11-11:50am **Midterm Exam 1**

Fall 2012 Math 0220

100 points total **S o l u t i o n s**

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1. (10 points) Evaluate the limit, if it exists. If it does not exist explain why.

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{3x-15}$$

In your work mention what Rules, Laws, Theorems or Formulas you use.

Solution:

$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{3x-15} = \lim_{x\to 5} \frac{\sqrt{x-1}-2}{3x-15} \cdot \frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}$$

$$= \lim_{x\to 5} \frac{x-1-4}{3(x-5)(\sqrt{x-1}+2)} = \lim_{x\to 5} \frac{x-5}{3(x-5)(\sqrt{x-1}+2)} = \lim_{x\to 5} \frac{1}{3(\sqrt{x-1}+2)}$$

$$\overset{DSP}{=} \frac{1}{3(\sqrt{5-1}+2)} = \frac{1}{12}.$$

2. (10 points) Find the first and second derivatives of the function

$$g(x) = \sqrt{2x^2-1}$$

Simplify your answer. In your work mention what Rules, Laws, Theorems or Formulas you use.

Solution:

$$g(x) = (2x^2-1)^{1/2}$$

$$g'(x) = \text{[Chain and Power rules]}\ \frac{1}{2}(2x^2-1)^{-1/2}(4x) = 2x(2x^2-1)^{-1/2} = \frac{2x}{\sqrt{2x^2-1}}$$

$$g''(x) = \left(2x(2x^2-1)^{-1/2}\right)' \quad \text{[Product, Chain, and Power rules]}$$

$$= 2(2x^2-1)^{-1/2} + 2x\left(-\frac{1}{2}\right)(2x^2-1)^{-3/2}(4x)$$

$$= 2(2x^2-1)^{-1/2} - 4x^2(2x^2-1)^{-3/2}$$

$$= \frac{2}{\sqrt{2x^2-1}} - \frac{4x^2}{(\sqrt{2x^2-1})^3} = \frac{2(2x^2-1)-4x^2}{(\sqrt{2x^2-1})^3} = -\frac{2}{(\sqrt{2x^2-1})^3}$$

3. (10 points) Use implicit differentiation to find an equation of the tangent line to the curve $x^{2/3}+y^{2/3}=5$ at the point $(8,1)$. Write the answer in the slope-intercept form.

Solution: Tangent line equation is $y = 1 + y'(8)(x-8)$. To find $y'(x)$ we differentiate the curve equation w.r.t. x (remember y is a function of x):

$$\left(\frac{2}{3}\right)\left(x^{-1/3}\right) + \left(\frac{2}{3}\right)\left(y^{-1/3}\right) y'(x) = 0.$$

At the point $(8,1)$ we have $\left(\frac{2}{3}\right)\left(8^{-1/3}\right) + \left(\frac{2}{3}\right)\left(1^{-1/3}\right) y'(8) = 0,$

$\frac{2}{3}\cdot\frac{1}{2} + \frac{2}{3}\cdot y'(8) = 0,\ \frac{1}{2} + y'(8) = 0.$

Then $y'(8) = -\frac{1}{2}.$

Tangent line equation is $y = 1 - \frac{1}{2}(x-8) = 1 - \frac{1}{2}x + 4$ or $y = -\frac{1}{2}x + 5.$

4. (10 points) Sketch the graph of an example of a function $g(x)$ if it satisfies all the given conditions

$$g(0) = -2, \quad g'(0) = 1, \lim_{x\to 1^-} g(x) = -1, \quad \lim_{x\to 1^+} g(x) = 1, \quad g(1) = 0$$
$$g(2) = 1, \quad g'(2) = -1, \quad g'(5) = 1 \text{ and } \lim_{x\to\infty} g(x) = 0.$$

Mark all the essential points on the axes.

Solution: See the graph.

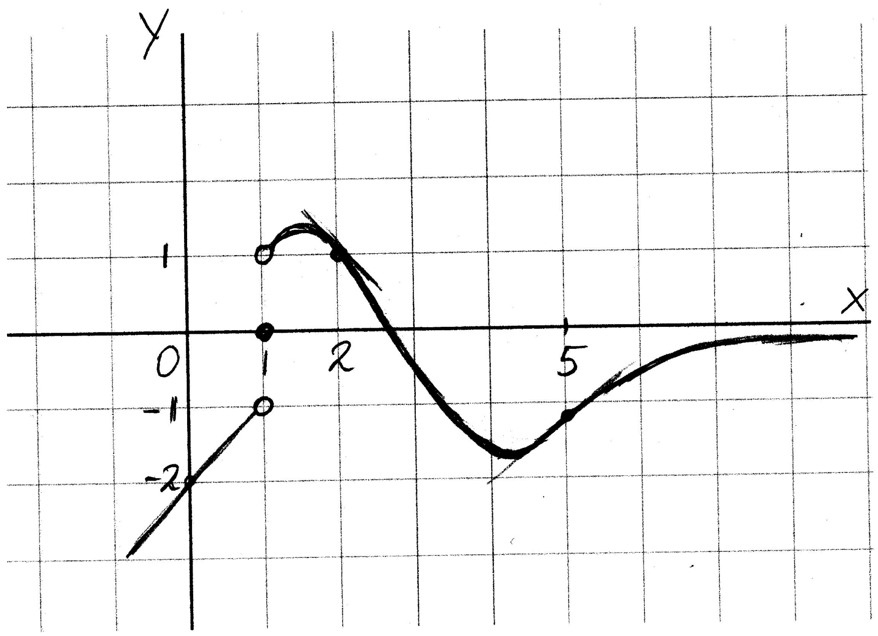


Figure 1: Problem 4

5. (10 points) Use the Intermediate Value Theorem to show that there is root of the equation $4 - x^2 = \sin x$ in the interval $(0, \pi)$. Support every step of your proof.

Solution: The given equation is equivalent to $4 - x^2 - \sin x = 0$. Let $f(x) = 4 - x^2 - \sin x$. The function $f(x)$ is continuous on the interval $[0, \pi]$, $f(0) = 4 > 0$, $f(\pi) = 4 - \pi^2 - 0 = 4 - \pi^2 < 0$. Nence $f(\pi) < 0 < f(0)$. By the Intermediate Value Theorem with $N = 0$ there is $c \in (0, \pi)$ such that $f(c) = N = 0$ or $4 - c^2 - \sin c = 0$, or $4 - c^2 = \sin c$. The last means that c is the root of the equation $4 - x^2 = \sin x$ inside the interval $(0, \pi)$.

6. (10 points) Find all horizontal asymptotes of the curve

$$y = \frac{\sqrt{4x^2 + 2}}{2x + 6}$$

Justify your answer by calculating corresponding limits. [Use $\sqrt{x^2} = -x$ when $x < 0$.]

Solution: We need to find limits at positive and negative infinities:

$$\lim_{x\to\infty}\frac{\sqrt{4x^2+2}}{2x+6}=\lim_{x\to\infty}\frac{\sqrt{4x^2+2}}{2x+6}\cdot\frac{1/x}{1/x}=\lim_{x\to\infty}\frac{\sqrt{4+2/x^2}}{2+6/x}=\frac{\sqrt{4}}{2}=1,$$

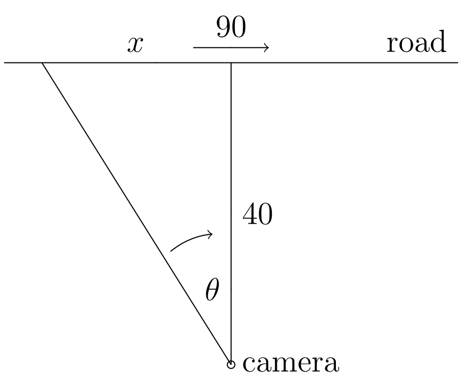
$$\lim_{x\to-\infty}\frac{\sqrt{4x^2+2}}{2x+6}=\lim_{x\to-\infty}\frac{\sqrt{4x^2+2}}{2x+6}\cdot\frac{1/(-x)}{1/(-x)}=\lim_{x\to\infty}\frac{\sqrt{(4x^2+2)/x^2}}{(2x+6)/(-x)}$$

$$=\lim_{x\to\infty}\frac{\sqrt{4+2/x^2}}{-2-6/x}=\frac{\sqrt{4}}{-2}=-1$$

Hence horizontal asymptotes are $y=1$ and $y=-1$.

7. (15 points) A camera is located 40 feet away from a straight road along which a car is traveling with a constant speed of 90 feet per second. The camera turns so that it is pointed at the car at all times. In radians per second, how fast is the camera turning as the car passes closest to the camera?

Solution:



We need to find $\frac{d\theta}{dt}$.

The relation between $x(t)$ and $\theta(t)$ is given by $\tan\theta=\frac{x}{40}$. Differentiate it with respect to t:

$\sec^2\theta\cdot\frac{d\theta}{dt}=\frac{1}{40}\cdot\frac{dx}{dt}$. It is given, that $\frac{dx}{dt}=90$.

When the car passes closest to the camera $\theta=0$ and $\sec 0=1$.

Then $\frac{d\theta}{dt}=\frac{90}{40}$ or $\frac{d\theta}{dt}=\frac{9}{4}$ rad/sec.

8. (10 points) Use a linear approximation to estimate the number $\sqrt{3.99}$.

Solution: Consider the function $f(x) = \sqrt{x}$. Its linearization at the point $a = 4$ is $L(x) = f(4) + f'(4)(x-4)$, where $f(4) = \sqrt{4} = 2$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $f'(4) = \dfrac{1}{2 \cdot 2} = \dfrac{1}{4}$.

Hence $L(x) = 2 + \dfrac{1}{4}(x-4)$.
Using the linear approximation we find:

$$\sqrt{3.99} = f(3.99) \approx L(3.99) = 2 + \frac{1}{4}(3.99-4) = 2 + \frac{1}{4}\left(-\frac{1}{100}\right) = 2 - \frac{1}{400} = 1\,\frac{399}{400}.$$

9. (15 points) Use differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a hemispherical dome with diameter 120 cm.

Solution: Let V be the volume of the semisphere. The amount of paint needed is ΔV which is approximately dV.

$$V = \frac{1}{2} \cdot \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3. \qquad dV = \frac{dV}{dr}\,dr = \frac{2}{3}\pi\,(3r^2)\,dr = 2\pi r^2 dr,$$

where $dr \approx \Delta r = 0.02$ cm and $r = 120/2 = 60$ cm.

Hence the amout of paint needed is approximately

$$dV = 2\pi \cdot 60^2 \cdot \frac{2}{100} = 2\pi \cdot 36 \cdot 100 \cdot \frac{2}{100} = 2\pi \cdot 36 \cdot 2 = 144\pi \text{ cm}^3.$$

bonus problem [10 pts] Find the 21st derivative $f^{(21)}(x)$ of the function $f(x) = \sin^2 x - \cos^2 x$.

Solution: $f(x) = -\cos 2x$, $f'(x) = 2\sin 2x$, $f''(x) = 2^2 \cos 2x$, $f'''(x) = -2^3 \sin 2x$, $f^{(4)}(x) = -2^4 \cos 2x = 2^4 f(x)$.

This means that every fourth derivative is 2^4 times the function itself. Since $20 = 5 \cdot 4$ then $f^{(20)}(x) = 2^{20} f(x)$ and $f^{(21)}(x) = 2^{20} f'(x) = 2^{20} \cdot 2\sin 2x$ or $f^{(21)}(x) = 2^{21}\sin 2x$.