Exam 2

Your name: Solutions

100 points total

Your TA's name:

1. Find the derivative of the function by any appropriate method.

(a) [10 points] $f(x) = \dfrac{1 + 3\ln x}{1 - 3\ln x}$

$$f'(x) = \frac{\frac{3}{x}(1-3\ln x) - \left(-\frac{3}{x}\right)(1+3\ln x)}{(1-3\ln x)^2} =$$

$$= \frac{\frac{3}{x}(1-3\ln x + 1 + 3\ln x)}{(1-3\ln x)^2} =$$

$$= \frac{6}{x(1-3\ln x)^2}$$

(b) [10 points] $y = (\sin x)^x$

$$\ln y = \ln(\sin x)^x = x \cdot \ln(\sin x)$$

$$\frac{y'}{y} = \ln(\sin x) + \frac{x}{\sin x} \cdot \cos x$$

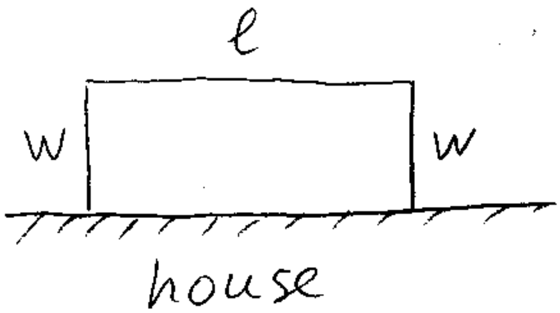
$$y' = (\sin x)^x \left[\ln(\sin x) + x \cot x\right]$$

(c) [10 points] $f(x) = \tan^{-1}(e^{2x})$ (Chain Rule)

$$f'(x) = \frac{1}{1+(e^{2x})^2} \cdot 2e^{2x}$$

$$f'(x) = \frac{2e^{2x}}{1+e^{4x}}$$

2. [15 points] Jim has 20 feet of fencing and wishes to make a rectangular fence for his dog. If he uses his house for one side of the fence what is maximum area?



$$\ell + 2w = 20$$

$$A = \ell w$$

$$\ell = 20 - 2w$$

$$A = (20-2w)w = 20w - 2w^2$$

$$A'(w) = 20 - 4w = 0 \Rightarrow w = 5$$

$$A''(w) = -4 < 0 \Rightarrow A(w) \text{ is CD } \left(\begin{array}{c}\text{for all}\\ w>0\end{array}\right)$$

$\Rightarrow$ A attains its abs. max

at $w = 5 \Rightarrow \ell = 20 - 2\cdot 5 = 10$

$$A = 5\cdot 10 = \boxed{50 \text{ sq. feet}}$$

3. Evaluate the limits

(a) [10 points] $\lim_{x\to 0} \frac{\tan x}{\sqrt{x}} \overset{H}{\underset{\text{"}\frac{0}{0}\text{"}}{=}} \lim_{x\to 0} \frac{\sec^2 x}{\frac{1}{2\sqrt{x}}} =$

$$= \lim_{x\to 0} \frac{2\sqrt{x}\cdot \sec^2 x}{1} = \frac{2\cdot 0\cdot 1^2}{1} = 0$$

(b) [10 points] $\lim_{x\to\infty} \frac{3^x}{x^3}$ $\underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{H}{=}}$ $\lim_{x\to\infty} \frac{\ln 3 \cdot 3^x}{3x^2} =$

$$= \frac{\ln 3}{3} \cdot \lim_{x\to\infty} \frac{3^x}{x^2} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{H}{=}} \frac{\ln 3}{3} \cdot \lim_{x\to\infty} \frac{\ln 3 \cdot 3^x}{2x}$$

$$= \frac{(\ln 3)^2}{6} \lim_{x\to\infty} \frac{3^x}{x} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{H}{=}} \frac{(\ln 3)^2}{6} \lim_{x\to\infty} \frac{\ln 3 \cdot 3^x}{1}$$

$$= \infty$$

4. [10 points] Find $f(x)$ in the most general form if $f''(x) = \sin x + 6$.

$$f'(x) = -\cos x + 6x + C_1$$

$$f(x) = -\sin x + 3x^2 + C_1 x + C_2$$

5. [10 points] Prove that there is a number c inside the interval $[1,4]$ where the derivative of the function $f(x) = \dfrac{x}{x+2}$ attains the value $\dfrac{1}{9}$.

$f(x)$ is continuous on $[1,4]$

(the point of discontinuity is $x = -2$

which is not in $[1,4]$)

$f(x)$ is differentiable on $[1,4]$

By MVT there is $c \in [1,4]$

such that

$$f'(c) = \frac{f(4)-f(1)}{4-1}$$

$$f(4) = \frac{4}{6} = \frac{2}{3}, \quad f(1) = \frac{1}{3}$$

$$f'(c) = \frac{\frac{2}{3}-\frac{1}{3}}{3} = \frac{\frac{1}{3}}{3} = \frac{1}{9}$$

6. [15 points] For the function

$$f(x) = \frac{x-2}{x^2} = x^{-1} - 2x^{-2}$$

find its domain, intercepts, asymptotes, intervals of increase or decrease, local maximum and minimum values, concavity, inflection points, and sketch its graph.

$D = (-\infty, 0) \cup (0, \infty)$

no y-intercepts because $x \neq 0$

x-intercept: $y = 0 \iff x - 2 = 0$, $\boxed{x = 2}$

h.a.: $\lim\limits_{x\to\infty} \frac{x-2}{x^2} \overset{H}{\underset{\text{"}\frac{\infty}{\infty}\text{"}}{=}} \lim\limits_{x\to\infty} \frac{1}{2x} = 0$

v.a.: $\lim\limits_{x\to 0^-} \frac{x-2}{x^2} = \frac{-2}{\text{small positive}} = -\infty \Rightarrow x = 0$ is v.a.

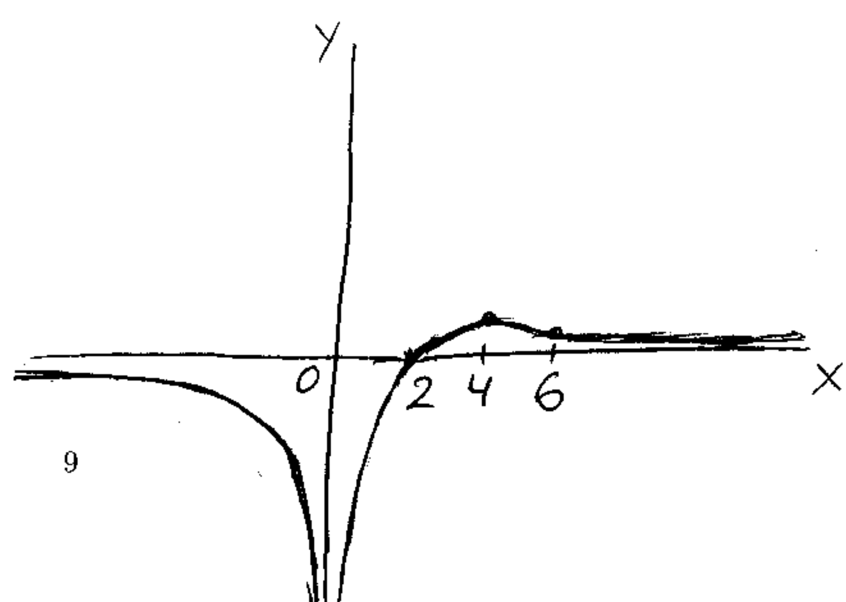
$\lim\limits_{x\to 0^+} \frac{x-2}{x^2} = \frac{-2}{\text{small positive}} = -\infty$

$f'(x) = -x^{-2} + 4x^{-3} = \frac{-x+4}{x^3}$

f'	$-$	0	$+$	4	$-$
f	$\searrow$ incr.		$\nearrow$ decr.	loc max	$\searrow$ incr.

$f''(x) = 2x^{-3} - 12x^{-4} = \frac{2(x-6)}{x^4}$

f''	$-$	0	$-$	6	$+$
f	CD $\cap$		CD $\cap$	IP	CU $\cup$



Bonus problem. [10 points extra] What approximately is $\sqrt{15.98}$? Use linear approximation to find the value.

$$f(x) = \sqrt{x} \quad , \quad a = 16$$

$$f(x) \approx L(x) = f(16) + f'(16)(x-16)$$

$$f(16) = 4 \quad , \quad f'(x) = \frac{1}{2\sqrt{x}} \quad , f'(16) = \frac{1}{8}$$

$$L(x) = 4 + \frac{1}{8}(x-16) = \frac{1}{8}x + 2$$

$$\sqrt{15.98} = f(15.98) \approx L(15.98) = 4 + \frac{1}{8}(15.98-16)$$

$$= 4 + \frac{1}{8}(-0.02) = 4 - \frac{1}{8} \cdot \frac{2}{100} = \boxed{4 - \frac{1}{400}}$$