Final Exam

Math 0220 (evening) **Spring 2011**

100 points total **Student's name:** Solutions

---

1. Evaluate the integrals.

(a) [10 points] $\int\limits_0^{\pi/2} \cos^3 x\, dx = \int\limits_0^{\pi/2} \cos^2 x \cdot \cos x dx =$

$$= \int\limits_0^{\pi/2} (1 - \sin^2 x)\, d(\sin x) = \int\limits_0^{1} (1-u^2) du =$$

$u = \sin x$

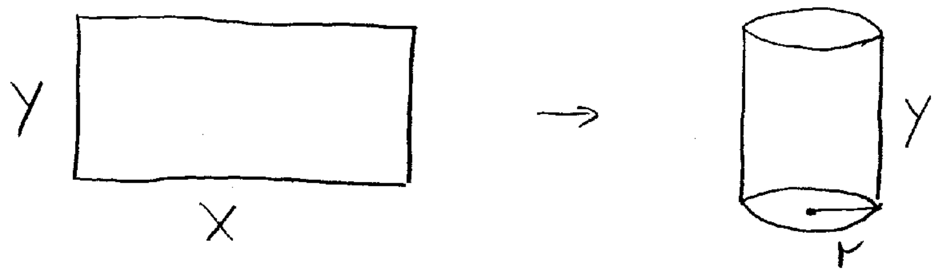
$$= \left[u - \frac{u^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

(b) [5 points] $\int 8x\, e^{2x^2}\, dx$ $= 2\int e^u du = 2e^u + C = 2e^{2x^2} + C$

$u = 2x^2$, $du = 4x\,dx$

Way 1

2. [15 points] A rectangular sheet of paper with perimeter 36 cm is to be rolled into a cylinder. What are the dimensions of the sheet that give the greatest volume?



$x + y = 18$

$y = 18 - x$

$x = 2\pi r \Rightarrow r = \frac{x}{2\pi}$

$$V = \pi r^2 y = \pi \cdot \left(\frac{x}{2\pi}\right)^2 \cdot (18 - x) = \pi \frac{x^2}{4\pi^2}(18 - x)$$

$$V(x) = \frac{1}{4\pi}(18x^2 - x^3)$$

$$V'(x) = \frac{1}{4\pi}(36x - 3x^2) = \frac{3}{4\pi}x(12 - x) = 0$$

CP's $x = 0$, $x = 12$

$0 \le x \le 18$

$V(0) = 0$

$V(12) = \frac{1}{4\pi} \cdot 144(18 - 12) = \frac{1}{4\pi} \cdot 144 \cdot 6 \rightarrow$ max

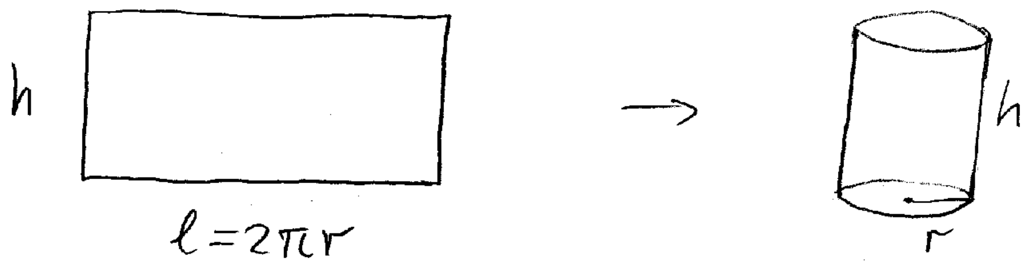
$V(18) = 0$

Max of V occurs when $x = 6 \Rightarrow y = 18 - 6 = 12$

$\boxed{\text{Answer: } 6 \text{ cm and } 12 \text{ cm}}$

Way 2

2. [15 points] A rectangular sheet of paper with perimeter 36 cm is to be rolled into a cylinder. What are the dimensions of the sheet that give the greatest volume?



$$36 = 2(h+\ell) \qquad V = \pi r^2 h$$

$$18 = h + 2\pi r \qquad V(r) = \pi r^2 (18 - 2\pi r)$$

$$h = 18 - 2\pi r \qquad V(r) = 2\pi (9r^2 - \pi r^3)$$

$$V'(r) = 2\pi(18r - 3\pi r^2) = 2\pi \cdot 3r(6 - \pi r) = 0$$

CP's: $r = 0$ or $r = \frac{6}{\pi}$

$$0 \le \ell \le 18 \Rightarrow 0 \le r \le \frac{18}{2\pi} = \frac{9}{\pi}$$

$$V(0) = 0$$

$$V\left(\frac{6}{\pi}\right) = \pi \cdot \frac{36}{\pi^2}\left(18 - 2\pi \cdot \frac{6}{\pi}\right) = \frac{36}{\pi}(18-12) = \frac{36}{\pi} \cdot 6 \rightarrow \text{max}$$

$$V\left(\frac{9}{\pi}\right) = \pi \cdot \frac{81}{\pi^2}\left(18 - 2\pi \cdot \frac{9}{\pi}\right) = \frac{81}{\pi}(18-18) = 0$$

$r = \frac{6}{\pi}$ provides the greatest volume. Then

$\ell = 2\pi \cdot \frac{6}{\pi} = 12$, $h = 18 - \ell = 6$

$\boxed{\text{Answer: 6 cm and 12 cm}}$

3. Find the limit, if it exists. If the limit does not exist explain why. Show all the necessary steps and justify your solution. You may use any method.

(a) [5 points] $\lim_{\theta \to 0^+} (1 + 3\theta)^{\cot \theta}$ It is "1^{∞}"

Let $y = (1+3\theta)^{\cot\theta}$ Then

$$\ln y = \cot\theta \cdot \ln(1+3\theta) = \frac{\ln(1+3\theta)}{\tan\theta}$$

$$\lim_{\theta\to 0^+} \ln y = \lim_{\theta\to 0^+} \frac{\ln(1+3\theta)}{\tan\theta} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{H}{=}} \lim_{\theta\to 0^+} \frac{\frac{3}{1+3\theta}}{\sec^2\theta} = \frac{\frac{3}{1}}{1} = 3$$

Hence $\lim_{\theta\to 0^+} (1+3\theta)^{\cot\theta} = e^{\lim_{\theta\to 0^+} \ln y} = \boxed{e^3}$

(b) [5 points] $\lim_{t\to -2} \frac{t+2}{3|t+2|}$

$$\frac{t+2}{3|t+2|} = \begin{cases} \frac{t+2}{3(t+2)}, & t+2>0 \\ \frac{t+2}{-3(t+2)}, & t+2<0 \end{cases} = \begin{cases} \frac{1}{3}, & t>-2 \\ -\frac{1}{3}, & t<-2 \end{cases}$$

$$\lim_{t\to -2^-} \frac{t+2}{3|t+2|} = -\frac{1}{3} \neq \frac{1}{3} = \lim_{t\to -2^+} \frac{t+2}{3|t+2|}$$

Hence the limit $\boxed{\text{DNE}}$

(c) [5 points] $\lim_{x\to\infty} \frac{\cos(2x^2)}{x}$

$$-1 \le \cos(2x^2) \le 1 \Rightarrow -\frac{1}{x} \le \frac{\cos(2x^2)}{x} \le \frac{1}{x}$$

$$\lim_{x\to\infty}\left(-\frac{1}{x}\right)=0, \quad \lim_{x\to\infty}\frac{1}{x}=0$$

By Squeeze Thm

$$\lim_{x\to\infty}\frac{\cos(2x^2)}{x} = \boxed{0}$$

4. (a) [5 points] Find the derivative of the function $g(x) = \int_2^{\sqrt{x}} \sqrt{t^2 - 1}\, dt$

$$g'(x) = \sqrt{(\sqrt{x})^2 - 1} \cdot (\sqrt{x})' = \boxed{\frac{\sqrt{x-1}}{2\sqrt{x}}} =$$

$$= \boxed{\frac{1}{2}\sqrt{\frac{x-1}{x}}} = \boxed{\frac{1}{2}\sqrt{1 - \frac{1}{x}}}$$

(All answers are good.)

(b) [10 points] Find an equation of the tangent line to the curve $y = \ln(2\tan x)$ when $x = \pi/4$. Write the answer in the form $y = mx + b$. No full credit if the form is different.

tan line: $y = y(\frac{\pi}{4}) + y'(\frac{\pi}{4})(x - \frac{\pi}{4})$

$y(\frac{\pi}{4}) = \ln(2 \cdot \tan\frac{\pi}{4}) = \ln 2$

$y'(x) = \frac{1}{2\tan x} \cdot \sec^2 x \cdot 2 \left(= \frac{1}{\sin x \cdot \cos x} = \frac{2}{\sin 2x} \right)$

$y'(\frac{\pi}{4}) = \frac{1}{2} \cdot 2 \cdot 2 = 2$

tan line: $y = \ln 2 + 2(x - \frac{\pi}{4})$

$$\boxed{y = 2x + \ln 2 - \frac{\pi}{2}}$$

$\left(m = 2, \ b = \ln 2 - \frac{\pi}{2}\right)$

5. For the function $f(x) = \dfrac{e^x}{x}$

(a) [2 points] Find its domain.

$D: \ x \neq 0$ or $x \in (-\infty, 0) \cup (0, \infty)$

(b) [5 points] Find vertical and horizontal asymptotes (if any).

v.a. $\lim\limits_{x \to 0^-} \dfrac{e^x}{x} = -\infty$, $\lim\limits_{x \to 0^+} \dfrac{e^x}{x} = \infty$

h.a. $\lim\limits_{x \to +\infty} \dfrac{e^x}{x} = +\infty$ No asymptote

$\lim\limits_{x \to -\infty} \dfrac{e^x}{x} = 0$

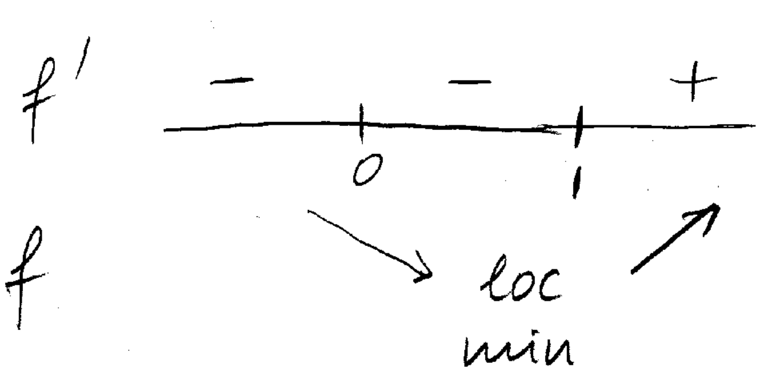
v.a. $\boxed{x=0}$, h.a. $\boxed{y=0}$ when $x \to -\infty$

(c) [5 points] Find the intervals on which f is increasing or decreasing.

$f'(x) = (x^{-1}e^x)' = -x^{-2}e^x + x^{-1}e^x = \dfrac{e^x}{x^2}(x-1)$

$f'(x)$ DNE when $x=0$

$f'(x) = 0$ when $x = 1$ CP's $x=0, x=1$



increasing on $(1, \infty)$

decreasing on $(-\infty, 0) \cup (0, 1)$

(d) [3 points] Find the local maximum and minimum values of f (f, not x!)

Loc min is at $x=1$

$f(1) = \frac{e}{1} = e$

There is no loc. max

(e) [5 points] Find the intervals of concavity [Hint: $x^2 - 2x + 2 > 0$ for any x].

$$f'(x) = e^x\left(\frac{x-1}{x^2}\right) = e^x(x^{-1} - x^{-2})$$

$$f''(x) = e^x(x^{-1} - x^{-2}) + e^x(-x^{-2} + 2x^{-3}) =$$

$$= e^x(x^{-1} - x^{-2} - x^{-2} + 2x^{-3}) = e^x(x^{-1} - 2x^{-2} + 2x^{-3})$$

$$= e^x\,\frac{x^2 - 2x + 2}{x^3}\,, \quad x^2 - 2x + 2 > 0\,, \quad x^3 = 0 \Leftrightarrow x = 0$$

f'' $-$ | $+$

        0

f CD   CU

CD on $(-\infty, 0)$

CU on $(0, \infty)$

6. (a) [15 points] Find the average value f_{ave} of the function $f = x \sin \pi x$ on the interval $[-1, 1]$.

$$f_{ave} = \frac{1}{1-(-1)} \int_{-1}^{1} x \sin \pi x \, dx = \frac{1}{2} \int_{-1}^{1} x \sin \pi x \, dx$$

$u = x \quad dv = \sin \pi x \, dx$

$du = dx \quad v = -\frac{1}{\pi} \cos \pi x$

$$f_{ave} = \frac{1}{2} \left[-\frac{x}{\pi} \cos \pi x \Big|_{-1}^{1} + \frac{1}{\pi} \int_{-1}^{1} \cos \pi x \, dx \right] =$$

$$= \frac{1}{2} \left[-\frac{1}{\pi} \cos \pi - \frac{1}{\pi} \cos(-\pi) + \frac{1}{\pi^2} \sin \pi x \Big|_{-1}^{1} \right] =$$

$$= \frac{1}{2} \left[\frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi^2} \cdot 0 \right] = \boxed{\frac{1}{\pi}}$$

(b) [5 points] Is there a number c inside the interval $(-1, 1)$ such that $f(c) = f_{ave}$? Support your answer.

Yes, since $f(x)$ is a continuous function on $(-1, 1)$ and by applying the Mean Value Thm for Integrals