Exam 1

Math 0220 (evening) **Spring 2011**

100 points total **Student's name:** Solution

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1. Find the limit, if it exists. If the limit does not exist explain why. Show all the necessary steps. You can use any method.

(a) [5 points] $\lim_{x\to -2} \frac{\sqrt{x^2-4}-x}{2x} = \frac{\sqrt{(-2)^2-4}-(-2)}{2(-2)} = \frac{2}{-4} = -\frac{1}{2}$

(Domain of the function $f(x) = \frac{\sqrt{x^2-4}-2}{2x}$ is $x \le -2$ or $x \geq 2$. Hence $x \to -2$ means $x \to -2^-$ in this case. If your answer is DNE then it is considered as right).

(b) [5 points] $\lim_{\theta\to\pi^-} \cot\theta = \lim_{\theta\to\pi^-} \frac{\cos\theta}{\sin\theta} = -\infty$

because $\lim_{\theta\to\pi^-} \cos\theta = \cos\pi = -1$

and $\sin\theta$ is a small positive number that approaches 0 as $\theta \to \pi^-$

(c) [5 points] $\lim_{h\to 2} \dfrac{h-2}{2|h-2|}$

$$|h-2| = \begin{cases} h-2, & h \geq 2 \\ -(h-2), & h < 2 \end{cases}$$

$$\frac{h-2}{|h-2|} = \begin{cases} 1, & h \geq 2 \\ -1, & h < 2 \end{cases}$$

$$\lim_{h\to 2^-} \frac{h-2}{2|h-2|} = -\frac{1}{2} \neq \lim_{h\to 2^+} \frac{h-2}{2|h-2|}$$

Hence the limit DNE

(d) [5 points] $\lim_{x\to\infty} \dfrac{x\sin x}{x^2+1}$

$$-1 \leq \sin x \leq 1$$

$$-\frac{x}{x^2+1} \leq \frac{x\sin x}{x^2+1} \leq \frac{x}{x^2+1}$$

$$\lim_{x\to\infty}\left(-\frac{x}{x^2+1}\right) = 0, \quad \lim_{x\to\infty} \frac{x}{x^2+1} = 0$$

By the Squeeze Theorem

$$\lim_{x\to\infty} \frac{x\sin x}{x^2+1} = \boxed{0}$$

(Do not use the "degree" argument. It is not a rational function of $\dfrac{Q(x)}{P(x)}$)

2. [10 points] Calculate y'' if $y^2 - x^2 = 2x$. Do not leave y' in your answer.

Differentiate both sides w.r.t. x (implicit differentiation)

$$2yy' - 2x = 2 \quad \Rightarrow \quad y' = \frac{x+1}{y}$$

Quotient Rule: $y'' = \left(\frac{x+1}{y}\right)' = \frac{y-(x+1)y'}{y^2}$

$$y'' = \frac{y-(x+1)\frac{x+1}{y}}{y^2} = \frac{y^2-(x+1)^2}{y \cdot y^2} = \boxed{\frac{y^2-x^2-2x-1}{y^3}}$$

this answer is fine

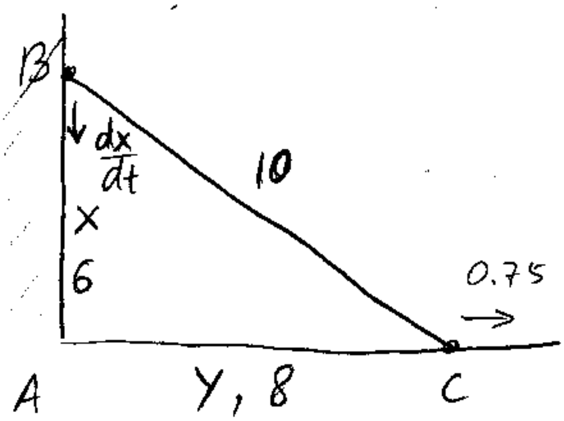
But, $y^2 - x^2 = 2x \Rightarrow y^2 - x^2 - 2x = 0$

Hence $y'' = \frac{(y^2-x^2-2x)-1}{y^3}$

$$\boxed{y'' = -\frac{1}{y^3}}$$

(No 3-story fractions in the answer)

3. [15 points] A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at the rate of 0.75 ft/sec. At what rate is the top of the ladder changing when the bottom of the ladder is 8 feet from the wall?



$$\frac{dx}{dt} = ?$$

$$0.75 \text{ ft/sec} = \frac{dy}{dt}$$

$$x^2 + y^2 = 10$$

Implicit dif-tion. DBS:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$x\frac{dx}{dt} = -y\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{y}{x}\cdot\frac{dy}{dt} \qquad y = 8, \quad x = 6 \quad \left(\sqrt{10^2 - 8^2} = 6\right)$$

$$\frac{dx}{dt} = -\frac{8}{6}\cdot 0.75 = -\frac{4}{3}\cdot\frac{3}{4} = \boxed{-1 \text{ ft/sec}}$$

4. [15 points] Find an equation of the tangent line to the curve $y = \frac{\sqrt{x}}{x+1}$ at the point $(4, 0.4)$. Write the answer in the form $y = mx + b$.

$$y' = \frac{\frac{1}{2\sqrt{x}}(x+1) - \sqrt{x}}{(x+1)^2}, \quad y'(4) = \frac{\frac{1}{4}\cdot 5 - 2}{5^2}$$

(do not simplify, go to numbers)

$$y'(4) = \frac{5-8}{4\cdot 25} = \frac{-3}{100} = -0.03$$

tan line $\quad y = 0.4 - 0.03(x-4)$

$$y = 0.4 - 0.03x + 0.12$$

$$\boxed{y = -0.03x + 0.52}$$

5. [10 points] Find the derivative of the function $y = \sin t + \pi \cos(3t^2)$.

$$\frac{dy}{dt} = \cos t + \pi \cdot 3 \cdot 2t \cdot (-\sin(3t^2))$$

$$\boxed{\frac{dy}{dt} = \cos t - 6\pi t \sin(3t^2)}$$

6. [10 points] Find the derivative of the function using the definition(!) of derivative (no credit will be given if you do not use the definition) if $f(x) = \frac{x^2}{2} - 1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{\frac{(x+h)^2}{2} - 1 - \left(\frac{x^2}{2} - 1\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(x+h)^2}{2} - 1 - \frac{x^2}{2} + 1}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(x+h)^2}{2} - \frac{x^2}{2}}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{2h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{2h} = \lim_{h \to 0} \frac{2xh + h^2}{2h}$$

$$= \lim_{h \to 0} \left(x + \frac{h}{2}\right) = x$$

$$\boxed{f'(x) = x}$$

7. Consider the function

$$f(x) = \begin{cases} \cos(\pi x), & \text{when } 0 \le x < \frac{1}{2} \\ -2x+1, & \text{when } \frac{1}{2} \le x \le 1 \end{cases}$$

(a) [10 points] Show that $f(x)$ is continuous on $[0,1]$

$\cos(\pi x)$ and $-2x+1$ are continuous functions

$\Rightarrow$ $f(x)$ is continuous if $\lim\limits_{x\to\frac{1}{2}^-} f(x) = \lim\limits_{x\to\frac{1}{2}^+} f(x) = f(\frac{1}{2})$

$\lim\limits_{x\to\frac{1}{2}^-} f(x) = \lim\limits_{x\to\frac{1}{2}^-} \cos(\pi x) = \cos(\frac{\pi}{2}) = 0$

$\lim\limits_{x\to\frac{1}{2}^+} f(x) = \lim\limits_{x\to\frac{1}{2}^+} (-2x+1) = -2\cdot\frac{1}{2}+1 = -1+1 = 0$

$f(\frac{1}{2}) = -2\cdot\frac{1}{2}+1 = 0$. Hence $f(x)$ is continuous at $\frac{1}{2}$ $\Rightarrow$ it is continuous on $[0,1]$

(b) [10 points] Using the Intermediate Value Theorem prove that there is a number c in the interval $(0,1)$ such that $f(c) = \frac{1}{2}$.

$f(x)$ is continuous on $[0,1]$ (see part (a))

$f(0) = \cos(\pi\cdot 0) = \cos 0 = 1$, $f(1) = -2\cdot 1+1 = -1$

Let $N = \frac{1}{2}$. Then $f(1) < N < f(0)$

By IVT there is c in $(0,1)$ such that

$f(c) = N = \frac{1}{2}$

Bonus problem. [10 points extra] A particle moves according to a law of motion $s = t^3 - 12t^2 + 36t$, where t is measured in seconds and s in feet, and $t \geq 0$. Find the total distance traveled during the first 8 seconds if $s(8) = 32$.

$$v(t) = s'(t) = 3t^2 - 24t + 36$$

velocity $= 0$ gives turning points

$$v(t) = 0 \iff t^2 - 8t + 12 = 0$$

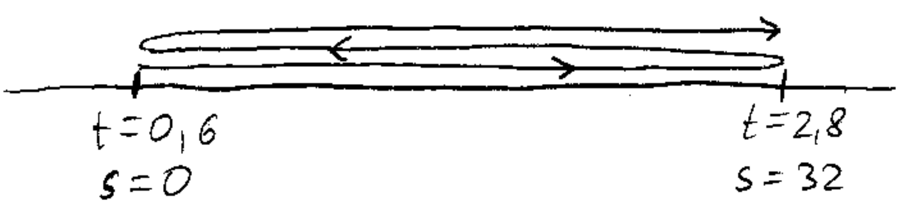
$$(t-2)(t-6) = 0 \Rightarrow t = 2,\ t = 6$$

$$s(0) = 0$$

$$s(2) = 8 - 48 + 72 = 32$$

$$s(6) = 6 \cdot 6^2 - 12 \cdot 6^2 + 6 \cdot 6^2 = 12 \cdot 6^2 - 12 \cdot 6^2 = 0$$

$$s(8) = 32 \quad (\text{given})$$



total distance is $3 \times 32 = \boxed{96\ ft}$