Exam 2 **Your name:** Solutions

100 points total **Your TA's name:**

1. Find the derivative of the function by any appropriate method

(a) [10 points] $f(x) = \ln \dfrac{e^x - 1}{e^x + 1}$

$$f'(x) = \frac{e^x+1}{e^x-1} \cdot \frac{e^x(e^x+1) - e^x(e^x-1)}{(e^x+1)^2} =$$

$$= \frac{e^x(e^x+1-e^x+1)}{(e^x-1)(e^x+1)} = \boxed{\frac{2e^x}{e^{2x}-1}}$$

or

$$f(x) = \ln(e^x-1) - \ln(e^x+1)$$

$$f'(x) = \frac{e^x}{e^x-1} - \frac{e^x}{e^x+1}$$

(b) [10 points] $y = x^{\ln x}$

$$\ln y = \ln x \cdot \ln x = (\ln x)^2$$

$$\frac{y'}{y} = 2 \ln x \cdot \frac{1}{x}$$

$$y' = 2 \cdot x^{\ln x} \cdot \frac{1}{x} \cdot \ln x$$

$$\boxed{y' = 2x^{\ln x - 1} \cdot \ln x} \quad \left(= x^{\ln x} \cdot \frac{2 \ln x}{x} \right)$$

2. [15 points] Find 2 numbers whose sum is 20 and whose product is as large as possible. For a proof use the optimization method.

Let these numbers be x and y.

Then $x+y=20 \Rightarrow y=20-x$

$$p=xy=x(20-x)=20x-x^2 \rightarrow \max$$

$p'(x)=20-2x=0 \Rightarrow x=10$

$p''(x)=-2<0$

Hence the graph of $p(x)$ is concave downward everywhere $\Rightarrow$ $p(x)$ attains its absolute maximum at the critical number $x=10$.

$y=20-x=20-10=10$

Answer: 10 and 10

3. [15 points] Find $f(x)$ if $f'(x) = x\left(3x - \frac{2}{x^2}\right)$ and $f(1) = 5$.

$$f'(x) = 3x^2 - \frac{2}{x}$$

$$f(x) = x^3 - 2\ln|x| + C$$

$$f(1) = 1 - 2 \cdot 0 + C = 1 + C = 5 \Rightarrow C = 4$$

$$\boxed{f(x) = x^3 - 2\ln|x| + 4}$$

4. [15 points] What, approximately, is $\sqrt{9.01}$? Use linear approximation to find the answer.

Consider $f(x) = \sqrt{9+x}$ with $a = 0$.

and x is near a.

$f(0) = 3$, $f'(x) = \frac{1}{2\sqrt{9+x}}$, $f'(0) = \frac{1}{6}$

Then the lin. approximation of $f(x)$ is

$$L(x) = 3 + \frac{1}{6}(x-0) = 3 + \frac{1}{6}x$$

Then

$$\sqrt{9.01} = f(0.01) \approx L(0.01) = 3 + \frac{1}{6} \cdot 0.01$$

$$= 3 + \frac{1}{600} = 3\frac{1}{600}$$

i.e. $\boxed{\sqrt{9.01} \approx 3\frac{1}{600}}$

Another way: $f(x) = \sqrt{x}$, $a = 9$, $f' = \frac{1}{2\sqrt{x}}$, $f(9) = \frac{1}{6}$

$f(9) = 3$

$L(x) = 3 + \frac{1}{6}(x-9) \left(= \frac{x+9}{6}\right)$. $\sqrt{9.01} \approx L(9.01) = 3 + \frac{9.01-9}{6} = 3 + \frac{1}{600}$

5. [20 points] Suppose that the population of a colony of bacteria increases exponentially. At the start of an experiment, there are 400 bacteria, and one hour later, the population has increased to 600. How long will it take for the population to reach 900?

$$p(t) = p_0 e^{\kappa t} = 400 (e^{\kappa})^t$$

$$p(1) = 400 (e^{\kappa})^1 = 400 (e^{\kappa}) = 600$$

$$\Rightarrow e^{\kappa} = \frac{600}{400} = \frac{3}{2}$$

$$p(t) = 400 \cdot \left(\frac{3}{2}\right)^t$$

$$400 \cdot \left(\frac{3}{2}\right)^t = 900, \quad \left(\frac{3}{2}\right)^t = \frac{900}{400} = \frac{9}{4}$$

$$t = 2 \quad \text{since} \quad \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

Answer: it will take 2 hours.

6. [15 points] Find the absolute maximum and the absolute minimum values of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ on the interval $[-2; 1]$.

$f'(x) = 12x^3 - 12x^2 - 24x =$

$= 12x(x^2 - x - 2) = 12x(x+1)(x-2) = 0$

critical numbers in $[-2, 1]$

are $x = 0$ and $x = -1$

$f(-2) = 3 \cdot 16 - 4 \cdot (-8) - 12 \cdot 4 + 5 = 48 + 32 - 48 + 5$

$= 37$ abs max

$f(-1) = 3 + 4 - 12 + 5 = 0$

$f(0) = 5$

$f(1) = 3 - 4 - 12 + 5 = -8$ abs min

Abs max value is 37

Abs min value is -8

Bonus problem. [10 points extra] Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

Let $f(x) = 2x - 1 - \sin x$

$f(x)$ is continuous and differentiable for all x.

$f(0) = -1$, $f(1) = 2 - 1 - \sin 1 = 1 - \sin 1 > 0$

since $|\sin x| \le 1$

and $\sin 1 \neq 1$

By IVT there is a root of $f(x)$ on the interval $(0,1)$

Assume $f(x)$ has two roots a and b $(a < b)$ i.e. $f(a) = f(b) = 0$

By MVT there is $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

On the other hand $f'(x) = 2 - \cos x > 0$

since $\cos x \le 1$. A contradiction to the assumption about two roots.

Hence $f(x)$ has exactly one root

$\Rightarrow$ the equation $2x - 1 - \sin x$ has exactly one root.