Exam 1

100 points total

Your name:

Your TA's name:

1. [10 points] Find an equation of the tangent line to the curve $y = 6\sin^2 x$ at the point $(\pi/4, 3)$. (Write the answer in the form $y = mx + b$.)

$$y' = 12 \sin x \cos x = 6 \sin 2x$$

$$y'\left(\frac{\pi}{4}\right) = 6 \cdot \sin \frac{\pi}{2} = 6 \cdot 1 = 6$$

tan line:

$$y = 3 + 6\left(x - \frac{\pi}{4}\right)$$

$$y = 6x + 3 - \frac{6}{4}\pi$$

$$\boxed{y = 6x + 3 - \frac{3}{2}\pi} \quad = 6x + \frac{6 - 3\pi}{2}$$

Just in case $\quad \sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$$12 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = 12 \cdot \frac{2}{4} = 12 \cdot \frac{1}{2} = 6$$

2. (a) [5 points] Find the constant c such that the function

$$f(x) = \begin{cases} c\sqrt{x} - 2, & \text{if} \quad x < 1 \\ x^3 - c, & \text{for} \quad x \geq 1 \end{cases}$$

is continuous on $(-\infty, \infty)$.

f is continuous at 1 if

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\left.\begin{aligned} &f(1) = 1^3 - c = 1 - c \\ &\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (c\sqrt{x} - 2) = c - 2 \\ &\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^3 - c) = 1 - c \end{aligned}\right\} \Rightarrow \quad c - 2 = 1 - c \qquad \boxed{c = \frac{3}{2}}$$

They all must be equal

(b) [5 points] Using the definition of the continuity of a function at a point prove, that for the found c the function is certainly continuous.

$$f(x) = \begin{cases} \frac{3}{2}\sqrt{x} - 2 & \text{if } x < 1 \\ x^3 - \frac{3}{2} & \text{if } x \geq 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(\frac{3}{2}\sqrt{x} - 2\right) = \frac{3}{2} - 2 = -\frac{1}{2}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(x^3 - \frac{3}{2}\right) = 1 - \frac{3}{2} = -\frac{1}{2}$$

$\Rightarrow$ the limit exists and $\lim_{x\to 1} f(x) = -\frac{1}{2}$

Also $f(1) = 1^3 - \frac{3}{2} = -\frac{1}{2}$. We get:

$\boxed{\lim_{x\to 1} f(x) = f(1). \quad \text{Hence } f(x) \text{ is continuous.}}$

The definition

3. [15 points] Calculate y' if $\cos(xy) = \sqrt{x} - y$.

Differentiate both sides (using implicit differentiation):

$$-\sin(xy)\cdot(y + xy') = \frac{1}{2\sqrt{x}} - y'$$

$$-y\cdot\sin(xy) - x\cdot\sin(xy)\cdot y' + y' = \frac{1}{2\sqrt{x}}$$

$$(1 - x\cdot\sin(xy))\,y' = \frac{1}{2\sqrt{x}} + y\sin(xy)$$

$$\boxed{y' = \frac{\frac{1}{2\sqrt{x}} + y\sin(xy)}{1 - x\sin(xy)}}$$

$$\boxed{y' = \frac{1 + 2y\sqrt{x}\sin(xy)}{2\sqrt{x}(1 - x\sin(xy))}}$$

(both answers are o.k.)

4. Find the limit, probably infinite, if it exists. If the limit does not exist explain why. If it is infinity, then find its sign. Provide all the necessary steps or explanations. You can use any method.

(a) [10 points] $\lim_{t \to 4} \frac{4-t}{|4-t|}$,

$$|4-t| = \begin{cases} 4-t, & 4-t \geq 0 \\ -(4-t), & 4-t < 0 \end{cases}$$

$$|4-t| = \begin{cases} 4-t, & t \leq 4 \\ -(4-t), & t > 4 \end{cases}$$

Hence

$$\lim_{t \to 4^-} \frac{4-t}{|4-t|} = \lim_{t \to 4^-} \frac{4-t}{4-t} = 1$$

$$\lim_{t \to 4^+} \frac{4-t}{|4-t|} = \lim_{t \to 4^+} \frac{4-t}{-(4-t)} = -1$$

$$\lim_{t \to 4^-} \frac{4-t}{|4-t|} \neq \lim_{t \to 4^+} \frac{4-t}{|4-t|}$$

Hence the limit DNE.

(b) [10 points] $\lim_{s\to\infty}\left(\sqrt{s^2+4s+1}-s\right)$. $=$

$$= \lim_{s\to\infty}\left(\sqrt{s^2+4s+1}-s\right)\cdot\frac{\sqrt{s^2+4s+1}+s}{\sqrt{s^2+4s+1}+s} =$$

$$= \lim_{s\to\infty}\frac{s^2+4s+1-s^2}{\sqrt{s^2+4s+1}+s} = \lim_{s\to\infty}\frac{4s+1}{\sqrt{s^2+4s+1}+s}\cdot\frac{\frac{1}{s}}{\frac{1}{s}}$$

$$= \lim_{s\to\infty}\frac{4+\frac{1}{s}}{\sqrt{1+\frac{4}{s}+\frac{1}{s^2}}+1} = \frac{4}{\sqrt{1}+1} = \frac{4}{2} = \boxed{2}$$

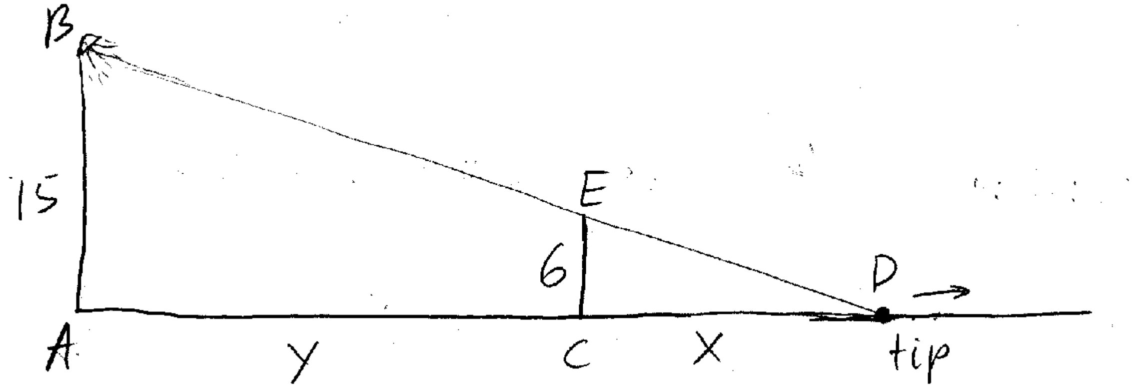
(c) [10 points] $\lim_{h\to 0^-}\frac{(h-2)^3+4}{h}$. $=$ $\frac{\lim_{h\to 0^-}(h-2)^3+4}{\lim_{h\to 0^-} h} =$

$$= \frac{-8+4}{\lim_{h\to 0^-} h} = \frac{-4}{\lim_{h\to 0^-} h} = \lim_{h\to 0^-}\frac{-4}{h} = +\infty$$

$\left(\text{"}\frac{\text{negative}}{\text{negative}} = \text{positive"}\right)$

$\left(\text{It is not "}\frac{0}{0}\text{" limit but "}\frac{\text{a number}}{0}\text{"}\right)$

5. [15 points] A light is on the top of a 15 ft tall pole and a 6ft tall person is walking away from the pole at a rate of 2 ft/sec. At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?



Let $AC = Y$, $CD = X$

Given: $AB = 15$, $CE = 6$, $\frac{dy}{dt} = 2$ ft/sec

Need to find $\frac{dx}{dt}$ when $y = 25$ ft

Similarity of the triangles ADB and CDE gives $\frac{X}{6} = \frac{X+Y}{15}$ or $\frac{X}{10} = \frac{Y}{15}$ or $3X = 2Y$

Differentiate both sides: $3\frac{dx}{dt} = 2\frac{dy}{dt}$

Hence $\frac{dx}{dt} = \frac{2}{3} \cdot \frac{dy}{dt} = \frac{2}{3} \cdot 2 = \boxed{\frac{4}{3} \text{ ft/sec}}$

(The result does not depend on the value of Y)

6. [10 points] Find all vertical and horizontal asymptotes of the function $y = \frac{2x^2+x-1}{x^2+x-2}$. Sketch its graph (1 point out of 10 for the graph).

$$y = \frac{(2x-1)(x+1)}{(x-1)(x+2)}$$ can not be reduced

v.a. candidates are $x=1$ and $x=-2$

$$\lim_{x\to 1^+} \frac{(2x-1)(x+1)}{(x-1)(x+2)} = +\infty \Rightarrow$$ $x=1$ is a vertical asymptote

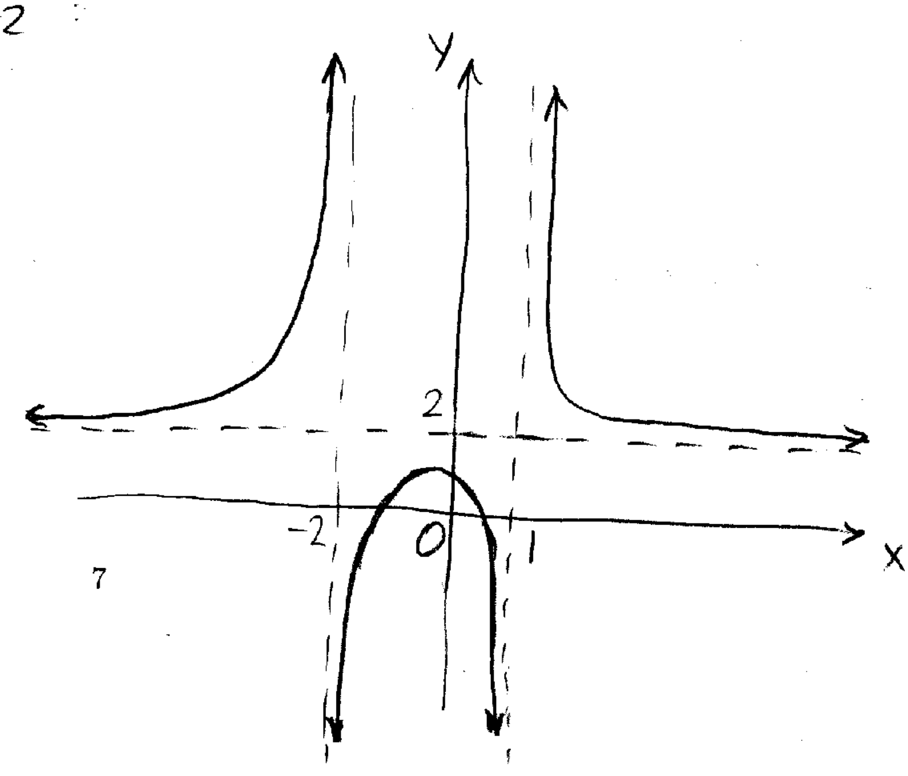
$$\lim_{x\to -2^-} \frac{(2x-1)(x+1)}{(x-1)(x+2)} = +\infty \Rightarrow$$ $x=-2$ is a vertical asymptote

h.a.: $$\lim_{x\to\infty} \frac{2x^2+x-1}{x^2+x-2} = \lim_{x\to\infty} \frac{2+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x}} = \frac{2}{1} = 2$$

$$\lim_{x\to -\infty} \frac{2x^2+x-1}{x^2+x-2} = 2$$

Also, $$\lim_{x\to 1^-} \frac{(2x-1)(x+1)}{(x-1)(x+2)} = -\infty$$

$$\lim_{x\to -2^+} \frac{(2x-1)(x+1)}{(x-1)(x+2)} = -\infty$$



7. [10 points] If $f(x) = \sqrt{3x-5}$ find $f''(3)$.

$$f'(x) = \frac{3}{2\sqrt{3x-5}} = \frac{3}{2}(3x-5)^{-1/2}$$

$$f''(x) = \frac{3}{2}\left(-\frac{1}{2}\right)(3x-5)^{-3/2} \cdot 3 = -\frac{9}{4}\,\frac{1}{(3x-5)^{3/2}}$$

$$f''(3) = -\frac{9}{4} \cdot \frac{1}{(9-5)^{3/2}} = -\frac{9}{4}\,\frac{1}{4^{3/2}}$$

$$4^{3/2} = \left(\sqrt{4}\right)^3 = 2^3 = 8$$

$$f''(3) = -\frac{9}{4} \cdot \frac{1}{8} = \boxed{-\frac{9}{32}}$$