Math 0031 **Sample Exam 2**

Spring 2017 **S o l u t i o n s**

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1. Find a polynomial function $f(x)$ of lowest degree with rational coefficients that has the numbers $1+i$ and $\sqrt{3}$ as its zeros.

Solution: Since the coefficients are rational, the conjugates of $1+i$ and $\sqrt{3}$ are also zeros. These conjugates are $1-i$ and $-\sqrt{3}$. Then

$f(x) = (x-(1+i))(x-(1-i))(x-\sqrt{3})(x-(-\sqrt{3}))$

$= ((x-1)-i)((x-1)+i)(x-\sqrt{3})(x+\sqrt{3})$

$= ((x-1)^2-i^2)(x^2-(\sqrt{3})^2) = (x^2-2x+1+1)(x^2-3) = (x^2-2x+2)(x^2-3)$

$= x^4-2x^3+2x^2-3x^2+6x-6$

Answer: $f(x) = x^4-2x^3-x^2+6x-6$.

2. Consider the function $f(x) = \dfrac{x^3+x^2-2x}{x^2-1}$.

(a) Determine the domain of the function.

Solution: This is a rational function. Therefore it is defined everywhere except points where the denominator is zero: $x^2-1=0 \quad \Leftrightarrow \quad (x+1)(x-1)=0 \quad \Leftrightarrow$ $x=-1,\ x=1$

The domain of the function is $D = (-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

(b) Determine the vertical asymptotes of the graph of the function, if any.

Solution: We factor numerator and denominator and try to simplify the function:

$$f(x) = \frac{x(x^2+x-2)}{(x+1)(x-1)} = \frac{x(x+2)(x-1)}{(x+1)(x-1)} = \frac{x(x+2)}{x+1}.$$

The graph of the function has vertical asymptotes when the denominator is zero, i.e. $x+1=0$. Therefore $x=-1$ is the vertical asymptote.

(c) Determine the horizontal asymptotes of the graph of the function, if any.

Solution: After simplification $f(x) = \dfrac{x(x+2)}{x+1} = \dfrac{x^2+2x}{x+1}$.

The degree of the numerator is not the same as the degree of the denominator. Therefore the graph of $f(x)$ has no horizontal asymptotes.

(d) Determine the oblique asymptotes of the graph of the function, if any.

Solution: The degree of the numerator is 1 greater than the degree of the denominator. Therefore the graph of $f(x)$ has an oblique asymptote. We use synthetic division to find the quotient:

$$f(x) = \frac{x^2+2x}{x+1} = (x^2+2x) \div (x+1)$$

$$\begin{array}{r|rrr} -1 & 1 & 2 & 0 \\ & & -1 & -1 \\ \hline & 1 & 1 & \text{-1} \end{array}$$

Therefore $f(x) = x + 1 - \dfrac{1}{x+1}$ and $y = x + 1$ is the oblique asymptote.

[An alternative solution:

$$f(x) = \frac{x^2+2x}{x+1} = \frac{x^2+2x+1-1}{x+1} = \frac{(x+1)^2-1}{x+1} = x+1-\frac{1}{x+1}$$]

(e) Sketch the graph of the function $f(x)$. Draw all asymptotes. Mark the asymptotes and points outside the domain, if any.

Solution:

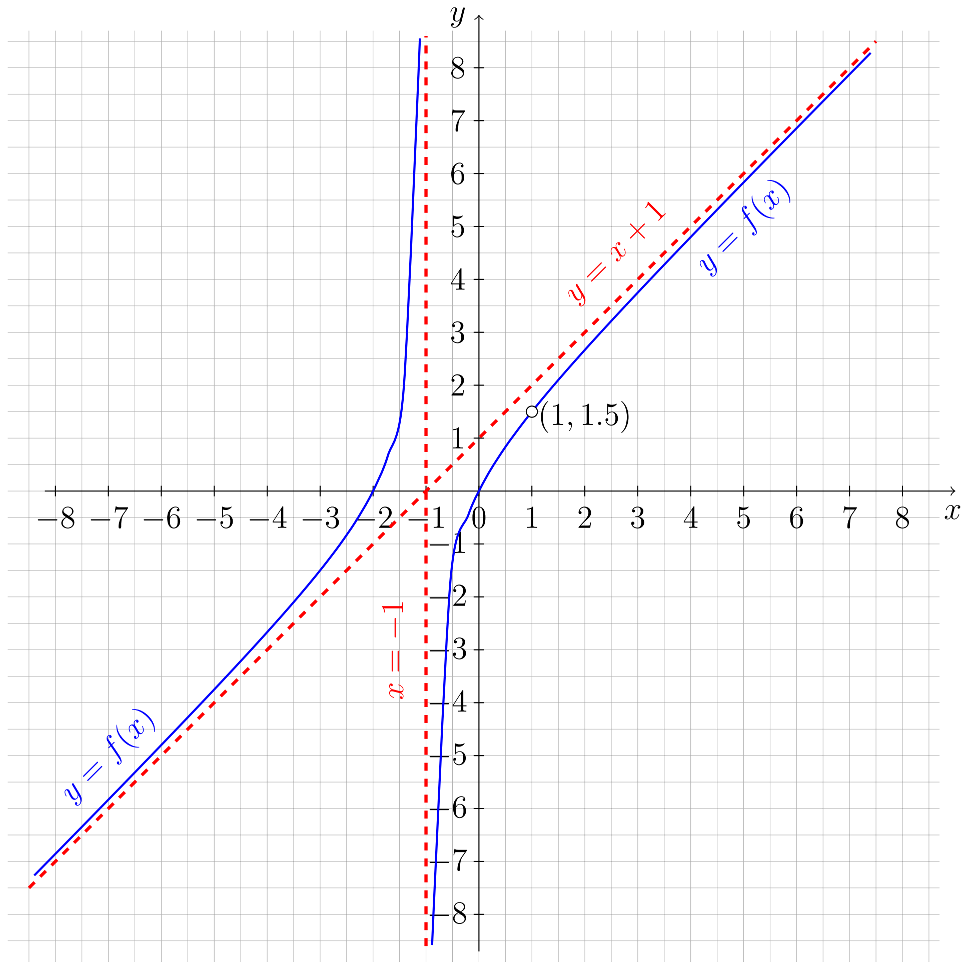


Figure 1: Graph of the function $f(x)$.

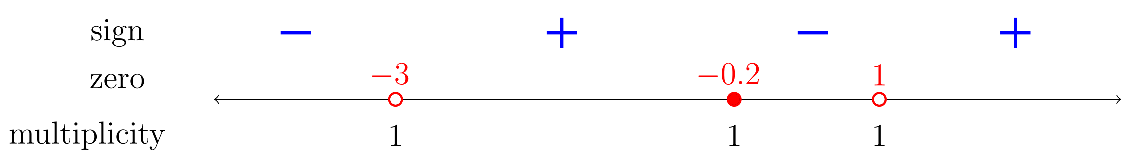
3. Solve the inequality $\dfrac{x-4}{x+3} \le \dfrac{x+2}{x-1}$.

Solution: $\dfrac{x-4}{x+3} \le \dfrac{x+2}{x-1} \quad \Leftrightarrow \quad \dfrac{x-4}{x+3} - \dfrac{x+2}{x-1} \le 0$

$$\Leftrightarrow \quad \frac{(x-4)(x-1)-(x+2)(x+3)}{(x+3)(x-1)} \le 0 \quad \Leftrightarrow \quad \frac{x^2-5x+4-(x^2+5x+6)}{(x+3)(x-1)} \le 0$$

$$\Leftrightarrow \quad \frac{x^2-5x+4-x^2-5x-6}{(x+3)(x-1)} \le 0 \quad \Leftrightarrow \quad \frac{-10x-2}{(x+3)(x-1)} \le 0$$

$$\Leftrightarrow \quad \frac{-10(x+0.2)}{(x+3)(x-1)} \le 0 \quad \Leftrightarrow \quad \frac{x+0.2}{(x+3)(x-1)} \ge 0$$



Answer: $x \in (-3, -0.2] \cup (1, \infty)$.

4. Using the definition of one-to-one function show that $h(x) = x^4 + 2x^2 - 5$ is not one-to-one function.

Solution: Let $a = 1$ and $b = -1$. Obviously $a \ne b$. On the other hand

$h(a) = h(1) = 1^4 + 2 \cdot 1^2 - 5 = 1 + 2 - 5 = -3$,

$h(b) = h(-1) = (-1)^4 + 2 \cdot (-1)^2 - 5 = 1 + 2 - 5 = -3 = h(1) = h(a)$.

So we found two numbers $a \ne b$ for which $h(a) = h(b)$. By the definition of one-to-one function if $a \ne b$ then $h(a) \ne h(b)$. Therefore $h(x)$ is not one-to-one function.

5. The function $f(x) = \dfrac{x-1}{x+2}$, $x > -2$ is one-to-one. Find its inverse when $f(x) < 1$.

Solution: 1. $y = \dfrac{x-1}{x+2}$ 2. $x = \dfrac{y-1}{y+2}$

3. $x = \dfrac{y-1}{y+2}$, $x(y+2) = y - 1$, $xy + 2x = y - 1$, $xy - y = -2x - 1$,,

$(x-1)y = -(2x+1)$, $y = -\dfrac{2x+1}{x-1}$

4. $f^{-1}(x) = -\dfrac{2x+1}{x-1}, \quad x < 1.$

6. Find

(a) $\log_3 \frac{1}{27}$ *Solution:* $\log_3 \frac{1}{27} = \log_3 3^{-3} = -3.$

(b) $\log_{64} 4$ *Solution:* $\log_{64} 4 = \log_{64} 64^{1/3} = \frac{1}{3}.$

(c) $\log \sqrt{100,000}$ *Solution:* $\log \sqrt{100,000} = \log \sqrt{10^5} = \log 10^{5/2} = \frac{5}{2}.$

(d) $\dfrac{\log_5 81}{\log_5 3}$ *Solution:* Using the Change-of-Base formula:

$$\frac{\log_5 81}{\log_5 3} = \log_3 81 = \log_3 3^4 = 4.$$

7. Simplify

(a) $\log_5(\frac{1}{25} \cdot \frac{1}{125})$ *Solution:* $\log_5(\frac{1}{25} \cdot \frac{1}{125}) = \log_5 \frac{1}{25} + \log_5 \frac{1}{125}$

$= -2 + (-3) = -5.$

(b) $\log t^3 + \log \dfrac{x}{t\sqrt{t}}$ *Solution:* $\log t^3 + \log \dfrac{x}{t\sqrt{t}} = \log \dfrac{t^3 x}{t^{3/2}} = \log t^{3/2} x$

$= \frac{3}{2} \log t + \log x.$

(c) $4 \ln x^{3/2} + 5 \ln \sqrt[5]{y^2}$ *Solution:* $4 \ln x^{3/2} + 5 \ln \sqrt[5]{y^2} = 4 \ln x^{3/2} + 5 \ln y^{2/5}$

$= 4 \cdot \frac{3}{2} \ln x + 5 \cdot \frac{2}{5} \ln y = 6 \ln x + 2 \ln y.$

8. Solve equations

(a) $3^{2x-8} = 9^{2-x}$ *Solution:* $3^{2x-8} = 9^{2-x} \quad \Leftrightarrow \quad 3^{2x-8} = 3^{2(2-x)} \quad \Leftrightarrow$

$2x - 8 = 4 - 2x \quad \Leftrightarrow \quad 4x = 12 \quad \Leftrightarrow \quad x = 3.$

(b) $\ln(2x - x^2) = 0$

Solution: $\ln(2x - x^2) = 0 \quad \Leftrightarrow \quad 2x - x^2 = 1 \quad \Leftrightarrow \quad x^2 - 2x + 1 = 0$

$\Leftrightarrow \quad (x-1)^2 = 0 \quad \Leftrightarrow \quad x = 1$ is a possible solution.

Domain: $2x - x^2 > 0 \quad \Leftrightarrow \quad x(2-x) > 0 \quad \Leftrightarrow \quad x \in (0, 2).$

Answer: $x = 1$.

(c) $\log_3(x+6) - \log_3(x+2) = \log_3 x$

Solution: $\log_3(x+6) - \log_3(x+2) = \log_3 x \quad \Leftrightarrow \quad \log_3 \dfrac{x+6}{x+2} = \log_3 x$

$\Leftrightarrow \quad \dfrac{x+6}{x+2} = x \quad \Leftrightarrow \quad x+6 = x(x+2) \quad \Leftrightarrow \quad x+6 = x^2+2x \quad \Leftrightarrow \quad x^2+x-6=0.$

$\Leftrightarrow \quad (x+3)(x-2)=0 \quad \Leftrightarrow \quad x=-3$ or $x=2$ are possible solutions.

Domain: $x+6>0,\ x+2>0$, and $x>0 \quad \Leftrightarrow \quad x \in (0,\infty)$.

Answer: $x = 2$.

9. A country's population doubled in 40 years. What is the exponential growth rate? Leave your answer in exact form.

Solution: Let r be the exponential growth rate, t - time, and $P(t)$ - population.

Then $P(t) = P_0\, e^{rt}$, where P_0 is the initial population. We have

$P(40) = P_0\, e^{40r} = 2P_0 \quad \Leftrightarrow \quad e^{40r} = 2 \quad \Leftrightarrow \quad 40r = \ln 2 \quad \Leftrightarrow \quad r = \dfrac{\ln 2}{40}$

Answer: The exponential growth rate is $r = \dfrac{\ln 2}{40}$.