Exam 1

Your name:

100 points total

Your TA's name:

1. [15 points] If $f(x) = \cos x$ and $g(x) = \sqrt[3]{5-x^2}$, find the functions $f \circ f$, $g \circ f$ and their derivatives. Do not simplify the results.

$f \circ f = f(f(x)) = \cos(\cos x)$

$(f(f(x)))' = -\sin(\cos x) \cdot (-\sin x) = \sin x \cdot \sin(\cos x)$

$G = g \circ f = g(f(x)) = \sqrt[3]{5-\cos^2 x} = (5-\cos^2 x)^{1/3}$

$G' = \frac{1}{3}(5-\cos^2 x)^{-2/3} \cdot (-2\cos x)(-\sin x)$

$G' = \frac{2}{3}\cos x \sin x (5-\cos^2 x)^{-2/3}$

2. Find the limit, if it exists. If the limit does not exist explain why. Show all the necessary steps. You can use any method.

(a) [10 points] $\lim_{x\to -2} \frac{\sqrt{x+3}-1}{x+2}$, $= \lim_{x\to -2} \frac{\sqrt{x+3}-1}{x+2} \cdot \frac{\sqrt{x+3}+1}{\sqrt{x+3}+1}$

$$= \lim_{x\to -2} \frac{x+3-1}{(x+2)(\sqrt{x+3}+1)} = \lim_{x\to -2} \frac{\cancel{x+2}}{(\cancel{x+2})(\sqrt{x+3}+1)}$$

$$= \lim_{x\to -2} \frac{1}{\sqrt{x+3}+1} = \frac{1}{\sqrt{-2+3}+1} = \frac{1}{\sqrt{1}+1} = \boxed{\frac{1}{2}}$$

(b) [10 points] $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, if $f(x) = 3x^2 - 5x$.

The given limit is the definition of the derivative of $f(x)$. Hence

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x) = \boxed{6x-5}$$

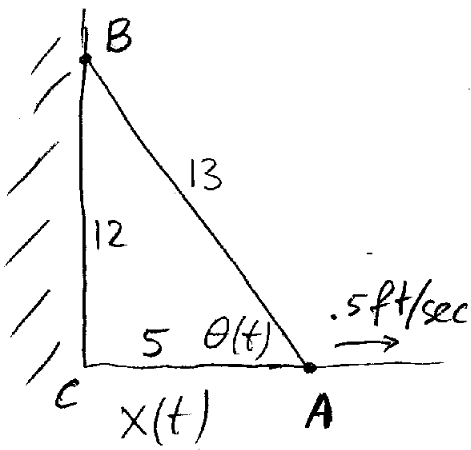
("any method" was used here)

or $\frac{f(x+h)-f(x)}{h} = \frac{3(x+h)^2-5(x+h)-(3x^2-5x)}{h} =$

$$= \frac{\cancel{3x^2}+6xh+3h^2\cancel{-5x}-5h\cancel{-3x^2}\cancel{+5x}}{h} = \frac{6xh+3h^2-5h}{h} = 6x+3h-5$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(6x+3h-5) = 6x+3\cdot 0-5 = \boxed{6x-5}$$

3. [20 points] A ladder 13 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at the rate of 0.5 ft/sec. At what rate is the angle between the ladder and the ground changing when the bottom of the ladder is 5 feet from the wall?



AB is the ladder

$AB = 13$

$$\frac{x}{13} = \cos\theta$$

Differentiate both sides!

$$\frac{1}{13}\frac{dx}{dt} = -\sin\theta \cdot \frac{d\theta}{dt}$$

The triangle ABC is right with sides 13, 12 and 5 $\Rightarrow$ $\sin\theta = \frac{BC}{BA} = \frac{12}{13}$

$$\frac{dx}{dt} = 0.5$$

So, $\frac{1}{13} \cdot 0.5 = -\frac{12}{13} \cdot \frac{d\theta}{dt}$

$$0.5 = -12\frac{d\theta}{dt} \Rightarrow \boxed{\frac{d\theta}{dt} = -\frac{1}{24} \text{ rad/sec}}$$

4. [20 points] Find an equation of the tangent line to the curve $\sqrt{y+x} - \sqrt{y-x} = 2$ at the point $(4, 5)$.

Differentiate both sides:

$$\frac{y'+1}{2\sqrt{y+x}} - \frac{y'-1}{2\sqrt{y-x}} = 0$$

plug in $x = 4$, $y = 5$ to get

$\sqrt{y+x} = \sqrt{5+4} = 3$, $\sqrt{y-x} = \sqrt{5-4} = 1$

So, $$\frac{y'+1}{6} - \frac{y'-1}{2} = 0$$

$$2(y'+1) - 6(y'-1) = 0$$

$$2y' - 6y' + 2 + 6 = 0$$

$$-4y' = -8 \qquad y' = 2$$

the tan. line: $y = 5 + 2(x-4)$

$$y = 2x + 5 - 8$$

$$\boxed{y = 2x - 3}$$

5. [10 points] Find the second derivative of the function $f(x) = x \sin x$.

We use the product rule:

$$\boxed{f'(x) = \sin x + x\cos x}$$

$$f''(x) = \cos x + \cos x + x(-\sin x)$$

$$\boxed{f''(x) = 2\cos x - x\sin x}$$

6. [15 points] Find all vertical and horizontal asymptotes of the function $y = \frac{x^2-9}{x^3+x^2-6x}$. Sketch its graph (2 points out of 15 for the graph).

$$y = \frac{(x-3)(x+3)}{x(x-2)(x+3)}, \quad y = \frac{x-3}{x(x-2)} \quad \text{when } x \neq -3$$

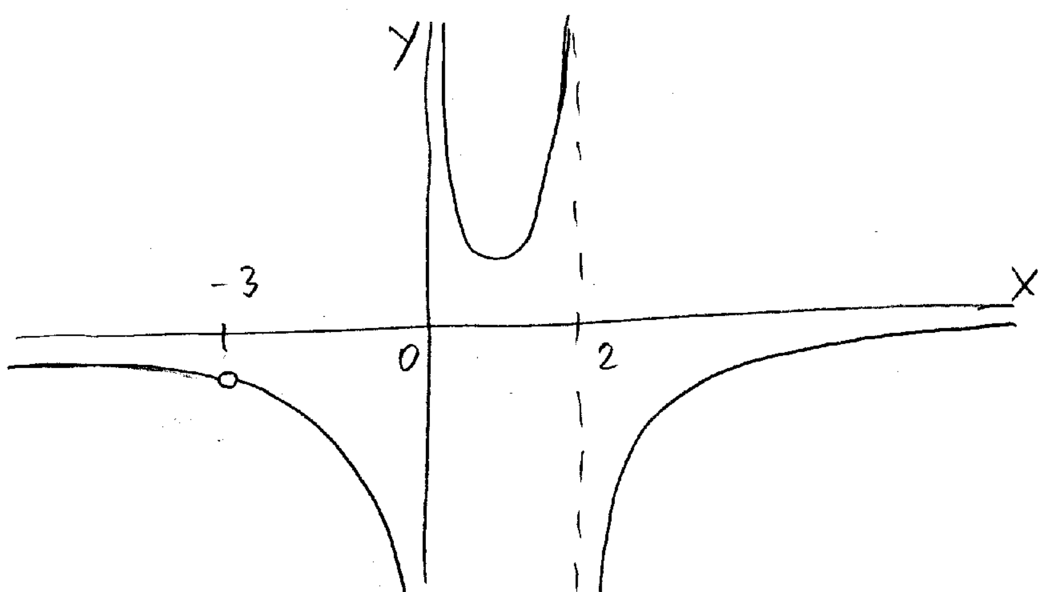
v.a. $\boxed{x=0}$ since $\lim_{x\to 0^+} \frac{x-3}{x(x-2)} = \infty$

$\lim_{x\to 0^-} \frac{x-3}{x(x-2)} = -\infty$

$\boxed{x=2}$ since $\lim_{x\to 2^-} \frac{x-3}{x(x-2)} = \infty$, $\lim_{x\to 2^+} \frac{x-3}{x(x-2)} = -\infty$

h.a. $\lim_{x\to \pm\infty} \frac{x-3}{x(x-2)} = \lim_{x\to\pm\infty} \frac{x-3}{x^2-2x} = \lim_{x\to\pm\infty} \frac{\frac{1}{x}-\frac{3}{x^2}}{1-\frac{2}{x}} = 0$

$\boxed{y=0}$



Bonus pb (8 pts) $f(x) = x\sin x$. Find $f^{(50)}(x)$.

Answer: $f^{(50)}(x) = 50\cos x - x\sin x$