. Each column that contains a leading 1 has 0's everywhere else.

We can obtain row-echelon form by performing row operations, which include:

1. Interchanging any two rows.
2. Multiplying each entry in a row by the same nonzero constant.
3. Adding a nonzero multiple of one row to another.

These will manipulate the system in a way that doesn't change the solution set of the original linear system.

Matrix Multiplication. Let $A = (a_{i,j})$ be $m \times \underline{n}$ and $B = (b_{i,j})$ be $\underline{n} \times p$. Then the product is $A\,B = (c_{i,j})$ an $m \times p$ matrix, where

$$c_{i,j} = a_{i,1}\,b_{1,j} + a_{i,2}\,b_{2,j} + a_{i,3}\,b_{3,j} + \cdots + a_{i,n}\,b_{n,j}$$

This looks complicated, but this just says, in order to obtain the (i,j) entry of $A\,B$, we take the ith row of A and j^{th} column of B, multiply the corresponding entries and add them. This is why you must check that the 'inner' dimensions match. That is, the number of columns in A must be equal to the number of rows in B.

The Matrix Inverse.

For an $n \times n$ matrix A, if there is a matrix A^{-1} so that $A^{-1}\,A = I$ (or $A\,A^{-1} = I$), then A^{-1} is the **inverse** of A. Here I is the $n \times n$ identity matrix (i.e. 1's on the diagonal, 0's elsewhere).

For any matrix A or vector x, the identity matrix satisfies

$$I(x) = I\,x = x \qquad \text{and} \qquad A\,I = I\,A = A\,.$$

Using the inverse to solve systems of linear equations. For n linear equations and n unknowns, we can consider the system as a matrix equation,

$$A\,X = B$$

where A is the $(n \times \underline{n})$ coefficient matrix, X is the $(\underline{n} \times 1)$ vector of unknowns, and B is the $(\underline{n} \times 1)$ vector of constants on the right. If A is invertible, we can multiply both sides *on the left* by A^{-1} to solve the system:

$$A\,X = B$$

$$\Longleftrightarrow\ A^{-1}\,(A\,X) = A^{-1}\,B$$

$$\Longleftrightarrow\ (A^{-1}\,A)\,X = A^{-1}\,B$$

$$\Longleftrightarrow\ I\,X = A^{-1}\,B$$

$$\Longleftrightarrow\ \boxed{X = A^{-1}\,B}$$

To find the inverse, one can apply *Guass-Jordan Elimination* and keep track of the identity matrix (in a 'super' augmented matrix) as she performs the row-equivalent operations. Once the matrix A is in reduced row-echelon form $(A \to I)$, the other matrix is the inverse of A !

Determinants. Remark. This is an operation on square matrices that gives a real number associated with a matrix A. If the determinant of the matrix is <u>nonzero</u>, then the matrix is <u>nonsingular</u> or <u>invertible</u>.

The 2×2 case. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the determinant of A is

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\,d - b\,c$$

Consider a square matrix $(n \times n)$ $A = (a_i, j)$.

Then the **Minor** with respect to the entry $a_{i,j}$, denoted $M_{i,j}$, is obtained by taking the determinant of the matrix formed by deleting the ith row and j^{th} column of A. Then the **Cofactor** of matrix entry $a_{i,j}$, denoted $A_{i,j}$, is given by

$$A_{i,j} = (-1)^{i+j} M_{i,j}$$

where $M_{i,j}$ is the **minor** described above.

One can obtain the determinant of a square matrix by **Cofactor Expansion** about a row or column. In working on specific matrices, one should choose a row or column with the most zeros (to reduce computation). For the 3×3 case, we show the expansion about the first column of the matrix $A = (a_{i,j})$. See the example on the next page.

Consider the matrix **A** given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The determinant of the matrix, denoted $|A|$, can be found by multiplying each element of the first column by its cofactor and adding:

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}.$$

Because

$$A_{11} = (-1)^{1+1}M_{11} = M_{11},$$
$$A_{21} = (-1)^{2+1}M_{21} = -M_{21},$$

and $\quad A_{31} = (-1)^{3+1}M_{31} = M_{31},$

we can write

$$|A| = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \cdot \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}.$$

It can be shown that we can determine $|A|$ by choosing *any* row or column, multiplying each element in that row or column by its cofactor, and adding. This is called *expanding* across a row or down a column. We just expanded down the first column. We now define the determinant of a square matrix of any order.