E

Quiz 4 **Your name:** Solutions

Your TA's name:

1. Find the derivatives of the given functions

(a) [10 points] $f(x) = 2^{\sin 3x}$

$f'(x) = \ln 2 \cdot 2^{\sin 3x} \cdot \cos 3x \cdot 3$

(b) [10 points] $f(t) = e^{\sec^{-1} t}$

$f'(t) = e^{\sec^{-1} t} \cdot \frac{1}{t\sqrt{t^2-1}}$

2. [10 points] A certain radioactive isotope has a half-life of 5 years. How many years will it take for 100 grams to decay to 25 grams?

$m(t) = m_0 e^{kt}$

half-life 5 years $\Rightarrow m(5) = \frac{1}{2} m_0 \Rightarrow$

$\Rightarrow m_0 e^{5k} = \frac{1}{2} m_0 \Rightarrow e^{5k} = \frac{1}{2}$

$5k = \ln \frac{1}{2}$, $k = \frac{1}{5} \ln \frac{1}{2} = \ln \left(\frac{1}{2}\right)^{\frac{1}{5}}$

$m(t) = m_0 \cdot e^{\ln\left(\frac{1}{2}\right)^{\frac{1}{5}} \cdot t} = \left(\frac{1}{2}\right)^{t/5}$

$m_0 = 100$

$m(t) = 25 \Leftrightarrow 100 \left(\frac{1}{2}\right)^{t/5} = 25$

$\left(\frac{1}{2}\right)^{t/5} = \frac{1}{4} = \left(\frac{1}{2}\right)^2 \Rightarrow \frac{t}{5} = 2 \Rightarrow t = 10$

Answer: 10 years

3. [10 points] Use a linear approximation (or differentials) to estimate the number $\sqrt{99.8}$.

$f(x) = \sqrt{x}$, $a = 100$

$f'(x) = \frac{1}{2\sqrt{x}}$, $f'(a) = \frac{1}{20}$

tan. line (linear approximation) is

$y = \sqrt{100} + \frac{1}{20}(x-100) = 10 + \frac{x}{20} - 5$

$y = \frac{x}{20} + 5$

$\sqrt{99.8} \approx \frac{99.8}{20} + 5 = 4.99 + 5 = \boxed{9.99}$

Another way : $f(x) = \sqrt{100+x}$, $a = 0$

$x = -0.2$

$f' = \frac{1}{2\sqrt{100+x}}$, $f'(0) = \frac{1}{20}$

tan line: $y = \sqrt{100} + \frac{1}{20}(x-0) = 10 + \frac{x}{20}$

$\sqrt{99.8} \approx 10 + \frac{-0.2}{20} = 9.99$.

Bonus problem for the first midterm [10 points]

Use the Intermediate Value Theorem to show that there is a root of the equation $\sin x - 1 = -2x$ in the interval $\left(0, \frac{\pi}{2}\right)$. Provide all the necessary steps and details.

$\sin x - 1 = -2x \iff f(x) = \sin x + 2x - 1 = 0$

Need to show that there is a root

$c \in (0, \frac{\pi}{2})$ s.t. $f(c) = 0$

1. $f(x)$ is continuous on $[0, \frac{\pi}{2}]$
2. $f(0) = -1 < 0$
3. $f(\frac{\pi}{2}) = \pi - 1 > 0$

Take $N = 0$ $\left(f(0) < N < f(\frac{\pi}{2})\right)$

By IVT there is $c \in (0, \frac{\pi}{2})$ s.t.

$f(c) = 0$ and hence c is a root

of the equation

$\sin x - 1 = -2x$ in the interval $(0, \frac{\pi}{2})$