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Midterm Exam 2

Fall 2012 Math 0220

100 points total

S o l u t i o n s

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1. Find $(f^{-1})'(2)$ if $f(x) = x^5 - x^3 + 2x$. [Hint: $f(1) = 2$].

Solution: $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))}$. To use the formula we need to find $f'(x)$ and $f^{-1}(2)$.

$f'(x) = 5x^4 - 3x^2 + 2$. Denote $f^{-1}(2) = x$. This means $f(x) = 2 \;\Rightarrow\; x = 1$.

Then $f'(f^{-1}(2)) = f'(1) = 5 - 3 + 2 = 4$

and $(f^{-1})'(2) = \dfrac{1}{4}$

2. Use logarithmic differentiation to find y' if $y = \dfrac{\sqrt{x}\, e^{3x}}{(x^3 - x)^9}$.

Solution: $\ln y = \ln\left(\dfrac{\sqrt{x}\, e^{3x}}{(x^3 - x)^9}\right) = \ln\left(x^{1/2}\right) + \ln\left(e^{3x}\right) - \ln\left((x^3 - x)^9\right)$.

$$\ln y = \frac{1}{2}\ln x + 3x - 9\ln(x^3 - x).$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\frac{1}{2}\ln x + 3x - 9\ln(x^3 - x)\right).$$

$$\frac{y'}{y} = \frac{1}{2x} + 3 - 9\cdot\frac{3x^2 - 1}{x^3 - x}, \quad y' = y\left(\frac{1}{2x} + 3 - 9\cdot\frac{3x^2 - 1}{x^3 - x}\right).$$

$$y' = \frac{\sqrt{x}\, e^{3x}}{(x^3 - x)^9}\left(\frac{1}{2x} + 3 - 9\cdot\frac{3x^2 - 1}{x^3 - x}\right).$$

3. A puppy weighs 2 pounds at birth and 4 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 5 months old? Simplify your answer.

Solution: Let $m(t)$ be the mass of the puppy after t months. Then

$m(t) = m(0)\, e^{kt}$, $m(0) = 2$, and $m(2) = m(0)\, e^{2k} = 2e^{2k} = 4$.

Solve the last equation for e^k: $e^{2k} = \dfrac{4}{2}$, $(e^k)^2 = 2$, $e^k = 2^{1/2}$.

Hence, $e^{kt} = 2^{t/2}$ and $m(t) = 2 \cdot 2^{t/2}$.

$m(3) = 2 \cdot 2^{5/2} = 2 \cdot 4\sqrt{2} = 8\sqrt{2}$ pounds.

4. Verify that the function $f(x) = \dfrac{x}{x+1}$ satisfies the hypotheses of the MVT on the interval $[0, 8]$. Then find all numbers c that satisfy the conclusion of the MVT.

Solution: The only discontinuity that $f(x)$ has is at -1. Hence $f(x)$ is continuous on $[0, 8]$. It is a rational function and hence differentiable on $(0, 8)$. Therefore, $f(x)$ satisfies the hypotheses of the MVT on the interval $[0, 8]$.

By the MVT there is a number c such that $f'(c) = \dfrac{f(8) - f(0)}{8 - 0} = \dfrac{8/9 - 0}{8} = \dfrac{1}{9}$.

$f'(x) = \dfrac{x + 1 - x}{(x+1)^2} = \dfrac{1}{(x+1)^2}$.

Therefore, to find c we need to solve the equation $\dfrac{1}{(c+1)^2} = \dfrac{1}{9} \Leftrightarrow (c+1)^2 = 9$.

Then $c + 1 = 3$ or $c + 1 = -3$. Two solutions $c = 2$ and $c = -4$. The last is not in $(0, 8)$.

Hence, $c = 2$.

5. For the function $f(x) = \dfrac{x^2 - 1}{x}$ find its domain, intervals and types of concavity.

Solution: The domain is $D = (-\infty, 0) \bigcup (0, \infty)$ (or $x \neq 0$).

$f(x) = \dfrac{x^2 - 1}{x} = x - x^{-1}$, $f'(x) = 1 + x^{-2}$, $f''(x) = -2x^{-3} = -\dfrac{2}{x^3}$.

The function is concave upward when $x < 0$ or x is in $(-\infty, 0)$ ($f''(x) > 0$).

The function is concave downward when $x > 0$ or x is in $(0, \infty)$ ($f''(x) < 0$).

6. A rectangular box with a square base and no top is to have a volume of 4 cubic inches. Find the dimensions for the box that require the least amount of material.

Solution: Let the bottom have sides x and x and y be the hight. Then the volume is $V = x^2 y = 4$ and the surface area that corresponds to the amount of material is

$A = f(x) = x^2 + 4xy$ (area of the bottom is x^2 and the area of four sides is $4xy$).

From the equation for the volume we obtain $y = \dfrac{4}{x^2}$. Then $f(x) = x^2 + 4x \cdot \dfrac{4}{x^2}$.

The problem is to minimize the function $f(x) = x^2 + \dfrac{16}{x}$, when $x > 0$.

CNs: $f'(x) = 2x - \dfrac{16}{x^2} = \dfrac{2(x^3 - 8)}{x^2}$. $f'(x) = 0$ whenever $x^3 - 8 = 0$ or $x = 2$.

$f'(x)$ DNE when $x = 0$ which is not in the domain of the function. Hence $f(x)$ has the only CN at 2.

The second derivative test: $f''(x) = 2 + \dfrac{32}{x^3} > 0$ for all $x > 0$. Specifically $f''(2) > 0$. Hence there is a local minimum at 2. The function is concave up on its domain and has the only local minimum. Therefore this local minimum at $x = 2$ must be the absolute minimum.

Hence $x = 2$ in. and $y = \dfrac{4}{2^2} = 1$ in.

The dimensions for the box that require the least amount of material are 2 in. $\times$ 2 in. of the bottom and 1 in. of the hight.

7. For the equation $x^2 = 6$ use Newton's method with the initial approximation $x_1 = 2$ to find the third approximation x_3 to the positive root.

Solution: $x^2 = 6 \Leftrightarrow x^2 - 6 = 0$. Let $f(x) = x^2 - 6$. To find an approximation of a root of the equation $x^2 = 6$ which is the root of the equation $f(x) = 0$ we apply Newton's method.

$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, with $x_1 = 2$.

$$f'(x) = 2x \quad \Rightarrow \quad x_{n+1} = x_n - \frac{x_n^2 - 6}{2x_n} = \frac{x_n^2 + 6}{2x_n} = \frac{x_n}{2} + \frac{3}{x_n}$$

$$x_2 = \frac{x_1}{2} + \frac{3}{x_1} = \frac{2}{2} + \frac{3}{2} = \frac{5}{2}$$

$$x_3 = \frac{x_2}{2} + \frac{3}{x_2} = \frac{5}{4} + \frac{3 \cdot 2}{5} = \frac{49}{20} = 2.45$$

bonus problem. Does there exist a differentiable everywhere function f such that $f(1) = 10$, $f(5) = -3$, and $f'(x) \geq -3$ for all x?

Solution: Such a function does not exist.

Indeed, assume it exists. It is given that the function is differentiable everywhere and hence it is continuous. Specifically it is continuous on $[1, 5]$ and differentiable on $[1, 5]$. By the MVT the equality $f(5) - f(1) = f'(c)(5 - 1)$ or $f(5) = f(1) + 4f'(c)$ must hold for some c in $(1, 5)$. It is given that $f'(x) \geq -3$ for all x and hence $f'(c) \geq -3$. Then it must be true that $f(5) = f(1) + 4f'(c) \geq f(1) + 4(-3) = 10 - 12 = -2$ or $f(5) \geq -2$. This contradcts to the fact that $f(5) = -3$. The last means that the assumtion was wrong and the function does not exist.

Another way: By MVT there must be c in $(1, 5)$ such that

$f'(c) = \dfrac{-3-10}{5-1} = -\dfrac{13}{4} = -3.25 < -3$ which contradicts to $f(x) \geq -3$ for all x.