Exam 2

Math 0220 (evening) Spring 2011

100 points total **Student's name:** Solutions

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1. [10 points] Use a linear approximation to estimate $\frac{1}{1001}$.

$f(x) = \frac{1}{x}$. Linear approximation at $a = 1000$

$$f(1000) = \frac{1}{1000}$$

$$f'(x) = -\frac{1}{x^2}, \quad f'(1000) = -\frac{1}{1{,}000{,}000}$$

$$f(x) \approx L(x) = f(1000) + f'(1000)(x-1000)$$

$$L(x) = \frac{1}{1000} - \frac{x-1000}{1{,}000{,}000}$$

Hence $\frac{1}{1001} = f(1001) \approx L(1001) = \frac{1}{1000} - \frac{1}{1{,}000{,}000}$

$$= 0.001 - 0.000001 = 0.000999$$

$$\left[= \frac{999}{1{,}000{,}000} \right]$$

2. [15 points] The half-life of cesium-137 is 30 years. Suppose you have a 300-mg sample. After how long will only 2 mg remain?

$$m(t) = m_0 e^{kt}, \quad m_0 = 300$$

$$m(30) = \frac{1}{2} m_0 \iff 300 e^{k \cdot 30} = \frac{1}{2} \cdot 300$$

$$(e^k)^{30} = \frac{1}{2}$$

$$e^k = \left(\frac{1}{2}\right)^{1/30}$$

Hence $m(t) = 300 \cdot \left(\frac{1}{2}\right)^{t/30}$

$$m(t) = 2 \iff 300 \cdot \left(\frac{1}{2}\right)^{t/30} = 2$$

$$\left(\frac{1}{2}\right)^{t/30} = \frac{1}{150} \iff 2^{t/30} = 150$$

$$\log_2 2^{t/30} = \log_2 150 \iff \frac{t}{30} = \log_2 150$$

$$\boxed{t = 30 \log_2 150}$$

It also can be $t = 30 \cdot \frac{\ln 150}{\ln 2}$

If you found k then $k = \ln\left(\frac{1}{2}\right)^{1/30} =$

$$= \frac{1}{30} \ln\left(\frac{1}{2}\right) = -\frac{\ln 2}{30}$$

3. Find derivatives of the given functions.

(a) [6 points] $f(\theta) = \ln(2\sin\theta)$

$$f'(\theta) = \frac{1}{2\sin\theta} \cdot 2\cos\theta = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

(b) [6 points] $y = 3^{\cos(\pi x)}$

$$y' = \ln 3 \cdot 3^{\cos(\pi x)}(-\sin(\pi x)) \cdot \pi$$

$$y' = -\pi \cdot \ln 3 \cdot \sin(\pi x) \cdot 3^{\cos(\pi x)}$$

By the way, $\ln 3$ is a constant.

$\frac{d}{dx}(\ln 3) = 0$, not $\frac{1}{3}$!

(c) [6 points] $y = x^{\ln x}$ (log. differentiation)

$\ln y = \ln x \cdot \ln x = (\ln x)^2$

Differentiate both sides w.r.t. x:

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[(\ln x)^2]$$

$$\frac{y'}{y} = 2 \cdot \ln x \cdot \frac{1}{x}$$

$$y' = y \cdot 2 \cdot \ln x \cdot \frac{1}{x}$$

$$y' = x^{\ln x} \cdot \frac{2}{x} \ln x \quad [= 2x^{\ln x - 1} \cdot \ln x]$$

4. For the function $f(x) = \frac{2x^2}{x^2+3} = \frac{2x^2+6-6}{x^2+3} = 2 - \frac{6}{x^2+3} = 2-6(x^2+3)^{-1}$

(a) [5 points] Find the intervals on which f is increasing or decreasing.

$$f'(x) = -6(-1)(x^2+3)^{-2}\cdot 2x = \frac{12x}{(x^2+3)^2} \quad \left[= 12x(x^2+3)^{-2}\right]$$

$(x^2+3)^2 > 0$. The sign of f' depends on $12x$ only.

$12x = 0 \Leftrightarrow x = 0$

f': $-$ | $+$ at 0

f is increasing on $(0, \infty)$

f is decreasing on $(-\infty, 0)$

(b) [5 points] Find the local maximum and minimum values of f.

CP: $x = 0$. f is decreasing for $x < 0$ and increasing for $x > 0$ $\Rightarrow$ $f(x)$ has a loc. min. at 0.

$f(0) = \frac{0}{0+3} = 0$ is the loc. min value

(c) [5 points] Find the intervals of concavity and the inflection points.

$$f''(x) = \left[12x(x^2+3)^{-2}\right]' = 12(x^2+3)^{-2} + 12x(-2)(x^2+3)^{-3}\cdot 2x$$

$$f''(x) = \frac{12}{(x^2+3)^3}\left[x^2+3-4x^2\right] = \frac{12(3-3x^2)}{(x^2+3)^3} = \frac{36(1-x^2)}{(x^2+3)^3}$$

$1 - x^2 = 0 \Leftrightarrow x = -1$ or $x = 1$

f'': $-$ | $+$ | $-$ at -1, 1

CD IP CU IP CD

concave up on $(-1, 1)$

concave down on $(-\infty, -1) \cup (1, \infty)$

IP's $(-1, \frac{1}{2})$ and $(1, \frac{1}{2})$

5. The function $f = (x^2 - 1)^3$ is defined on the interval $[-1, 2]$.

(a) [5 points] Explain why the function attains its absolute maximum and absolute minimum values on the given interval.

1. The interval $[-1,2]$ is closed.
2. The function $f(x)=(x^2-1)^3$ is a polynomial and hence it is continuous everywhere, in particular it is continuous on $[-1,2]$.

(b) [10 points] Find the absolute maximum and the absolute minimum values of f on the interval.

$f'(x) = 3(x^2-1)^2 \cdot 2x = 6x(x^2-1)^2$

CP's : $f'(x) = 0 \iff x=0$ or $x^2-1=0$

$x=0$ or $x=-1,\ x=1$

$f'(x)$ exists everywhere

CP's are $x=0$, $x=-1$, and $x=1$

$f(-1) = 0$

$f(0) = \boxed{-1 \text{ is abs. min value}}$

$f(1) = 0$

$f(2) = \boxed{27 \text{ is abs. max value}}$

(we evaluated f at all CP's and endpoints: $-1, 0, 1, 2$)

6. For each limit define the type of indeterminate form [1 point]. Them find the limit [5 points].

(a) [6 points] $\lim_{x \to -\infty} x^2 e^x$ It is "$\infty \cdot 0$"

$$\lim_{x \to -\infty} x^2 e^x = \lim_{x \to -\infty} \frac{x^2}{e^{-x}} \underset{"\frac{\infty}{\infty}"}{\overset{H}{=}} \lim_{x \to -\infty} \frac{2x}{-e^{-x}}$$

$$\underset{"\frac{\infty}{\infty}"}{\overset{H}{=}} \lim_{x \to -\infty} \frac{2}{e^{-x}} = \boxed{0.}$$

Note, this way $\lim_{x \to -\infty} \frac{e^x}{x^{-2}} \underset{"\frac{0}{0}"}{\overset{H}{=}} \lim_{x \to -\infty} \frac{e^x}{-2x^{-3}} = -\frac{1}{2} \lim_{x \to -\infty} x^3 e^x$

does not work because the power of x increases.

(b) [6 points] $\lim_{x \to 0}(1-2x)^{1/x}$ It is "1^∞". Use ln

Let $y = (1-2x)^{1/x}$. Then $\ln y = \frac{1}{x} \ln(1-2x)$

$$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln(1-2x)}{x} \underset{"\frac{0}{0}"}{\overset{H}{=}}$$

$$= \lim_{x \to 0} \frac{\frac{-2}{1-2x}}{1} = \lim_{x \to 0} \frac{-2}{1-2x} = -2$$

Hence $\lim_{x \to 0} (1-2x)^{1/x} = \boxed{e^{-2}}$

7. [15 points] Verify that the function $f(x) = x^3 + x - 1$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

$f(x)$ is a polynomial. Hence it is continuous on the closed interval $[0,2]$ and differentiable on the open interval $(0,2)$. Hence there is c in (a,b) s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}, \quad b=2,\ a=0$$

$$f(2) = 8+2-1 = 9, \quad f(0) = -1$$

$$\left.\begin{array}{l} f'(c) = \dfrac{9-(-1)}{2-0} = \dfrac{10}{2} = 5 \\ f'(x) = 3x^2+1 \end{array}\right\} \Rightarrow 3c^2+1 = 5$$

$$3c^2 = 4$$

$$c^2 = \frac{4}{3}$$

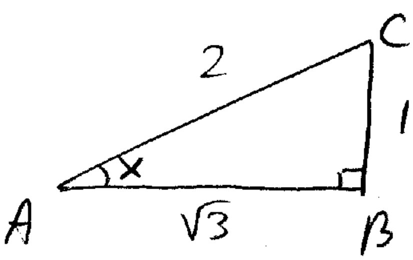
$$c = \pm \frac{2}{\sqrt{3}} = \pm \frac{2\sqrt{3}}{3}$$

$c = -\dfrac{2\sqrt{3}}{3}$ is not in the interval $(0,2)$

$$\boxed{c = \frac{2\sqrt{3}}{3}}$$

Bonus problem. [10 points extra] Find the exact value of the expression $2^{3\log_2 3} + \tan(\sin^{-1}\frac{1}{2})$.

$$2^{3\log_2 3} = 2^{\log_2 3^3} = 2^{\log_2 27} = 27$$



Let $x = \sin^{-1}\frac{1}{2} \iff \frac{1}{2} = \sin x$

(x is an angle)

Let $BC = 1$ and $AC = 2$

It makes $\sin x = \frac{opp}{hyp} = \frac{1}{2}$

(this is what we need)

Then $AB = \sqrt{2^2 - 1^2} = \sqrt{3}$

$$\tan(\sin^{-1}\tfrac{1}{2}) = \tan x = \frac{opp}{adj} = \frac{1}{\sqrt{3}}$$

(see the picture)

Answer: $\boxed{27 + \frac{1}{\sqrt{3}}}$ $\left(= 27 + \frac{\sqrt{3}}{3}\right)$

Another way: $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$