12-12:50pm

Midterm Exam 1

Fall 2012 Math 0220

100 points total

S o l u t i o n s

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1. (10 points) Find an equation of the tangent line to the curve $y = \dfrac{2}{(x-1)^2}$ at the point $(2, 2)$. Write the answer in the slope-intercept form.

   Solution: Tangent line equation is $y = 2 + y'(2)(x-2)$.
   $y = 2(x-1)^{-2}$, $y'(x) = -4(x-1)^{-3}$, $y'(2) = -4(2-1)^{-3} = -4$
   Tangent line equation is $y = 2 - 4(x-2)$ or $y = -4x + 10$.

2. (10 points) Evaluate the limit, if it exists. If it does not exist explain why.

$$\lim_{t \to -6} \frac{t+6}{2\,|t+6|}$$

   In your work mention what Rules, Laws, Theorems or Formulas you use.

   Solution:

$$|t+6| = \begin{cases} t+6, & t+6 \geq 0 \\ -(t+6), & t+6 < 0 \end{cases}$$

   or

$$|t+6| = \begin{cases} t+6, & t \geq -6 \\ -(t+6), & t < -6 \end{cases}$$

$$\lim_{t \to -6^-} \frac{t+6}{2\,|t+6|} = \lim_{t \to -6,\ t<-6} \frac{t+6}{-2(t+6)} = \lim_{t \to -6^-} \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

$$\lim_{t \to -6^+} \frac{t+6}{2\,|t+6|} = \lim_{t \to -6,\ t>-6} \frac{t+6}{2(t+6)} = \lim_{t \to -6^+} \frac{1}{2} = \frac{1}{2} \neq \lim_{t \to -6^-} \frac{t+6}{2\,|t+6|}$$

   Hence, the given limit does not exist.

3. (10 points) Sketch the graph of an example of a function $g(x)$ if it satisfies all the given conditions

$$g(0) = 2, \quad g'(0) = -1, \quad \lim_{x\to 1^-} g(x) = 1, \quad \lim_{x\to 1^+} g(x) = -\infty,$$
$$g(3) = 1, \quad g'(3) = 1, \quad g'(5) = -1/2 \text{ and } \lim_{x\to\infty} g(x) = 0.$$

Mark all the essential points on the axes.

Solution: See the graph.

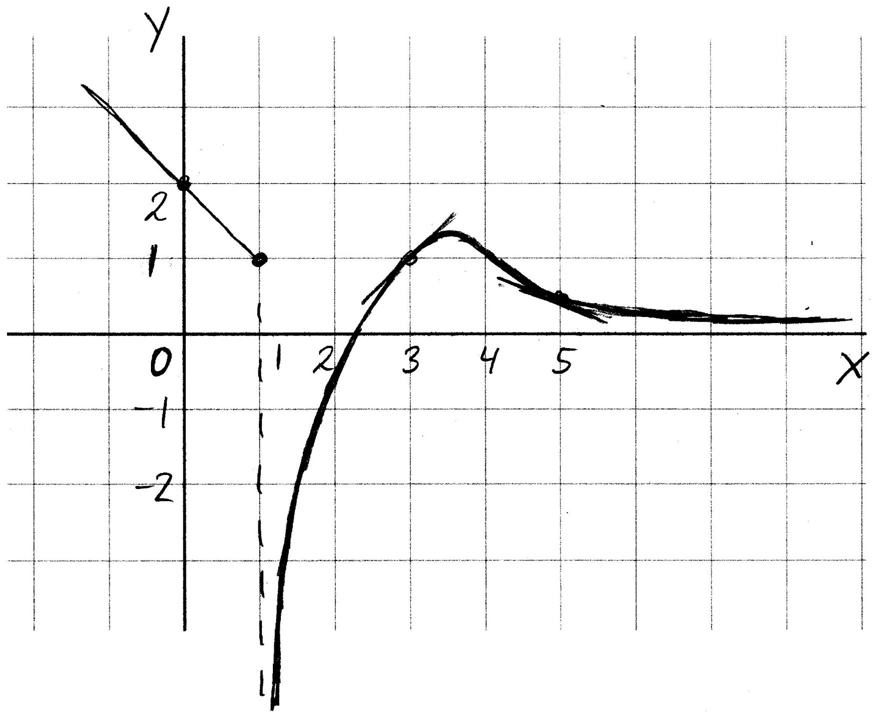


Figure 1: Problem 3

4. (10 points) Find the derivative of the function

$$f(x) = \frac{\sqrt{\cos x}}{2x^2 - 1}$$

Do not simplify your answer. In your work mention what Rules, Laws, Theorems or Formulas you use.

Solution: Quotient, Chain, and Power rules:

$$f'(x) = \frac{(\sqrt{\cos x})'(2x^2 - 1) - \sqrt{\cos x}(2x^2 - 1)'}{(2x^2 - 1)^2}$$

$$= \frac{\frac{-\sin x}{2\sqrt{\cos x}}(2x^2 - 1) - \sqrt{\cos x} \cdot 4x}{(2x^2 - 1)^2}$$

$$= \frac{-\frac{1}{2}(\cos x)^{-1/2} \sin x (2x^2 - 1) - 4x\sqrt{\cos x}}{(2x^2 - 1)^2}$$

$$= -\frac{(2x^2 - 1)\sin x + 8x \cos x}{2\sqrt{\cos x}(2x^2 - 1)^2}$$

All these answers are acceptable.

5. (10 points) A particle moves according to a law of motion $s(t) = 3t^2 - 12t + 2$, $t \geq 0$, where t is measured in seconds and s in feet.

(a) (5 points) What is its velocity after 3 seconds?

(b) (5 points) When is the particle at rest?

(c) (5 points) Find the total distance traveled during the first 5 seconds.

Solution: (a) $v(t) = s'(t) = 6t - 12$, $v(3) = 18 - 12 = 6$ ft/sec.

(b) $v(t) = 0 \quad \Leftrightarrow \quad 6t - 12 = 0 \quad \Leftrightarrow \quad t = 2$ sec.

(c) The only turning point is at $t = 2$.

The distance traveled is $d = |s(2) - s(0)| + |s(2) - s(0)|$

$s(0) = 2$, $s(2) = 3 \cdot 4 - 12 \cdot 2 + 2 = -10$, $s(5) = 75 - 60 + 2 = 17$

$d = |-10 - 2| + |17 - (-10)| = 12 + 27 = 39$ feet.

6. (10 points) Find all horizontal asymptotes of the curve

$$y = \frac{\sqrt{9x^2+5}}{x+6}$$

Justify your answer by calculating corresponding limits. [Use $\sqrt{x^2} = -x$ when $x < 0$.]

Solution: We need to find limits at positive and negative infinities:

$$\lim_{x\to\infty} \frac{\sqrt{9x^2+5}}{x+6} = \lim_{x\to\infty} \frac{\sqrt{9x^2+5}}{x+6} \cdot \frac{1/x}{1/x} = \lim_{x\to\infty} \frac{\sqrt{9+5/x^2}}{1+6/x} = \frac{\sqrt{9}}{1} = 3,$$

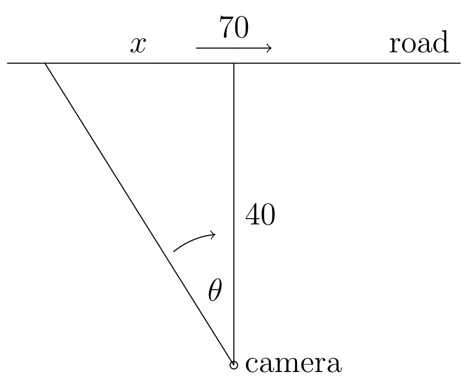
$$\lim_{x\to-\infty} \frac{\sqrt{9x^2+5}}{x+6} = \lim_{x\to-\infty} \frac{\sqrt{9x^2+5}}{x+6} \cdot \frac{1/(-x)}{1/(-x)} = \lim_{x\to\infty} \frac{\sqrt{(9x^2+5)/x^2}}{(x+6)/(-x)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{9+5/x^2}}{-1-6/x} = \frac{\sqrt{9}}{-1} = -3$$

Hence horizontal asymptotes are $y = 3$ and $y = -3$.

7. (15 points) A camera is located 40 feet away from a straight road along which a car is traveling with a constant speed of 70 feet per second. The camera turns so that it is pointed at the car at all times. In radians per second, how fast is the camera turning as the car passes closest to the camera?

Solution:



We need to find $\frac{d\theta}{dt}$.

The relation between $x(t)$ and $\theta(t)$ is given by $\tan\theta = \frac{x}{40}$. Differentiate it with respect to t:

$\sec^2\theta \cdot \dfrac{d\theta}{dt} = \dfrac{1}{40} \cdot \dfrac{dx}{dt}$. It is given, that $\dfrac{dx}{dt} = 70$.

When the car passes closest to the camera $\theta = 0$ and $\sec 0 = 1$.

Then $\dfrac{d\theta}{dt} = \dfrac{70}{40}$ or $\dfrac{d\theta}{dt} = \dfrac{7}{4}$ rad/sec.

8. (10 points) Use a linear approximation to estimate the number $(1.007)^8$.

Solution: Consider the function $f(x) = x^8$. Its linearization at the point $a = 1$ is $L(x) = f(1) + f'(1)(x-4)$, where $f(1) = 1$, $f'(x) = 8x^7$, $f'(1) = 8$.

Hence $L(x) = 1 + 8(x-1)$.

Using the linear approximation we find:

$(1.007)^8 = f(1.007) \approx L(1.007) = 1 + 8(1.007 - 1) = 1 + 8 \cdot 0.007 = 1 + 0.056 = 1.056$.

9. (15 points) Use differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a hemispherical dome with diameter 120 cm.

Solution: Let V be the volume of the semisphere. The amount of paint needed is ΔV which is approximately dV.

$V = \dfrac{1}{2} \cdot \dfrac{4}{3}\pi r^3 = \dfrac{2}{3}\pi r^3$. $\qquad dV = \dfrac{dV}{dr}\,dr = \dfrac{2}{3}\pi\,(3r^2)\,dr = 2\pi r^2 dr$,

where $dr \approx \Delta r = 0.02$ cm and $r = 120/2 = 60$ cm.

Hence the amout of paint needed is approximately

$dV = 2\pi \cdot 60^2 \cdot \dfrac{2}{100} = 2\pi \cdot 36 \cdot 100 \cdot \dfrac{2}{100} = 2\pi \cdot 36 \cdot 2 = 144\pi \text{ cm}^3$.

bonus problem [10 points extra] Find the 21st derivative $f^{(21)}(x)$ of the function $f(x) = \sin x \cos x$.

Solution: $f(x) = \frac{1}{2}\sin 2x$, $f'(x) = \cos 2x$, $f''(x) = -2\sin 2x$, $f'''(x) = -2^2\cos 2x$,

$f^{(4)}(x) = 2^3 \sin 2x = 2^4 f(x)$.

This means that every fourth derivative is 2^4 times the function itself. Since $20 = 5 \cdot 4$ then

$f^{(20)}(x) = 2^{20} f(x)$ and

$f^{(21)}(x) = 2^{20} f'(x)$ or $f^{(21)}(x) = 2^{20}\cos 2x = 2^{20}(\cos^2 x - \sin^2 x)$.